FROM MULTI-INSTANTON EXPANSION TO EXACT RESULTS

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Introduction

In quantum field theory. the main analytic tool to determine physical quantities is perturbation theory. However, simple arguments indicate that the perturbative expansion is always divergent. As a consequence, when the interaction is not weak enough the plain expansion is useless.

Unlike convergent series, divergent series do not determine, in general, a unique analytic function. The question is then how much does the perturbative expansion determine a field theory?

Borel summability. An exception is provided by the class of divergent Borel summable series to which correspond a unique function.

The questions then become, which field theories generate Borel summable expansions? What kind of additional information should one derive from the field theory to complement the expansion when it is not Borel summable?

Large order behaviour. Using the quartic anharmonic potential as a toy model, Bender and Wu were able to determine the large order behaviour of perturbation theory exactly but with methods that do not generalize to quantum field theory.

In the late seventies, following Lipatov idea, the large order behaviour of perturbation theory in quantum mechanics and a number of quantum field theories could then determined using functional integral methods.

In quantum field theory like in quantum mechanics with analytic potentials, the large order behaviour is related to the generalized barrier penetration amplitude in the semi-classical limit when the parameters in the interaction take physical or non-physical values (real or complex). The latter is determined by instantons, finite action solutions of the Euclidean equations of motion.

However, in renormalizable quantum field theories (like $\phi^4_{d=4}$), additional contributions are generated by UV or IR divergences (renormalons).

The large order behaviour gives only indications of possible Borel summability, but can show that a series is non-Borel summable: This happens in the case of physical barrier penetration.

The simplest examples of non-Borel summable series is provided by the spectrum of Hamiltonians involving analytic potentials with degenerate minima in quantum mechanics. The question then is how can one determine the exact spectrum by weak interaction expansions?

In this talk, the form of the hyperasymptotic semi-classical expansion of low-lying energy levels in the example of the quartic double-well is described and it is indicated how it yields the exact spectrum.

Such an expansion involves an infinite number of perturbative series.

However, these series can be derived from generalized Bohr–Sommerfeld quantization formulae, which involve only a few spectral functions.

The properties of the hyperasymptotic expansion has been later linked to Ecalle's theory of resurgent functions, as shown by Pham's collaborators.

The form of the hyperasymptotic expansion has been initially conjectured on the basis of a semi-classical evaluation by the steepest descent method of the path integral representation of the partition function.

The infinite number of quasi-saddle points or multi-instantons, yields contributions that can be summed exactly at leading order.

The same strategy could still be useful in problems for which our present understanding is more limited.

Finally, in quantum mechanics these properties can also been understood within the framework of the complex WKB expansion (Voros) of the solutions of the Schrödinger equation.

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Hyperasymptotic expansions and generalized Bohr–Sommerfeld quantization formulae

Remark. The results we describe here, mainly generalize to polynomial potentials and, on a case by case study, to some other analytic potentials like periodic potentials of cosine potential type.

Perturbative knowledge. Perturbative expansions are obtained by first approximating the potential by a harmonic potential near its minimum. This leads to generically divergent expansions in powers of $\hbar$ for energy eigenvalues of order $\hbar$.

For potentials with degenerate minima, expansions in powers of $\hbar$ are non-Borel summable because all terms have, at large orders, the same sign and the Borel transform has singularities on the integration contour.

Moreover, additional contributions of order $\exp(-\text{const.}\ /\hbar)$, generated by quantum tunnelling between minima, have at least to be added.

The quartic double-well potential

We explain explicitly the conjecture in the example of the quartic double-well potential. The symbol g plays the role of $\hbar$ and energy eigenvalues are measured in units of $\hbar$, a normalization adapted to perturbative expansions.

The Hamiltonian corresponding to the double-well potential can be written as

$$H = -\frac{g}{2}\left(\frac{\mathrm{d}}{\mathrm{d}q}\right)^2 + \frac{1}{g}V(q), \quad V(q) = \frac{1}{2}q^2(1-q)^2.$$

The potential is symmetric in $q \leftrightarrow (1-q)$ and has two degenerate minima at $q = 0, 1$. The Hamiltonian thus commutes with the reflection operator,

$$P\psi(q) = \psi(1-q) \ \Rightarrow [H, P] = 0\,.$$

The eigenfunctions and eigenvalues of H satisfy

$$H\psi_{\epsilon,N}(q) = E_{\epsilon,N}(g)\psi_{\epsilon,N}(q), \quad P\psi_{\epsilon,N}(q) = \epsilon\psi_{\epsilon,N}(q),$$

where $\epsilon = \pm 1$ and $E_{\epsilon,N}(g) = N + 1/2 + O(g)$.

The initial conjectures (Zinn-Justin 1983)

We have conjectured that the eigenvalues $E_{\epsilon,N}(g)$ have an exact semi-classical expansion of the form

$$E_{\epsilon,N}(g) = \sum_{0}^{\infty} E^{(0)}_{N,l} g^l$$
$$+ \sum_{n=1}^{\infty} \left(\frac{2}{g}\right)^{Nn} \left(-\epsilon \frac{e^{-1/6g}}{\sqrt{\pi g}}\right)^n \sum_{k=0}^{n-1} \left(\ln(-2/g)\right)^k \sum_{l=0}^{\infty} \mathcal{E}_{N,nkl} g^l .$$

The series $\sum \mathcal{E}_{N,nkl} g^l$ in powers of g are not Borel summable for $g > 0$ but can be summed by Borel transformation for g negative, where $\ln(-g)$ is real.

