Theta dependence and anomaly matching.

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(Partially Based on works with Yuta Kikuchi, 1705.01949, 1708.01962)

Motivation

θ-dependence of Yang-Mills theory. (Gauge group : $SU(N)$).

$$S = -\frac{1}{2g^2}\int tr(G\wedge * G) + \underbrace{i\frac{\theta}{8\pi^2}\int tr(G\wedge G)}_{i\theta \underbrace{c_2(G)}_{\in \mathbb{Z}}}. \qquad G = dA + iA\wedge A.$$

$$Z(\theta) = \int \mathcal{D}A \exp\left(-\frac{1}{2g^2}\int tr(G\wedge * G) + i\frac{\theta}{8\pi^2}\int tr(G\wedge G)\right) = e^{-\beta V E(\theta)}$$

for $\beta V \to \infty$.

$Z(\theta)$ is 2π-periodic in θ. (Very clear).

However, 2π-periodicity of $E(\theta)$ is very tricky. (Witten, 1980).

Taken large-N limit : $\lambda = g^2 N$, $\tilde{\theta} = \frac{\theta}{N}$, $N \gg 1$.

$$Z(\theta) = \int \mathcal{D}A \exp\left[N\left(-\frac{1}{2\lambda}\int tr(G\wedge * G) + i\frac{\tilde{\theta}}{8\pi^2}\int tr(G\wedge G)\right)\right]$$
$$= e^{-\beta V N f(\tilde{\theta})}$$

$$\Rightarrow E(\theta) = N f\left(\frac{\theta}{N}\right)$$

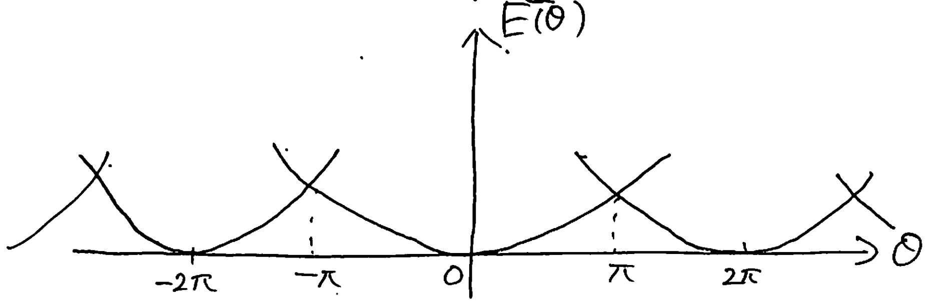
$$\underset{\text{Taylor exp. at } \theta=0}{=} N E_0 + \frac{\chi}{N}\theta^2 + \cancel{O\left(\frac{1}{N^3}\right)}$$

neglect this by taking $N \gg 1$. for $\cancel{\theta(\theta)}$ $\theta \sim O(1)$.

2π-periodicity of $E(\theta)$ is gone (?).

Ground-state energy must be a multi-branch function :

$$E(\theta) = \min_{k\in\mathbb{Z}}\left(N f\left(\frac{\theta}{N} + \frac{2\pi k}{N}\right)\right).$$

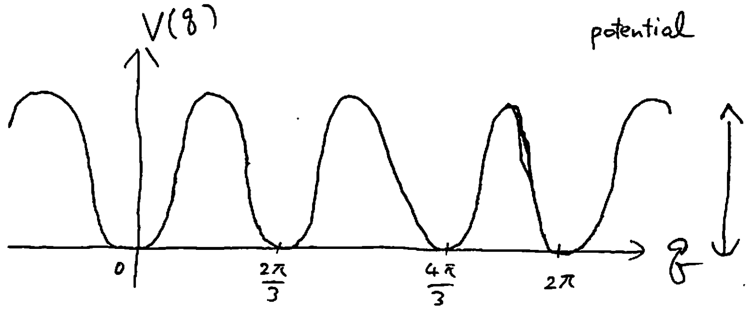


$\Rightarrow$ How can we obtain this information from the computation of partition functions?

