Exact WKB analysis of the Gauss hypergeometric differential equation

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RIMS-iTHEMS International Workshop on Resurgence Theory
September 6, 2017
RIKEN Kobe Campus

Plan of this talk:

1. Introduction
2. The hypergeometric differential equation
3. A quick review of the exact WKB analysis
4. Exact WKB analysis of the hypergeometric differential equation

1. Introduction

The aim of this talk is to relate the Gauss hypergeometric function and Borel resummed WKB solutions.

The Gauss hypergeometric function $_2F_1(a,b,c;z)$ is a standard solution of the hypergeometric differential equation.

If we introduce a large parameter in the hypergeometric differential equation suitably, we can construct WKB solutions of the equation.

These formal solutions are Borel summable under suitable generic conditions. Taking the Borel sum, we have analytic solutions of the hypergeometric differential equation.

$_2F_1(a,b,c;z)$ can be expressed explicitly as a linear combination of the Borel resummed WKB solutions.

As an application, we obtain asymptotic expansion formulas of the Gauss hypergeometric function with respect to the parameter.

2. The hypergeometric differential equation

- **The hypergeometric differential equation:**

$$x(1-x)\frac{d^2w}{dx^2} + (c-(a+b+1)x)\frac{dw}{dx} - abw = 0, \tag{2.1}$$

where $a, b, c \in \mathbb{C}$. Regular singular at $x = 0, 1, \infty$.

- **The hypergeometric series (or function):** $(c \neq 0, -1, -2, \dots)$

$$ {}_2F_1(a,b,c;x) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} x^n, \tag{2.2}$$

where $(a)_n = a(a+1)\cdots(a+n-1) = \dfrac{\Gamma(a+n)}{\Gamma(a)}$, **etc.**

- ⋄ **The radius of convergence = 1.**
  Thus ${}_2F_1(a,b,c;x)$ defines a holomorphic function on $\{x; |x| < 1\}$.
  (If a or $b \in \mathbb{Z}_{\leq 0}$, ${}_2F_1(a,b,c;x)$ is a polynomial of x.)

- ⋄ **${}_2F_1(a,b,c;x)$ defines a holomorphic function on the universal covering of $\mathbb{C} - \{0, 1\}$.**

- $\dfrac{1}{\Gamma(c)}{}_2F_1(a,b,c;x)$ **is an entire function of** a, b **and** c**.**

- **Characteristic exponents (Riemann scheme):**

$$\left\{ \begin{matrix} 0 & 1 & \infty \\ 0 & 0 & a \\ 1-c & c-a-b & b \end{matrix} \right\}$$

- **Standard solutions of** (2.1) **(notation of BMP):**

$$\begin{aligned}
u_1 &= {}_2F_1(a,b,c;x), \\
u_2 &= {}_2F_1(a,b,a+b+1-c;1-x), \\
u_3 &= (-x)^{-a}{}_2F_1\left(a,a+1-c,a+1-b;\frac{1}{x}\right), \\
u_4 &= (-x)^{-b}{}_2F_1\left(b,b+1-c,b+1-a;\frac{1}{x}\right), \\
u_5 &= x^{1-c}{}_2F_1(a+1-c,b+1-c,2-c;x), \\
u_6 &= (1-x)^{c-a-b}{}_2F_1(c-a,c-b,c+1-a-b;1-x).
\end{aligned}$$

(Six of Kummer's 24 solutions.)

- **Standard bases of solution space of (2.1):**

$$(u_1, u_5),\ (u_2, u_6),\ (u_3, u_4) \quad (a, b, c : \text{generic})$$

- **Connection formulas:**

$$(u_1, u_5) = (u_2, u_6) \begin{pmatrix} \dfrac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} & \dfrac{\Gamma(2-c)\Gamma(c-a-b)}{\Gamma(1-a)\Gamma(1-b)} \\ \dfrac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)} & \dfrac{\Gamma(2-c)\Gamma(a+b-c)}{\Gamma(a-c+1)\Gamma(b-c+1)} \end{pmatrix},$$

$$(u_1, u_5) = (u_3, u_4) \begin{pmatrix} \dfrac{\Gamma(c)\Gamma(b-a)}{\Gamma(c-a)\Gamma(b)} & \dfrac{\Gamma(2-c)\Gamma(b-a)}{\Gamma(1-a)\Gamma(b+1-c)} e^{i\pi(1-c)} \\ \dfrac{\Gamma(c)\Gamma(a-b)}{\Gamma(c-b)\Gamma(a)} & \dfrac{\Gamma(2-c)\Gamma(a-b)}{\Gamma(1-b)\Gamma(a+1-c)} e^{i\pi(1-c)} \end{pmatrix},$$

$$\cdots\cdots$$

3. A quick review of the exact WKB analysis

- **Consider the differential equation in the complex domain**

$$\left(-\frac{d^2}{dx^2}+\eta^2 Q\right)\psi = 0. \tag{3.1}$$

Here $\eta\,(=1/\hbar)$ is a positive large parameter and $Q=\sum_{j=0}^{N}\eta^{-j}Q_j(x)$ is a polynomial of η^{-1} with rational coefficients Q_j ($j=0,1,\ldots,N$).

- **Assume:**

$G(x)Q_j(x)$ ($j=1,2,\ldots,N$) are polynomials in x, where $Q_0(x)=\dfrac{F(x)}{G(x)}$ with coprime polynomials $F(x)$, $G(x)$.