One then proceeds by analytic continuation to $g > 0$ consistently for the series and $\ln(-g)$.

In the analytic continuation, the Borel sums become complex with imaginary parts exponentially smaller by about a factor $e^{-1/3g}$ than the real parts. These imaginary contributions are cancelled by the perturbative imaginary parts coming from the function $\ln(-2/g)$.

Generalized Bohr–Sommerfeld quantization formula. We have also conjectured that all the series are generated by an expansion for g small of a spectral equation or generalized Bohr–Sommerfeld quantization formula, which in the case of the double-well potential reads ($\epsilon = \pm 1$)

$$\frac{1}{\Gamma(\frac{1}{2} - B)} + \frac{\epsilon i}{\sqrt{2\pi}} \left(-\frac{2}{g}\right)^{B(E,g)} e^{-A(E,g)/2} = 0$$

with

$$B(E, g) = -B(-E, -g) = E + \textstyle\sum_{k=1} g^k b_{k+1}(E),$$
$$A(E, g) = -A(-E, -g) = \frac{1}{3g} + \textstyle\sum_{k=1} g^k a_{k+1}(E).$$

The coefficients $b_k(E)$, $a_k(E)$ are odd or even polynomials in E of degree k. The three first orders, for example, are

$$B(E,g) = E + g\left(3E^2 + \tfrac{1}{4}\right) + g^2\left(35E^3 + \tfrac{25}{4}E\right) + O\left(g^3\right),$$
$$A(E,g) = \tfrac{1}{3}g^{-1} + g\left(17E^2 + \tfrac{19}{12}\right) + g^2\left(227E^3 + \tfrac{187}{4}E\right) + O\left(g^3\right).$$

The function $B(E,g)$ can be inferred from the complex WKB perturbative expansion. The function $A(E,g)$ has initially been determined at this order by a combination of analytic and numerical calculations.

However, more recently, it has been proved (Dunne and Unsal) for the double well and cosine potentials, using differential equation techniques, the simple relation

$$\frac{\partial E}{\partial B} = -6Bg - 3g^2\frac{\partial A}{\partial g},$$

which reduces the determination of both functions to the determination of the perturbative spectral function $B(E,g)$.

The n-instanton contributions at leading order

Replacing the functions A and B by their leading terms, one obtains

$$\frac{e^{-1/6g}}{\sqrt{2\pi}}\left(-\frac{2}{g}\right)^{E} = -\frac{\epsilon i}{\Gamma(\frac{1}{2}-E)} \Leftrightarrow \frac{\cos\pi E}{\pi} = \epsilon i \frac{e^{-1/6g}}{\sqrt{2\pi}}\left(-\frac{2}{g}\right)^{E}\frac{1}{\Gamma(\frac{1}{2}+E)}\,.$$

Expanding the equation in powers of $e^{-1/6g}$, one recovers terms that, in the path integral representation, correspond to the successive multi-instanton contributions at leading order, as illustrated later.

For example, the term proportional to $e^{-1/6g}$,

$$E_N^{(1)}(g) = -\frac{\epsilon}{N!}\left(\frac{2}{g}\right)^{N+1/2}\frac{e^{-1/6g}}{\sqrt{2\pi}}\left(1+O(g)\right),$$

can be identified with the one-instanton contribution at leading order.

The next term, (the two-instanton contribution), is ($\psi = (\ln\Gamma)'$)

$$E_N^{(2)}(g) = \frac{1}{(N!)^2}\left(\frac{2}{g}\right)^{2N+1}\frac{e^{-1/3g}}{2\pi}\left[\ln(-2/g) - \psi(N+1) + O\left(g\ln g\right)\right].$$

More generally, the nth power, which can be identified with the n-instanton contribution at leading order, has the form

$$E_N^{(n)}(g) = (-1)^n \left(\frac{2}{g}\right)^{n(N+1/2)} \left(\frac{\mathrm{e}^{-1/6g}}{\sqrt{2\pi}}\right)^n \left[P_n^{(N)}(\ln(-2/g)) + O\left(g\,(\ln g)^{n-1}\right)\right],$$

in which $P_n^N(\sigma)$ is a polynomial of degree $(n-1)$. For example, for $N=0$ one finds (γ is Euler's constant)

$$P_1^{(0)}(\sigma) = 1\,, \quad P_2^{(0)}(\sigma) = \sigma + \gamma\,, \quad P_3^{(0)}(\sigma) = \frac{3}{2}\,(\sigma+\gamma)^2 + \frac{\pi^2}{12} \cdots .$$

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An application: Large order behaviour of perturbation series

After an analytic continuation from g negative to g positive, the Borel sums become complex with an imaginary part exponentially smaller by about a factor $\mathrm{e}^{-1/3g}$ than the real part.

Consistently, the function $\ln(-2/g)$ also becomes complex with an imaginary part $\pm i\pi$. Since the sum of all contributions is real, imaginary parts must cancel.

