An old story:
From the valley instantons to a sypersymmetric Q.M

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1. Introduction: "A long time ago…."

Back in 1992-1999…

1. A new method for digging deep into the non-perturbative arena was invented.
2. It gave insight into the interplay between the perturbative and non-perturbative parts.
3. A surprise finding of a new series of SUSY QM.

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NUCLEAR PHYSICS B

A new valley method for instanton deformation *

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Valley views: instantons, large order behaviors, and supersymmetry

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1. Introduction: "Historical background"

The origin of "the valley method"?

1. Instanton – Anti Instanton interaction is crucial for unitary amplitude in Baryon and Lepton number violation (via Sphareron) process in the standard model at TeV scale. (We miss you, SSC.)
2. Streamline method was applied.
3. But it is ambiguous and has difficulties.
4. "The new valley method" was invented in order to overcome all those problems.
5. It was applied to Q.M., which lead us to a finding on P and NP interplay.

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1. Introduction: "Who needs the valley method?"

And we wrote:

"The novel feature of our investigation is that the valley we introduce is not merely an improvement of the saddle point method but a unification of perturbative and non-perturbative methods, which opens a way to go beyond the dilute gas approximations. The process of unification will be discussed in detail in ……

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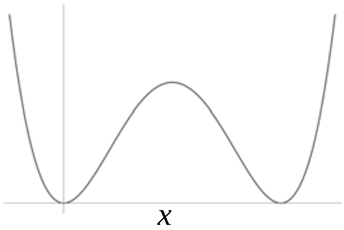
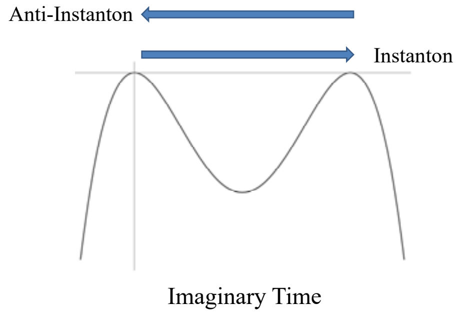
2. The Valley and the Functional Integral

Toy model: a double well potential for a particle in 1 dimensional space.





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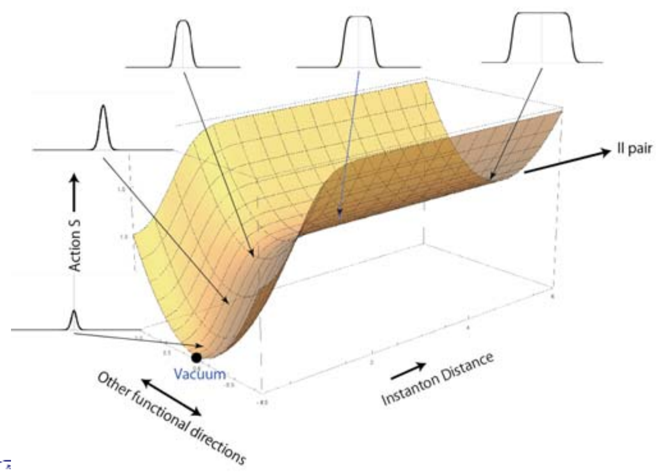
“Bird’s Eye View”



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The “Aoyama-Kikuchi” (AK) ☺ Valley Equation

Discrete Notation
$$\frac{\partial^2 S}{\partial\phi_i \partial\phi_j}\frac{\partial S}{\partial\phi_j} = \lambda_{\min}\frac{\partial S}{\partial\phi_i}$$

Continuous Notation
$$\int d\tau' \frac{\delta^2 S[q]}{\delta q(\tau)\delta q(\tau')}\frac{\delta S[q]}{\delta q(\tau')} = \lambda \frac{\delta S[q]}{\delta q(\tau)}.$$

which can be rewritten as
$$\frac{\delta}{\delta q(\tau)}\left[\frac{1}{2}\int\left(\frac{\delta S[q]}{\delta q(\tau')}\right)^2 d\tau' - \lambda S[q]\right] = 0.$$