Toy model : Quantum mechanics

$\xi : S^1 \to \mathbb{R}/2\pi\mathbb{Z}$

$$S = \int dt \left\{ \frac{1}{2}\left(\frac{d\xi}{dt}\right)^2 + \underbrace{\lambda\left(1-\cos(N\xi)\right)}_{\text{potential } V(\xi)} + i\frac{\theta}{2\pi}\frac{d\xi}{dt} \right\}.$$



Hamiltonian

$$H = \frac{1}{2}\left(p - \frac{\theta}{2\pi}\right)^2 + \lambda\left(1-\cos(N\xi)\right), \qquad p = \frac{1}{i}\frac{\partial}{\partial \xi}.$$

Hilbert space

$$\mathcal{H} = L^2\left(\begin{array}{c}2\pi\text{-periodic functions}\\ \text{of } \xi.\end{array}\right).$$

Partition function.

$$Z(\theta) = \mathrm{tr}\left(e^{-\beta H}\right)$$

$$= \int \mathcal{D}\xi \, \exp\left(-\int_0^\beta dt \left(\frac{1}{2}\left(\frac{d\xi}{dt}\right)^2 + \lambda\left(1-\cos(N\xi)\right) + i\frac{\theta}{2\pi}\frac{d\xi}{dt}\right)\right)$$

Since $\int dt \, i\frac{\theta}{2\pi}\frac{d\xi}{dt} = i\frac{\theta}{2\pi}\int d\xi = i\theta n, \quad \left(n = \frac{1}{2\pi}(\xi(\beta)-\xi(0)) \in \mathbb{Z}\right)$.

$Z(\theta)$ is (trivially) a 2π-periodic function.

What about the ground state energy?

Semiclassical calculations

Take $\beta\sqrt{\lambda} \gg 1$.

$$H = \frac{1}{2}\left(p - \frac{\theta}{2\pi}\right)^2 + \frac{\lambda N^2}{2} q^2 + O(q^4) \qquad q \simeq 0.$$

$\Rightarrow$ l-th excited state around $q = 0$.

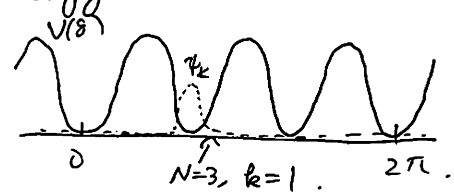
$$E_l = \hbar\sqrt{\lambda}N\left(l + \frac{1}{2}\right).$$

$l \geq 1$ does not contribute to $\mathrm{tr}\left(e^{-\beta H}\right)$ in this limit.

Pay attention only to $l = 0$ state.

We have N states with that perturbative energy.

$$\psi_k \simeq \exp\left(-\frac{\sqrt{\lambda}N}{2\hbar}\left(q - \frac{2\pi k}{N}\right)^2\right).$$



Within the one-instanton calculation, we get

$$\left(\langle\psi_i| e^{-\beta H}|\psi_j\rangle\right)_{ij} = e^{-\beta E_0}\begin{pmatrix} 1 & \beta e^{-S_{inst}}e^{i\frac{\theta}{N}} & 0 & 0 & \cdots & \beta e^{-S_{inst}}e^{-i\frac{\theta}{N}} \\ \beta e^{-S_{inst}}e^{-i\frac{\theta}{N}} & 1 & \beta e^{-S_{inst}}e^{i\frac{\theta}{N}} & & & 0 \\ 0 & \beta e^{-S_{inst}}e^{-i\frac{\theta}{N}} & \ddots & \ddots & & 0 \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ 0 & & & \ddots & \ddots & \beta e^{-S_{inst}}e^{i\frac{\theta}{N}} \\ \beta e^{-S_{inst}}e^{i\frac{\theta}{N}} & 0 & \cdots & 0 & \beta e^{-S_{inst}}e^{-i\frac{\theta}{N}} & 1 \end{pmatrix}.$$