For example, the non-perturbative imaginary part of the Borel sum of the perturbation series cancels the perturbative imaginary part of the two-instanton contribution. For the ground state,

$$\operatorname{Im} E^{(0)}(g) \underset{g\to 0}{\sim} \frac{1}{\pi g}\,\mathrm{e}^{-1/3g}\,\operatorname{Im}\left[P_2^{(0)}\bigl(\ln(-2/g)\bigr)\right] = -\frac{1}{g}\,\mathrm{e}^{-1/3g}\,.$$

The coefficients of the perturbative expansion

$$E^{(0)}(g) = \sum_k E_k^{(0)} g^k$$

of the ground state energy, are related to the imaginary part by a Cauchy integral ($k > 1$):

$$E_k^{(0)} = \frac{1}{\pi} \int_0^\infty \mathrm{Im}\left[E^{(0)}(g)\right] \frac{\mathrm{d}g}{g^{k+1}}.$$

For $k \to \infty$, the integral is dominated by small g values. Thus,

$$E_k^{(0)} \underset{k\to\infty}{\sim} -\frac{1}{\pi} \int_0^\infty \frac{\mathrm{e}^{-1/3g}}{g^{k+2}} \mathrm{d}g = -\frac{1}{\pi} 3^{k+1} k!\,.$$

Similarly, since $\operatorname{Im} E^{(1)}(g)$ and $\operatorname{Im} E^{(3)}(g)$ cancel at leading order,

$$\operatorname{Im} E^{(1)}(g) \sim 3\pi \left(\frac{\mathrm{e}^{-1/6g}}{\sqrt{\pi g}} \right)^3 \left[\ln(2/g) + \gamma + O(g \ln(g)) \right].$$

The coefficients of the expansion

$$E^{(1)}(g) = -\frac{1}{\sqrt{\pi g}} \, \mathrm{e}^{-1/6g} \left(1 + \sum^{\infty} E_k^{(1)} g^k \right)$$

are given by the dispersion integral

$$E_k^{(1)} = -\frac{1}{\pi} \int_0^\infty \left\{ \operatorname{Im} \left[E^{(1)}(g) \right] \sqrt{\pi g} \, \mathrm{e}^{1/6g} \right\} \frac{\mathrm{d}g}{g^{k+1}}.$$

Combining both equations, one finds

$$E_k^{(1)} \sim -\frac{3}{\pi} \int_0^\infty \left(\ln \frac{2}{g} + \gamma \right) \mathrm{e}^{-1/3g} \, \frac{\mathrm{d}g}{g^{k+2}} \sim -\frac{3^{k+2}}{\pi} k! \, (\ln 6k + \gamma) \, .$$

Both results have been verified numerically by calculating many terms of the corresponding series.

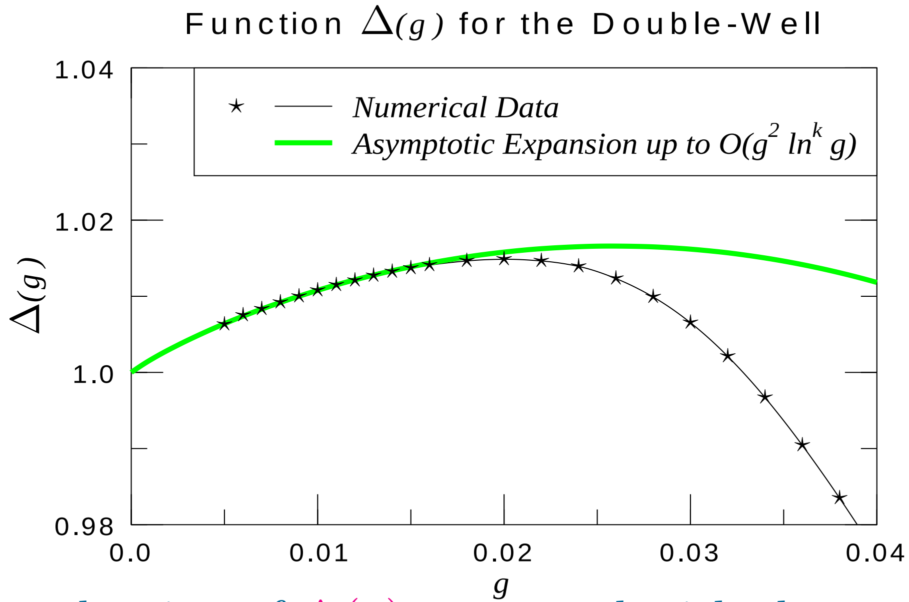

Fig. 1 Numerical evaluation of $\Delta(g)$ compared with the asymptotic expansion.

The real part of the two-instanton contribution

Another numerical test is provided by the evaluation of the ratio,

$$\Delta(g) = 4 \frac{\left\{\frac{1}{2}\left(E_{0,+} + E_{0,-}\right) - \mathrm{Re}\left[\text{Borel sum } E^{(0)}(g)\right]\right\}}{\left(E_{0,+} - E_{0,-}\right)^2 \left(\ln 2g^{-1} + \gamma\right)} = 1 + 3g + \cdots,$$

which is dominated for $g \ll 1$ by the two-instanton contribution.