Length of the gradient vector^2

Constraint on the action

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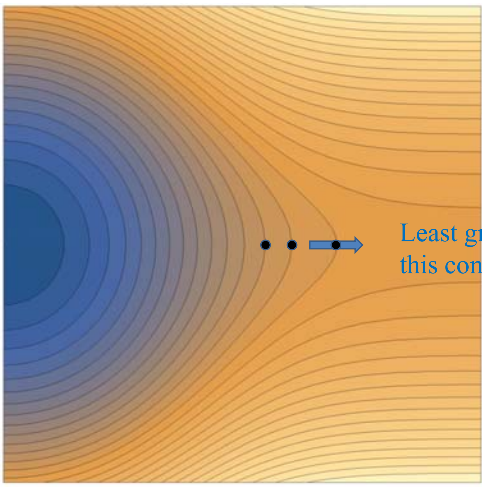
The AK-valley’s geography



Least gradient on this contour line

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The Fadeev-Popov and the AK valley

FP method of introduction of the collective coordinate

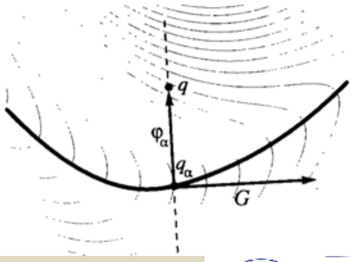
The valley line

$$1 = \int d\alpha \, \delta((\phi_i - \phi_i(\alpha))R_i(\alpha))\Delta(\phi(\alpha)),$$

where $R_i \equiv \frac{1}{\sqrt{A}} \frac{\partial S}{\partial \phi_i}, \quad A = \sum_i \left(\frac{\partial S}{\partial \phi_i} \right)^2.$

Remember that the valley is defined by

$$\frac{\partial^2 S}{\partial \phi_i \partial \phi_j} \frac{\partial S}{\partial \phi_j} = \lambda_{\min} \frac{\partial S}{\partial \phi_i}.$$



Therefore, the smallest eigenvalue is completely removed from the rest of the functional integration!

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Yet another form of the AK valley equation

"External Force"

$$\frac{\delta S[q]}{\delta q(\tau)} = F(\tau),$$

$$\int d\tau' \frac{\delta^2 S[q]}{\delta q(\tau)\delta q(\tau')} F(\tau') = \lambda F(\tau),$$

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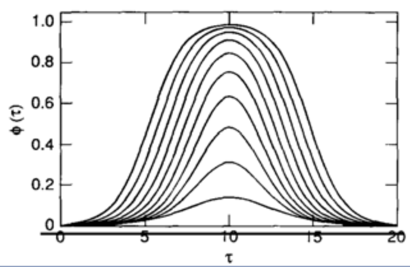
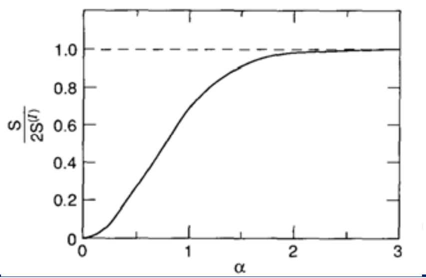
3. Symmetric double well, doubly well done!

Symmetric double well potential

$$S[q] = \int d\tau \left[\frac{1}{2}\left(\frac{dq}{d\tau}\right)^2 + V(q)\right],$$

$$V(q) = \frac{1}{2} q^2 (1 - gq)^2$$





Analytical constructions

$I\bar{I}$ valley

Configuration, action, determinant, were analytically constructed at large separation.

$$\tilde{S}(R) = \begin{cases} \sigma R^2 & \text{at } R \to 0; \\ \frac{1}{3} - 2\,e^{-R} & \text{at } R \to \infty, \end{cases}$$

"Interaction term"

The transition amplitude

Changing the integration variable R to $t = \tilde{S}(R)$,

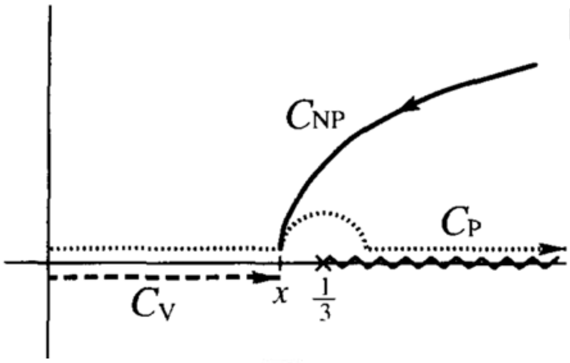
$$Z = \lim_{\lambda \to 1/3} |\Psi(0)|^2 \frac{e^{-T/2}}{\pi g^2} \int_{C_V} dt\, F(t)\, e^{-t/g^2}$$

where $x = \tilde{S}(T) = 1/3 - 2\,e^{-T}$, $C_V = [0, x]$.