We can diagonalize this matrix by

$$|E_k\rangle = \sum_{i=0}^{N-1}\omega^{ik}|\psi_i\rangle. \qquad \left(\omega = e^{\frac{2\pi i}{N}}\right)$$

$$e^{-\beta H}|E_k\rangle = e^{-\beta E_0}\left(1 + 2\beta A e^{-S_{inst}}\cos\left(\frac{\theta}{N} + \frac{2\pi k}{N}\right)\right)|E_k\rangle$$

$$\left(= \exp\left(-\beta\left(E_0 - 2\cancel{\beta}Ae^{-S_{inst}}\cos\left(\frac{\theta}{N} + \frac{2\pi k}{N}\right)\right)\right)\right)$$

One-instanton calc. is justified for $\beta A e^{-S_{inst}} \ll 1$, and this is possible because of the exponential suppression. $\left(S_{inst} \sim O(\sqrt{\lambda})\right)$.

$$H|E_k\rangle = \left(E_0 - 2\cancel{\beta}Ae^{-S_{inst}}\cos\left(\frac{\theta}{N} + \frac{2\pi k}{N}\right)\right)|E_k\rangle.$$

Thus, we get an N-branch solution for the ground-state energy:

$$E(\theta) = \min_{k\in\{0,\cdots,N-1\}}\left(E_0 - 2\cancel{\beta}Ae^{-S_{inst}}\cos\left(\frac{\theta}{N} + \frac{2\pi k}{N}\right)\right).$$

How do we decompose $Z(\theta)$ as follows?

$$Z(\theta) = \mathrm{tr}\left(e^{-\beta H}\right) = \sum_{k=0}^{N-1} e^{-\beta\left(\underbrace{E_0 - Ae^{-S_{inst}}\cos\left(\frac{\theta}{N}+\frac{2\pi k}{N}\right)}_{\equiv E_k(\theta)}\right)}$$

(This is not so trivial. For example, at the one-instanton level,

$$Z(\theta) = \sum_k e^{-\beta E_0} e^{\beta A e^{-S_{inst}}\cos\left(\frac{\theta}{N}+\frac{2\pi k}{N}\right)}$$

$$= \sum_{k=0}^{N-1} e^{-\beta E_0}\left(1+\beta A e^{-S_{inst}}\cos\left(\frac{\theta}{N}+\frac{2\pi k}{N}\right)\right) = N e^{-\beta E_0} \quad (N \geq 2),$$

No θ-dependence can be captured. Minimal # of instantons is N to get θ-dep.)

Twisted boundary condition

Basic idea: Instead of computing $Z(\theta) = \mathrm{tr}\left(e^{-\beta H}\right)$, we calculate

$$Z(\theta, U^n) = \mathrm{tr}\left(U^n e^{-\beta H}\right).$$

U: generator of the $\mathbb{Z}_N$ symmetry $q \mapsto q + \frac{2\pi}{N}$, i.e.,

$$U = e^{i\frac{2\pi}{N}p}.$$

$$U|\Psi_i\rangle = |\Psi_{i-1}\rangle. \Rightarrow U|E_k\rangle = \sum_{i=0}^{N-1} \omega^{ik} |\Psi_{i-1}\rangle$$

$$= \omega^k |E_k\rangle.$$

Therefore,

$$Z(\theta, U^n) = \sum_{k=0}^{N-1}\left(e^{-\beta E_k(\theta)} e^{i\frac{2\pi k}{N}n}\right).$$

This additional phase can be a book-keeping device for the state. Especially, we can perform the projection to $|E_k\rangle$:

$$\sum_{n=0}^{N-1} \omega^{-nk} Z(\theta, U^n) = \sum_{k'=0}^{N-1}\left(e^{-\beta E_{k'}(\theta)} \underbrace{\sum_{n=0}^{N-1} e^{\frac{2\pi i n}{N}(k'-k)}}_{=\delta_{k',k}}\right)$$

$$= e^{-\beta E_k(\theta)}.$$

Summing up twisted partition functions with a certain weight, we can show $2\pi N$-periodicity of each branch of the ground state.

Background gauge fields, 't Hooft anomaly, global inconsistency

Our theory has the $\mathbb{Z}_N$ shift symmetry : $q \to q + \frac{2\pi}{N}$.