Asymmetric wells

For a potential with two degenerate asymmetric wells,

$$V(q) = \tfrac{1}{2}\omega_1^2 q^2 + O(q^3), \quad V(q) = \tfrac{1}{2}\omega_2^2 (q-q_0)^2 + O\left((q-q_0)^3\right),$$

the conjectured spectral equation has the form

$$\frac{1}{\Gamma(\frac{1}{2}-B_1)\Gamma(\frac{1}{2}-B_2)} + \frac{1}{2\pi}\left(-\frac{2C_1}{g}\right)^{B_1(E,g)}\left(-\frac{2C_2}{g}\right)^{B_2(E,g)} \mathrm{e}^{-A(g,E)} = 0\,,$$

where $B_1(E,g)$ and $B_2(E,g)$ are determined by the perturbative expansions around the two minima of the potential

$$B_1(E,g) = E/\omega_1 + O(g), \quad B_1(E,g) = E/\omega_2 + O(g),$$

and the constants C_1 and C_2 are adjusted in such a way that

$$A(E,g) - a/g = O(g), \quad a = 2\int_0^{q_0} \mathrm{d}q\,\sqrt{2V(q)}\,.$$

From the poles of Γ-functions for $g \to 0$, one sees that the spectral equation yields two sets of energy eigenvalues,

$$E_N = \left(N + \tfrac{1}{2}\right) \omega_1 + O(g)\,, \quad E_N = \left(N + \tfrac{1}{2}\right) \omega_2 + O(g).$$

The same expression contains the instanton contributions to the two different sets of eigenvalues.

One verifies that multi-instanton contributions are singular for $\omega = 1$ but the spectral equation is regular in the symmetric limit.

One-instanton contribution and large order behaviour. The spectral equation can again be used to infer the large order behaviour of perturbation theory. Setting $\omega_1 = 1$, $\omega_2 = \omega$, for the energy $E_N(g) = N + \frac{1}{2} + O(g)$ one infers that the coefficients E_{Nk} of the perturbative expansion of $E_N(g)$ behave, for order $k \to \infty$, like

$$E_{Nk} \underset{k \to \infty}{=} K_N \frac{\Gamma(k + (N + 1/2)(1 + 1/\omega))}{a^{k + (N+1/2)(1+1/\omega)}} \left(1 + O\left(k^{-1}\right)\right).$$

Another analytic potential: the periodic cosine potential

The cosine potential is still an entire function but no longer a polynomial. On the other hand the periodicity of the potential simplifies the analysis, because eigenfunctions can be classified according to their behaviour under a translation of one period T,

$$\psi_\varphi(q+T) = \mathrm{e}^{i\varphi}\,\psi_\varphi(q).$$

For the potential $\frac{1}{16}(1-\cos 4q)$ (and thus $T=\pi/2$), the conjecture then takes the form

$$\left(\frac{2}{g}\right)^{-B}\frac{\mathrm{e}^{A(E,g)/2}}{\Gamma(\frac{1}{2}-B)}+\left(\frac{-2}{g}\right)^{B}\frac{\mathrm{e}^{-A(g,E)/2}}{\Gamma(\frac{1}{2}+B)}=\frac{2\cos\varphi}{\sqrt{2\pi}}.$$

Multi-Instanton contributions: Direct evaluation

The conjectures were initially motivated by a summation of leading order multi-instanton contributions. The method may be worth recalling since it could still be useful for other, less understood, problems.

Partition function and resolvent

The quantum partition function can be expressed as a path integral. The partition function for a Hamiltonian H with discrete spectrum, has the expansion

$$\mathcal{Z}(\beta) \equiv \operatorname{tr} \mathrm{e}^{-\beta H} = \sum_{N \geq 0} \mathrm{e}^{-\beta E_N}, \quad E_N \geq E_{N-1}.$$

The trace $G(E)$ of the resolvent of H (after analytic continuation and possible subtraction) is given by

$$G(E) = \operatorname{tr} \frac{1}{H-E} = \sum_N \frac{1}{E_N - E} = \int_0^\infty \mathrm{d}\beta \; \mathrm{e}^{\beta E} \, \mathcal{Z}(\beta)\,.$$

The Fredholm determinant $\mathcal{D}(E) \propto \det(H - E)$, which vanishes on the spectrum, follows

$$\frac{\partial}{\partial E} \ln \mathcal{D}(E) = -G(E).$$

For a symmetric double-well potential, it is convenient to consider the two projected partition functions,

$$\mathcal{Z}_{\pm}(\beta) = \mathrm{tr}\left[\tfrac{1}{2}(1 \pm P)\, \mathrm{e}^{-\beta H}\right] = \sum_{N=0} \mathrm{e}^{-\beta E_{\pm,N}},$$

where P is the reflection operator. The eigenvalues of H are then poles of

$$G_{\epsilon}(E) = \int_0^{\infty} \mathrm{d}\beta\ \mathrm{e}^{\beta E}\, \mathcal{Z}_{\epsilon}(\beta), \quad \epsilon = \pm 1$$

and zeros of $\mathcal{D}_{\epsilon}(E)$ with

$$\frac{\partial}{\partial E} \ln \mathcal{D}_{\epsilon}(E) = -G_{\epsilon}(E).$$

For the periodic cosine potential, one uses a generalized partition function with twisted boundary conditions depending on a rotation angle.