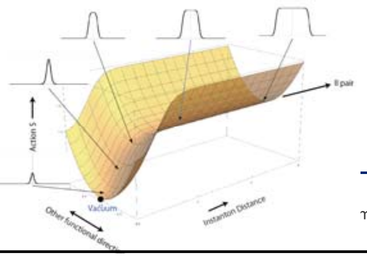
Note that this contains everything, both perturbative and non-perturbative parts.

Then, how do we separate them?

"Separation conjecture"



Equivalent to Bogomolny's analytic continuation to g^2 < 0.



Results

$$\alpha = e^{-1/6g^2} / \sqrt{\pi g^2}$$

$$E_{\varsigma,\mathrm{NP}}(\epsilon = 0, N_+ = 0) = \varsigma\alpha + \alpha^2 \left\{ \gamma + \ln\left(-\frac{2}{g^2}\right) \right\},$$

$$|\Psi_\varsigma(q_+)|^2_{\mathrm{NP}} = \frac{\alpha^2}{2} |\Psi(0)|^2 \left\{ \frac{\Gamma''(1)}{2} + \gamma \ln\left(-\frac{2}{g^2}\right) + \frac{1}{2}\left\{\ln\left(-\frac{2}{g^2}\right)\right\}^2 \right\}$$

No worries about the imaginary parts, by construction.

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3. Asymmetric double well: Valleys

Take the following;

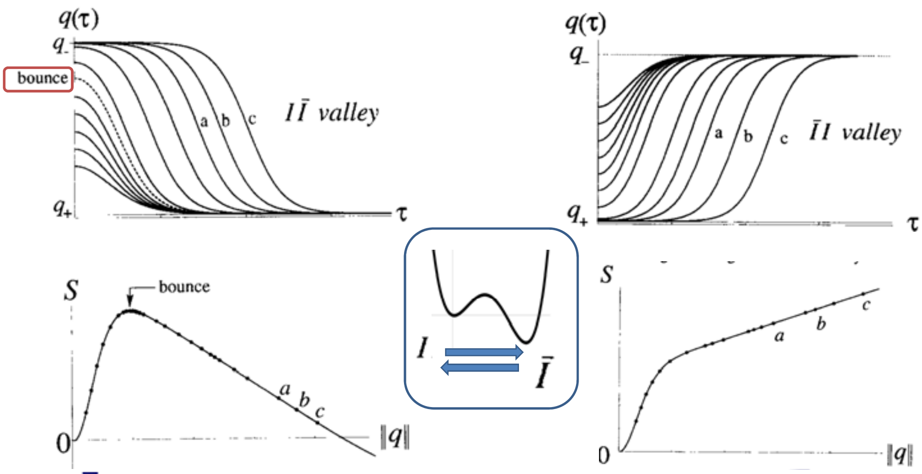
$$S[q] = \int d\tau \left[\frac{1}{2} \left(\frac{dq}{d\tau} \right)^2 + V(q) \right],$$

$$V(q) = \frac{1}{2} q^2 (1 - gq)^2 - \epsilon g q.$$

Bounce is the only (interesting) solution of the equation of motion.

Then, what can we do?

Valley instanton and anti-instanton comes into rescue



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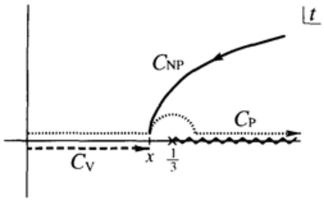
Analytical Construction

Scaling: $S[q]=\frac{1}{g^2}\tilde{S}[\tilde{q}]$, $\quad \tilde{S}[\tilde{q}]=\int d\tau\left[\frac{1}{2}\left(\frac{d\tilde{q}}{d\tau}\right)^2+\tilde{V}(\tilde{q})\right]$,

$$\tilde{V}(\tilde{q})=\tilde{V}_0(\tilde{q})+\epsilon g^2\tilde{V}_1(\tilde{q}),\quad \tilde{V}_0(\tilde{q})=\frac{1}{2}\tilde{q}^2(1-\tilde{q})^2,\quad \tilde{V}_1(\tilde{q})=-\tilde{q}.$$

We constructed valley instantons, actions and determinant at large separation for small ϵg^2, not small ϵ. Using the same separation of P and NP as before, we obtained…



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Energy and the Wavefunction for $\epsilon \notin \mathbb{Z}$