We introduce the $\mathbb{Z}_N$ gauge field as a background :

$$Z_\theta[A] = \int \mathcal{D}q \, \exp\left(-\int \left(|dq + A|^2 + \lambda\left(1 - \cos(Nq + \phi)\right) + \frac{i\theta}{2\pi}(dq + A)\right)\right).$$

A : $U(1)$ gauge field with the constraint $NA = d\phi$.

(The action is $\mathbb{Z}_N$-gauge inv. under the local transformation

$$q \to q - \lambda, \quad A \to A + d\lambda, \quad \phi \to \phi - N\lambda \quad).$$

Example of the $\mathbb{Z}_N$-gauge field (A, ϕ) :

$$A = \frac{2\pi n}{N}\delta(t - t_0)\, dt, \quad \phi = 2\pi n \Theta(t - t_0). \quad \left(\begin{array}{c}\text{This solves} \\ d\phi = NA\end{array}\right).$$

For this config., we find that

$$Z[A] = Z(\theta, U^n).$$

In order to obtain $e^{-\beta E_k(\theta)}$, all we have to do is to make A the dynamical $\mathbb{Z}_N$ gauge field :

$$e^{-\beta E_k(\theta)} = \sum_{n=0}^{N-1} \omega^{-nk} Z(\theta, U^n)$$

$$= \int \mathcal{D}A \, \underbrace{e^{-ik\int A} Z_\theta[A]}_{\equiv Z_{\theta,k}[A]}.$$

$$= \int \mathcal{D}A \, \mathcal{D}q \, \exp\left(-\int \left(|dq + A|^2 + \lambda(1 - \cos(Nq)) + \frac{i\theta}{2\pi}(dq + A) + ikA\right)\right)$$

In this process, we can designate the Chern-Simons level k, which corresponds to the projection into the states with $U = e^{2\pi i k/N}$.

Employing the knowledge of 't Hooft anomaly matching, we can show the degeneracy of ground states at $\theta = \pi$. (cf. Gaiotto, Kapustin, Komargodski, Seiberg, 1703.00501, YT, Kikuchi, 1705.01949, 1708.01962, etc.)

At $\theta = 0, \pm\pi, \pm 2\pi, \dots$,

S is invariant under the Charge Conjugation, $C: q \longrightarrow -q$.

(At $\theta=0$, this is trivial.
At $\theta = \pi$, the nontrivial term is $i\frac{\theta}{2\pi}\int dq$: $i\frac{\pi}{2\pi}\int dq \mapsto -i\frac{\pi}{2\pi}\int dq = i\frac{\pi}{2\pi}\int dq - \underbrace{i\int dq}_{\text{integer}}$
This is exponentiated, so it is invariant.)

Let us check the invariance of C for $Z_{\theta=\pi, k}[A]$.

$$C: q \to -q, \quad A \longrightarrow -A.$$

Then, we obtain.

$$Z_{\theta=\pi,k}[A] \to Z_{\theta=\pi,k}[-A] = Z_{\theta=\pi,k}[A] \exp\left(+i(2k+1)\int A\right).$$

The twisted partition function is not invariant under C at $\theta=\pi$ unless

$$2k+1 = 0 \mod N.$$

$\Rightarrow$ For even $\cancel{k}$ N, no solution exists. This is an 't Hooft anomaly.

This means that all the energy eigenstates must form a pair under C.

$\Rightarrow$ For odd N, we have a unique solution $k = -\frac{N+1}{2}$.

This means that the C-inv. energy eigenstate must have the specific $\mathbb{Z}_N$ charge $k_\pi = -\frac{N+1}{2}$. Other states form the pair.

Doing the same thing at $\theta=0$, we get

$$Z_{\theta=0,k}[A] \Rightarrow Z_{\theta=0,k}[-A] = Z_{\theta=0,k}[A] \exp\left(+i\, 2k\int A\right).$$

The C-inv. state at $\theta=0$ must have the $\mathbb{Z}_N$ charge $k_0 = 0$.

$k_0 \neq k_\pi$, so that
- Ground state at $\theta=0$ or $\theta=\pi$ is degenerate $\Leftarrow$ This is realized.
- Level crossing occurs between $\theta=0, \pi$.

This condition is global inconsistency.