Path integral and spectrum of the double-well potential

In the path integral formulation of quantum mechanics, the partition function is given by a summation over closed paths,

$$\mathcal{Z}(\beta) \propto \int_{q(-\beta/2)=q(\beta/2)} [\mathrm{d}q(t)] \exp\left[-\frac{1}{g}\int_{-\beta/2}^{\beta/2} \left[\tfrac{1}{2}\dot{q}^2(t) + V\big(q(t)\big)\right] \mathrm{d}t\right],$$

We need also

$$\mathcal{Z}_\mathrm{a}(\beta) \equiv \mathrm{tr}\left(P\,\mathrm{e}^{-\beta H}\right),$$

which is obtained by a path integral with the boundary conditions $q(-\beta/2) = P(q(\beta/2))$.

Then, eigenvalues corresponding to symmetric and antisymmetric eigenfunctions can be inferred from the combinations

$$\mathcal{Z}_\pm(\beta) = \mathrm{tr}\left[\tfrac{1}{2}(1 \pm P)\,\mathrm{e}^{-\beta H}\right] = \tfrac{1}{2}\big(\mathcal{Z}(\beta) \pm \mathcal{Z}_\mathrm{a}(\beta)\big).$$

Perturbative expansion: Steepest descent method

For $g \to 0$, the path integral can be evaluated by the steepest descent method. Saddle points are solutions $q_c(t)$ to the Euclidean equations of motion. When the potential has a unique minimum, for example, located at $q = 0$, the leading saddle point is $q_c(t) \equiv 0$. A systematic expansion around the saddle point then yields the perturbative expansion of the eigenvalues of the Hamiltonian.

In the case of the symmetric double-well potential, one must sum over the two saddle points. To each saddle point corresponds one eigenvalue and thus all eigenvalues are twice degenerate to all orders in perturbation theory:

$$E_{\pm,N}(g) = E_N^{(0)}(g) \equiv \sum_{n=0}^{\infty} E_{N,n}^{(0)} g^n .$$

One-Instanton

Eigenvalues can be inferred from the large β expansion. For $\beta \to \infty$, leading contributions to the path integral come from finite action solutions of the Euclidean equations of motion.

In the case of $\mathcal{Z}_{\mathrm{a}}(\beta)$, constant solutions do not satisfy the boundary conditions. Finite action solutions (instantons) necessarily correspond to paths that connect the two minima of the potential (see Fig. 2).

In the example of the quartic double-well potential, such solutions are

$$q_{\mathrm{c}}(t) = \left(1 + \mathrm{e}^{\pm(t-t_0)}\right)^{-1} \;\Rightarrow\; \mathcal{S}(q_{\mathrm{c}}) = 1/6\,.$$

Since the two solutions depend on an integration constant t_0 (the instanton position), one finds two one-parameter families of degenerated saddle points.

The corresponding contribution to the path integral is proportional, at leading order for $g \to 0$ and for $\beta \to \infty$, to $\mathrm{e}^{-1/(6g)}$ and thus is non-perturbative.

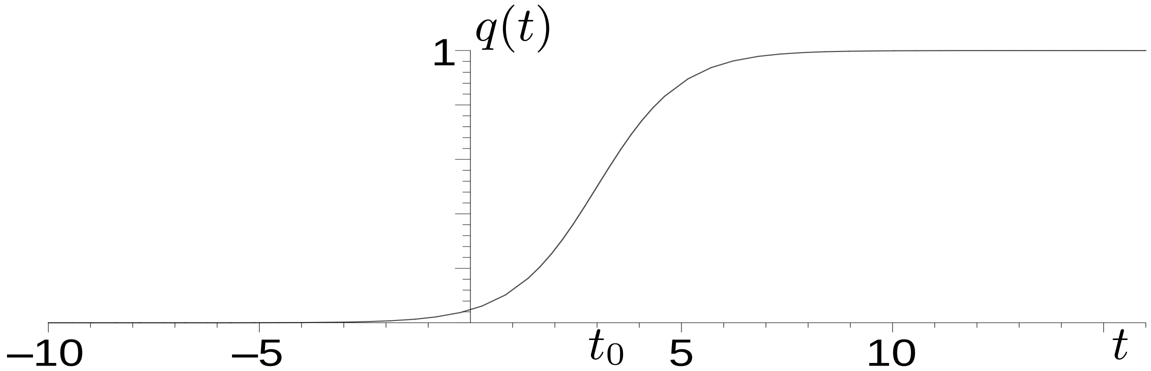

Fig. 2 The instanton configuration.

The complete calculation involves integrating exactly over the time t_0 (the collective coordinate), which for β finite varies in $[0, \beta]$, and over the remaining fluctuations in the Gaussian approximation. The two lowest eigenvalues are given by ($\epsilon = \pm 1$)

$$E_{\epsilon,0}(g) = \lim_{\beta\to\infty} -\frac{1}{\beta} \ln \mathcal{Z}_\epsilon(\beta) \underset{g\to 0,\beta\to\infty}{=} E_0^{(0)}(g) - \epsilon E_0^{(1)}(g),$$

$$E_0^{(1)}(g) = \frac{1}{\sqrt{\pi g}}\, \mathrm{e}^{-1/6g} \bigl(1 + O(g)\bigr).$$

Multi-instantons

For β finite, one finds subleading saddle points, which correspond to oscillations in the well of the potential $-V(q)$. For $\beta \to \infty$, the action of the solutions with n oscillations goes to $n \times 1/6$.