$I\bar{I}$ *valley*

$$E_{\rm NP}^{(+)}(\epsilon, N_+ = 0) = -\alpha^2 \left(-\frac{2}{g^2}\right)^{\epsilon} \Gamma(-\epsilon).$$

$$|\Psi(q_+)|^2_{\rm NP} = -\alpha^2 |\Psi(0)|^2 \left(-\frac{2}{g^2}\right)^{\epsilon} \left\{ \ln\left(-\frac{2}{g^2}\right) \Gamma(-\epsilon) - \Gamma'(-\epsilon) \right\}$$

$\bar{I}I$ *valley*

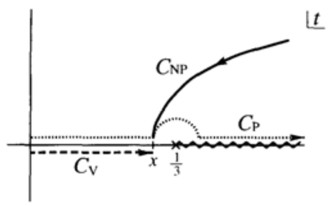
$$E_{\rm NP}^{(-)}(\epsilon, N_- = 0) = -\alpha^2 \left(-\frac{2}{g^2}\right)^{-\epsilon} \Gamma(\epsilon),$$

$$|\Psi(q_-)|^2_{\rm NP} = -\alpha^2 |\Psi(0)|^2 \left(-\frac{2}{g^2}\right)^{-\epsilon} \left\{ \ln\left(-\frac{2}{g^2}\right) \Gamma(\epsilon) - \Gamma'(\epsilon) \right\}$$

The role of ϵ

$I\bar{I}$ *valley* $\quad t = \tilde{S}(R) + \epsilon g^2 \tilde{S}_1(R) = \tilde{S}_0(R)$

$$\tilde{S}_0(R) = \begin{cases} \sigma R^2 & \text{for } R \to 0; \\ \frac{1}{3} - 2e^{-R} & \text{for } R \to \infty, \end{cases}$$



$$Z = \lim_{x \to 1/3} |\Psi(0)|^2 \frac{e^{-T/2}}{\pi g^2} \int_{C_V} dt\, F(t)\, e^{-t/g^2}$$

$$F(t) = \begin{cases} \frac{T}{\sqrt{4\sigma t}} & \text{for } t \to 0; \\ \frac{1}{1/3 - t} \left(\frac{1/3 - t}{2}\right)^{-\epsilon} \ln\left(\frac{1/3 - t}{1/3 - x}\right) & \text{for } t \to 1/3. \end{cases}$$

Summing over multi-valley

0-th order energy levels

$$E_0^{(+)}(\epsilon, N_+) = \frac{1}{2} + N_+,$$

$$E_0^{(-)}(\epsilon, N_-) = -\epsilon + \frac{1}{2} + N_-,$$

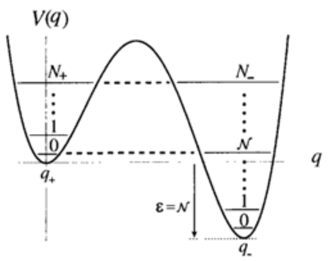
$\epsilon \notin \mathbb{Z}$

$$E_{\rm NP}^{(\pm)}(\epsilon, N_\pm) = \alpha^2 \frac{(-1)^{N_\pm+1}}{N_\pm!} \left(-\frac{2}{g^2}\right)^{\pm\epsilon+2N_\pm} \Gamma\left(\mp\epsilon - N_\pm\right) + O(\alpha^4).$$

Integer ϵ

0-th order energy levels: degeneracy

$$E_0(N_\pm) = \frac{1}{2} + N_+ = -\epsilon + \frac{1}{2} + N_-.$$



Energy Splitting:

$$E_{\varsigma,\rm NP}(\epsilon, N_+) = \varsigma\alpha\sqrt{\frac{1}{N_+!N_-!}\left(\frac{2}{g^2}\right)^{N_++N_-}} + \frac{\alpha^2}{2}\frac{1}{N_+!N_-!}\left(\frac{2}{g^2}\right)^{N_++N_-}$$
$$\times\left[2\ln\left(-\frac{2}{g^2}\right) - \psi(N_++1) - \psi(N_-+1)\right],$$

where $\varsigma = \pm 1$ and $N_- = N_+ + \epsilon = N_+ + \mathcal{N}$.