However, the amplitude of the saddle point contribution diverges for $\beta \to \infty$. Indeed, the classical solutions decompose into a succession of largely separated instantons and fluctuations that change the distances between instantons induce only infinitesimal variations of the action.

Therefore, one has to sum over all configurations of largely separated instantons, connected in a smooth way, which become solutions of the equation of motion only asymptotically, for infinite separation. They depend on n collective coordinates, the positions of instantons.

Because they are not exact solutions, the action has a dependence on the collective coordinates, called instanton interaction.

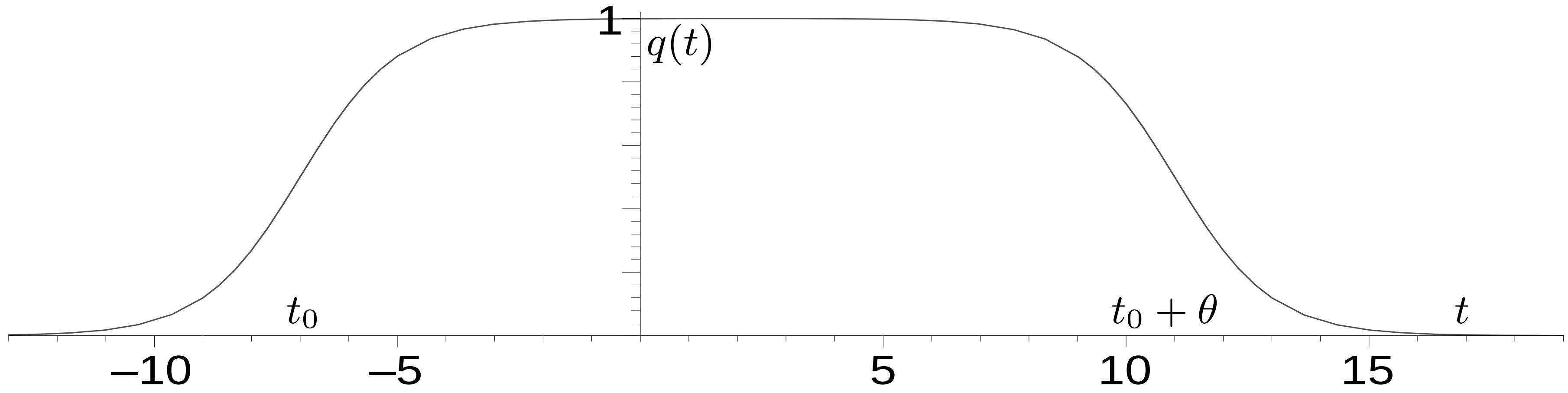

Fig. 3 A two-instanton configuration.

Example: the two-instanton configurations

In the infinite β limit, the one-instanton configuration can be written as

$$q_{\pm}(t) = f\left(\mp(t - t_0)\right), \; f(t) \equiv 1/\left(1 + \mathrm{e}^{t}\right) = 1 - f(-t),$$

where the constant t_0 characterizes the instanton position.

One verifies that a configuration $q_{\rm c}(t)$ that is the sum of instantons separated by a distance θ, up to an additive constant adjusted in such a way as to satisfy the boundary conditions (Fig. 3),

$$q_{\rm c}(t) = f(t-\theta/2) + f(-t-\theta/2) - 1 = f(t-\theta/2) - f(t+\theta/2),$$

has the required properties: it is differentiable and for θ large, but fixed, it minimizes the variation of the action. The corresponding action is

$$\mathcal{S}(q_c) = \tfrac{1}{3} - 2\,{\rm e}^{-\theta} + O\left({\rm e}^{-2\theta}\right).$$

The contributions to the classical action of order ${\rm e}^{-2\theta}$ give only a correction of order g.

For β large, but finite, symmetry between θ and $(\beta-\theta)$ implies

$$\mathcal{S}(q_c) = \tfrac{1}{3} - 2\,{\rm e}^{-\theta} - 2\,{\rm e}^{-(\beta-\theta)} + \text{negligible contributions.}$$

The n-instanton action

For a succession of n instantons (more precisely, alternatively instantons and anti-instantons) separated by times θ_i with

$$\sum_{i=1}^{n}\theta_i = \beta\,,$$

the classical action $\mathcal{S}_{\mathrm{c}}(\theta_i)$ is then

$$\mathcal{S}_{\mathrm{c}}(\theta_i) = \frac{n}{6} - 2\sum_{i=1}^{n} \mathrm{e}^{-\theta_i} + O\left(\mathrm{e}^{-(\theta_i+\theta_j)}\right).$$

At leading order, for $\theta_i \gg 1$, it is a sum of nearest-neighbour interactions. For n even, the n-instanton configurations contribute to $\operatorname{tr}\mathrm{e}^{-\beta H}$, while for n odd they contribute to $\operatorname{tr}\left(P\,\mathrm{e}^{-\beta H}\right)$ (P is the reflection operator).