$$\alpha = e^{-1/6g^2}/\sqrt{\pi g^2}$$

Contents

5. Asymmetric Double Well: Large Orders

Perturbative Expansion
$$E_{\mathrm{P}}^{(\pm)}(\epsilon, N) = \sum_{m=0}^{\infty} E_{\mathrm{P}}^{(\pm)}(\epsilon, N, m) g^{2m}.$$

$$E_{\mathrm{P}}^{(\pm)}(\epsilon, N, m) = -\frac{1}{\pi}\int_0^{\infty} dg^2 \frac{\mathrm{Im}(E_{\mathrm{NP}}^{(\pm)}(\epsilon, N))}{g^{2m+2}}$$

This yields

$$E_{\mathrm{P}}^{(\pm)}(\epsilon, N, m) = A^{(\pm)}(\epsilon, N) 3^m \Gamma(\pm\epsilon + 2N + m + 1)\left[1 + O\left(\frac{1}{m}\right)\right],$$

where the coefficient $A^{(\pm)}(\epsilon, N)$ is defined as

$$A^{(\pm)}(\epsilon, N) \equiv -\frac{3}{\pi}\frac{6^{\pm\epsilon+2N}}{N!\,\Gamma(\pm\epsilon + 1 + N)}.$$

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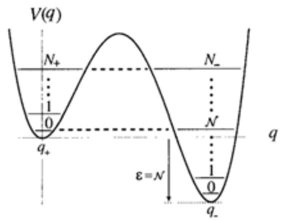
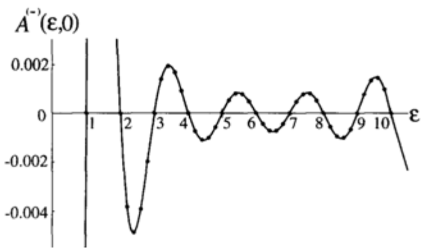
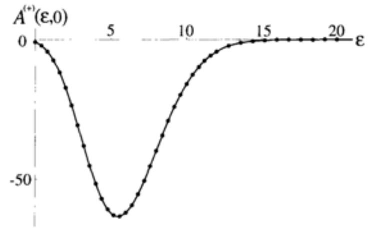
Numerical Check

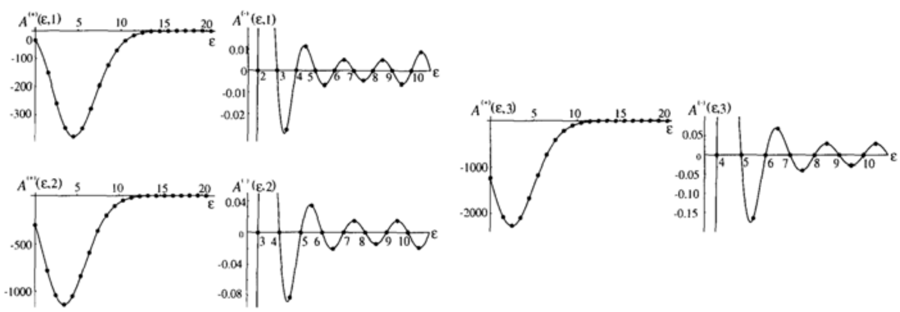
Exact calculation of Perturbative coefficients
(using *Mathematica*) was done for N=0…6
Up to 200th and 478th order at the Yukawa Institute.







Numerical Check: Higher levels



Great agreements between our P-NP
method and the Numerical (Exact)
Calculation.

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6. Asymmetric Double Well: N–fold SYSY

$\epsilon = 1$ has a supersymmetry.

$$Q^\dagger \equiv D\psi^\dagger, \qquad Q = D^\dagger\psi,$$

where

$$D = p - iW(q), \qquad D^\dagger = p + iW(q), \quad W(q) = q(1 - gq).$$

$$\mathbf{H} = \frac{1}{2}\{Q^\dagger, Q\} = \frac{1}{2}(p^2 + W^2(q)) + W'(q)\left(\psi^\dagger\psi - \frac{1}{2}\right).$$

The ground state (E=0) wave function in not normalizable and therefore the SUSY is dynamically broken, nonperturbatively.

$$\Psi_G(q) = \exp\left(-\int_0^q dq' W(q')\right) = \exp\left(-\frac{q^2}{2} + g\frac{q^3}{3}\right).$$

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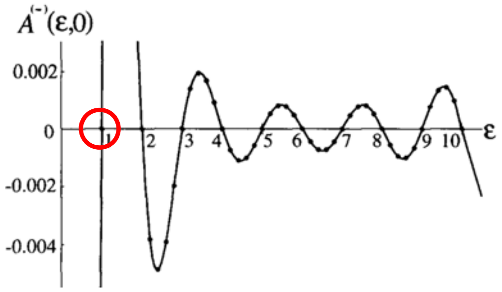
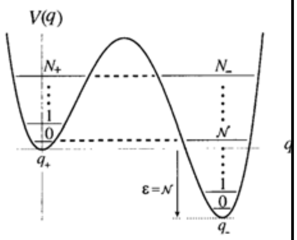
This explains

Perturbation theory does not realize the SUSY breaking and E=0 is protected by a non-renormalization theorem.