The n-instanton contribution

The evaluation, at leading order, of the contribution to the path integral of the neighbourhood of the n-instanton configuration is simple but slightly technical. One finds that the n-instanton contribution to the combination

$$\mathcal{Z}_\epsilon(\beta) = \tfrac{1}{2}\,\mathrm{tr}\left[(1+\epsilon P)\,\mathrm{e}^{-\beta H}\right],$$

($\epsilon = \pm 1$), can be written as

$$\mathcal{Z}_\epsilon^{(n)}(\beta) = \mathrm{e}^{-\beta/2}\,\frac{\beta}{n}\left(\epsilon\frac{\mathrm{e}^{-1/6g}}{\sqrt{\pi g}}\right)^n \int_{\substack{\theta_i \ge 0 \\ \sum \theta_i = \beta}} \prod_i \mathrm{d}\theta_i \exp\left(\frac{2}{g}\sum_{i=1}^{n} \mathrm{e}^{-\theta_i}\right).$$

Neglecting the instanton interaction and summing over n one recovers the one-instanton approximation to the energy eigenvalues.

Beyond the one-instanton approximation: a problem. If one examines the classical action for multi-instantons, one discovers that the interaction between instantons is attractive. Therefore, for g small, the dominant contributions to the integral come from configurations in which the instantons are close.

For such configurations, the concept of instanton is no longer meaningful, since the configurations cannot be distinguished from fluctuations around the constant or the one-instanton solutions.

Such a difficulty should have been expected.

In the case of potentials with degenerate minima the perturbative expansion is not Borel summable and the series determines eigenvalues only up to exponentially decreasing terms that are of the order of two-instanton contributions. But if the perturbative expansion is ambiguous at the two-instanton order, n-instanton contributions with $n \geq 2$ cannot be defined.

To proceed any further, it is necessary to first define the sum of the perturbative expansion.

In the example of the quartic double-well potential, one can show that the perturbation series is Borel summable for g negative and this is the way we define the sum of the perturbative expansion.

Simultaneously, for g negative, the interaction between instantons becomes repulsive and the multi-instanton contributions become meaningful.

Therefore, we first calculate, for g small and negative, both the sum of the perturbation series and the multi-instanton instanton contributions, and then perform an analytic continuation to $g = |g| \pm i0$ of all quantities consistently.

The sum of leading order instanton contributions

We assume that initially g is negative and calculate the sum of leading n-instanton contributions to the trace of the resolvents,

$$\mathcal{G}_\epsilon(E) = \sum_{n=1} \int_0^\infty \mathrm{d}\beta \ \mathrm{e}^{\beta E} \ \mathcal{Z}_\epsilon^{(n)}(\beta),$$

where

$$\mathcal{Z}_\epsilon^{(n)}(\beta) \sim \frac{\beta}{n} \mathrm{e}^{-\beta/2} \left(\frac{\epsilon}{\sqrt{2\pi}}\right)^n \mathrm{e}^{-n/6g} \int_{\sum \theta_i = \beta}^{\theta_i \geq 0} \prod_{i=1}^n \mathrm{d}\theta_i \exp\left[\frac{2}{g}\sum_{i=1}^n \mathrm{e}^{-\theta_i}\right].$$

The integration over β is immediate, the integrals over the θ_i then factorize. Evaluating the unique integral for $g \to 0_-$, summing over n and adding the resolvent of the harmonic oscillator, one finds for the resolvent $G_\epsilon(E)$ a result consistent with the conjectures:

$$G_\epsilon(E) = -\frac{\partial}{\partial E} \ln \mathcal{D}_\epsilon(E) \ \Rightarrow \ \mathcal{D}_\epsilon(E) = \frac{1}{\Gamma(\frac{1}{2} - E)} + \epsilon i \left(-\frac{2}{g}\right)^E \frac{\mathrm{e}^{-1/6g}}{\sqrt{2\pi}}.$$

Perturbative and WKB expansions from Schrödinger equations

These conjectures, motivated by semi-classical evaluations of path integrals (instanton calculus), have been confirmed by considerations based on the Schrödinger equation,

$$[H\psi](q) \equiv -\frac{g}{2}\psi''(q) + \frac{1}{g}V(q)\psi(q) = E\psi(q),$$

where the potential V is an entire function. This allows extending the Schrödinger equation and its solutions to the q-complex plane.

Setting

$$S(q) = -g\psi'(q)/\psi(q),$$

one derives a Riccati equation from the Schrödinger equation. It reads

$$gS'(q) - S^2(q) + 2V(q) - 2gE = 0\,.$$

One decomposes

$$S(q) = S_+(q) + S_-(q) \text{ where, formally, } S_\pm(q;g,E) = \pm S_\pm(q;-g,-E)\,.$$

Then,

$$gS'_- - S_+^2 - S_-^2 + 2V(q) - 2gE = 0\,,\ \ gS'_+ - 2S_+S_- = 0\,.$$

The quantization condition (or spectral equation) can then be written as

$$-\frac{1}{2i\pi g}\oint_C \mathrm{d}z\, S_+(z,E) = N + \tfrac{1}{2}\,,$$

where N is also the number of real zeros of the eigenfunction, and C is a contour that encloses them. This elegant formulation, restricted, however, to one dimension and analytic potentials, bypasses the difficulties generally associated with turning points.

It allows a smooth transition between WKB expansion ($g \to 0$, gE fixed), in our normalization, and perturbative expansion ($g \to 0$, E fixed), which can be derived by expanding the WKB expansion at E fixed.

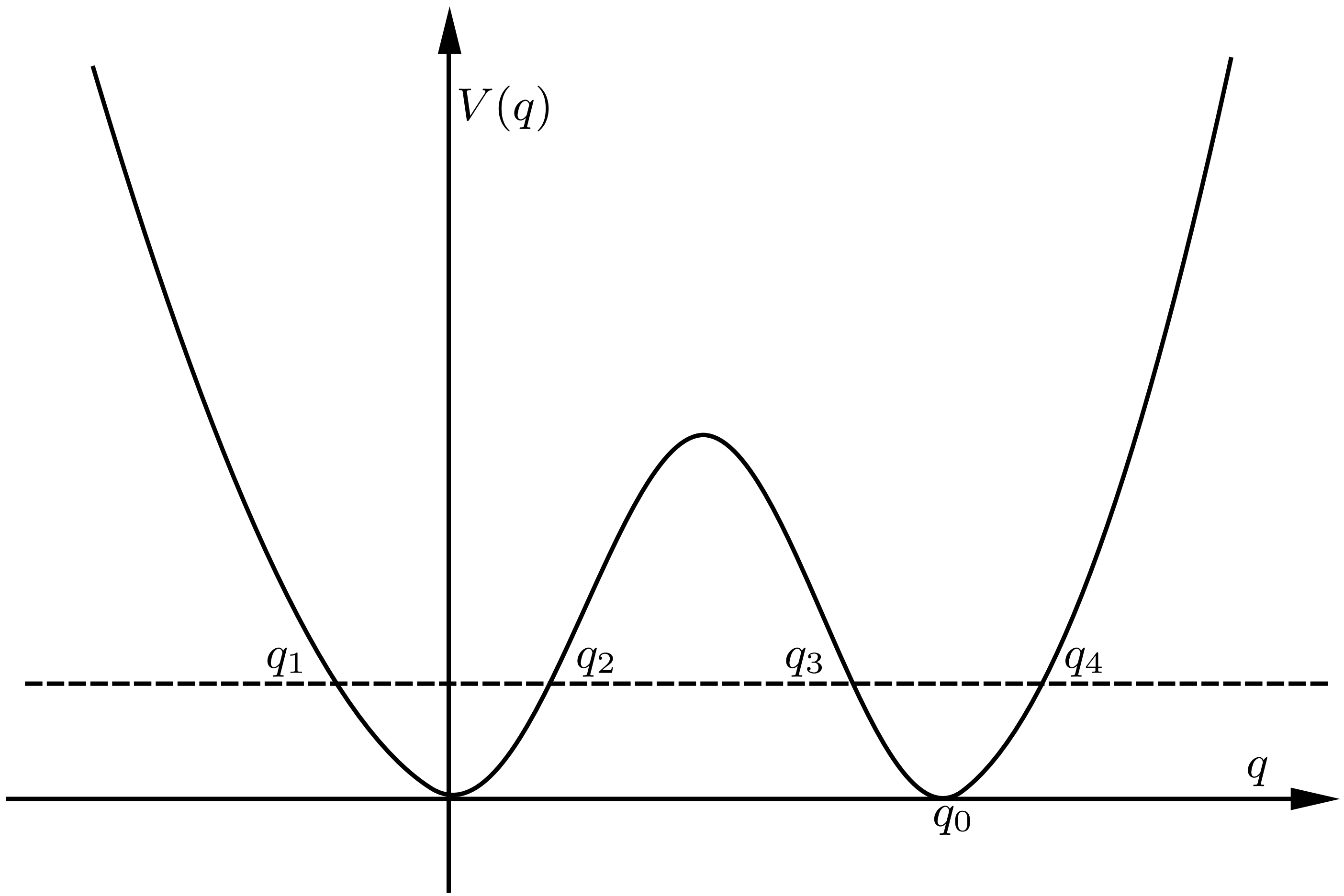

Fig. 4 Degenerate minima: The four turning points.

WKB expansion

At leading order in the WKB limit, the function S_+ reduces to

$$S_+(q) = S(q) = S_0(q), \quad S_0(q) = \sqrt{2V(q) - 2gE}$$

and the quantization condition becomes

$$N + \tfrac{1}{2} = B(E, g) = -\frac{1}{2i\pi g} \oint_C \mathrm{d}z \, S_0(z, E),$$

where the contour C encloses the cut of $S_0(q)$ which joins the turning points.

If the potential has two degenerate, non necessarily symmetric, minima, for E small enough, the function $S_0(q)$ has four branch points $q_1 < q_2 < q_3 < q_4$ on the real axis (Fig. 4).

One can define two functions $B_1(E, g)$ and $B_2(E, g)$ which, at leading order, correspond to contours enclosing the cuts $[q_1, q_2]$ and $[q_3, q_4]$.

Moreover, comparing with the conjecture, one infers the decomposition

$$\frac{1}{g}\oint_{C[q_2,q_3]} \mathrm{d}z\, S_+(z) = A(E,g) + \ln(2\pi) - \sum_{i=1}^{2} \ln\Gamma\bigl(\tfrac{1}{2} - B_i(E,g)\bigr) + B_i(E,g)\ln(-g/2C_i),$$

where, at leading order in the WKB expansion, the contour now encloses a cut $[q_2, q_3]$ and the constants C_i are chosen such that $A(E,g)$ has no term of order g^0.