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What about all the other zeros?

N-fold supersymmetry:

$$Q_{\mathcal{N}}^{\dagger} = \begin{pmatrix} 0 & D^{\mathcal{N}} \\ 0 & 0 \end{pmatrix}, \qquad Q_{\mathcal{N}} = \begin{pmatrix} 0 & 0 \\ (D^{\dagger})^{\mathcal{N}} & 0 \end{pmatrix}$$

$$\mathbf{H}_{\mathcal{N}} = \frac{1}{2}\left\{Q_{\mathcal{N}}, Q_{\mathcal{N}}^{\dagger}\right\} = \frac{1}{2}\begin{pmatrix} D^{\mathcal{N}} D^{\dagger\mathcal{N}} & 0 \\ 0 & D^{\dagger\mathcal{N}} D^{\mathcal{N}} \end{pmatrix}$$

We have shown:

$$\mathbf{H}_{\mathcal{N}} = \frac{1}{2} \det \mathbf{M}_{\mathcal{N}}(H_{\mathcal{N}})$$

Our Hamiltonian

$$\mathbf{M}_{\mathcal{N}}(E)_{ij} = 2g(i-\mathcal{N})\delta_{i,j+1} + (2E+\mathcal{N}-1-2i)\delta_{i,j} + (i+2)(i+1)\delta_{i,j-2},$$

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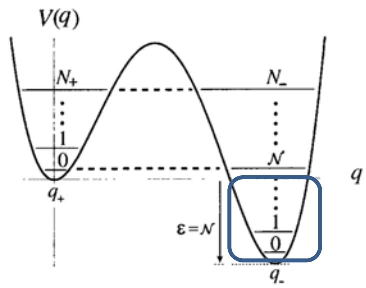
The Isolated levels

The Perturbative energy levels



$\mathcal{N} = 1 : E = 0,$

$\mathcal{N} = 2 : \left(E - \frac{1}{2}\right)\left(E + \frac{1}{2}\right) = 0,$

$\mathcal{N} = 3 : E(E-1)(E+1) + 2g^2 = 0,$

$\mathcal{N} = 4 : \left(E - \frac{3}{2}\right)\left(E - \frac{1}{2}\right)\left(E + \frac{1}{2}\right)\left(E + \frac{3}{2}\right) + 12Eg^2 = 0,$

$\mathcal{N} = 5 : E(E-2)(E-1)(E+1)(E+2) + 6(7E^2 - 52)g^2 = 0.$

Finite convergence radios for $N > 2$.

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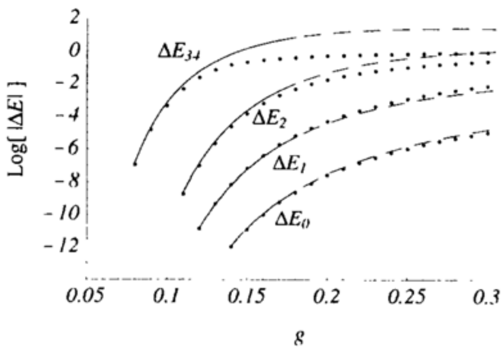
Numerical confirmation of the NP contribution

For isolated levels in N-fold SUSY, the perturbative part is completely known. This allowed us to numerically confirm our NP prediction by looking at differences between their numerical energy eigenvalues and the perturbative eigenvalues.

N=3



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Summary

- The AK valley method was proposed.
- Instanton solutions were constructed both numerically and analytically, and was confirmed to have desirable properties. “Instanton interactions” at large distances were identified.
- Separation of Perturbative and Non-Perturbative contribution to the path integral was proposed.
- Non-perturbative part was calculated for Asymmetric double well model.
- Large order behavior of the perturbation series were found, and numerically confirmed.
- N-fold supersymmetry was found. Perturbative part of the energy eigenvalues were exactly calculated for isolated states.
- Using N-fold supersymmetry, NP contribution was numerically confirms.

- In short, all checks out.

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