- **WKB solutions:**

$$\psi = \exp\left(\int S(x,\eta)dx\right). \tag{3.2}$$

- **Associated Riccati equation:**

(3.3) $$\frac{dS}{dx} + S^2 = \eta^2 Q.$$

- **Formal solutions:** $S = \sum_{j=-1}^{\infty} \eta^{-j} S_j$ **constructed recursively by**

(3.4) $$S_{-1}^2 = Q_0,$$

(3.5) $$S_{j+1} = -\frac{1}{2S_{-1}}\left(\frac{dS_j}{dx} + \sum_{k=0}^{j} S_{j-k}S_k - Q_{j+2}\right), \; j = -1, 0, 1, 2, \ldots.$$

($Q_j = 0$ for $j > N$)

According to the choice of the leading term $S_{-1} = S_{-1}^{(\pm)} = \pm\sqrt{Q_0}$, we have two formal solutions

$$S^{(\pm)} = \sum_{j=-1}^{\infty} \eta^{-j} S_j^{(\pm)}$$

to the Riccati equation.

- **Normalization**

(3.6) $$S_{\rm odd} := \frac{1}{2}(S^{(+)} - S^{(-)}) =: \sum_{j=-1}^{\infty} \eta^{-j} S_{{\rm odd},j},$$

(3.7) $$S_{\rm even} := \frac{1}{2}(S^{(+)} + S^{(-)}) =: \sum_{j=0}^{\infty} \eta^{-j} S_{{\rm even},j}.$$

Then we have $S^{(\pm)} = \pm S_{\rm odd} + S_{\rm even}$ and

(3.8) $$S_{\rm even} = -\frac{1}{2}\frac{d}{dx}\log S_{\rm odd}.$$

Thus we can take normalization of the integration of $S_{\rm even}$ as $-\frac{1}{2}\log S_{\rm odd}$.

WKB solution normalized at a generic point $x_0 \in \mathbb{C}$:

(3.9) $$\psi_{\pm}^{(x_0)} := \frac{1}{\sqrt{S_{\rm odd}}} \exp\left(\pm \int_{x_0}^{x} S_{\rm odd} dx\right).$$

- **Some basic notions:**

⋄ **a (simple) turning point $\Longleftrightarrow$ a (simple) zero of Q_0**

⋄ **a Stokes curve $\Longleftrightarrow$ an integral curve of $\mathrm{Im}\sqrt{Q_0}dx = 0$ emanating from a turning point**

⋄ **a Stokes region $\Longleftrightarrow$ a region surrounded by Stokes curves**

⋄ **a regular singular point $\Longleftrightarrow$ a singular point r such that $(x-r)^2 Q_0$ is regular at $x = r$**

WKB solution normalized at a simple turning point $a \in \mathbb{C}$:

(3.10)
$$\psi_{\pm} := \frac{1}{\sqrt{S_{\mathrm{odd}}}} \exp\left(\pm \int_a^x S_{\mathrm{odd}} dx\right),$$

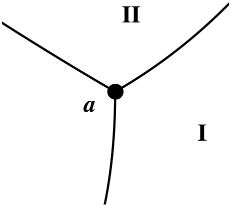
where the integration is understood as a half of the contour integral starting from x in the second sheet of the Riemann surface of $\sqrt{Q_0}$ going back to x in the first sheet detouring the turning point.

- **The Borel resummation**

⋄ **Under some conditions, a suitably normalized WKB solution ψ is Borel summable in each Stokes region (Koike-Schäfke).**

⋄ **The Borel sum of ψ in a Stokes region D is denoted by Ψ^D.**

- **Connection formula (Voros [V])**



⋄ **$\psi_{\pm}$: WKB solutions normalized at a simple turning point a**

⋄ **$\Psi^D_{\pm}$: The Borel sums of $\psi_{\pm}$ in $D = \mathrm{I}, \mathrm{II}$.**

⋄ **If $\mathrm{Re} \int_a^x \sqrt{Q_0}dx > 0$ on the boundary Stokes curve between I and II, then we have**

$$\Psi^{\mathrm{I}}_{+} = \Psi^{\mathrm{II}}_{+} + i\, \Psi^{\mathrm{II}}_{-},$$
$$\Psi^{\mathrm{I}}_{-} = \Psi^{\mathrm{II}}_{-}.$$

In this case, we say that ψ_+ is dominant (ψ_- is recessive) on the Stokes curve.

• WKB solutions normalized at a regular singular point

Assume that Q_0 has a double pole at $x = r$ and $(x-r)^2 Q_j$ ($j = 1, 2, \ldots, N$) are holomorphic at $x = r$.

⋄ **Define $\rho = \rho_0 + \eta^{-1}\rho_1 + \eta^{-2}\rho_2 + \cdots$ by**

$$\rho = \operatorname*{Res}_{x=r} \sqrt{Q} \qquad \left(Q = \sum_{j=0}^{N} \eta^{-j} Q_j \right).$$

By Proposition 3.6 in Kawai-Takei [KT], we have

$$\operatorname*{Res}_{x=r} S_{\mathrm{odd}} = \sigma\eta$$

with

$$\sigma = \rho \sqrt{1 + \frac{1}{4\rho^2\eta^2}}.$$

⋄ **WKB solutions normalized at the regular singular point $x = r$:**

$$\psi_{\pm}^{(r)} := \frac{(x-r)^{\pm\sigma\eta}}{\sqrt{S_{\mathrm{odd}}}} \exp\left(\pm \int_r^x \left(S_{\mathrm{odd}} - \frac{\sigma\eta}{x-r} \right) dx \right).$$

- **Recessive WKB solution at the regular singular point $x = r$**

We assume $\mathrm{Re}\,\rho_0 > 0$. Then $\psi_+^{(r)}$ is recessive on any Stokes curve flowing into $x = r$.

⋄ By the connection formula, the recessive WKB solution does not have Stokes phenomena on the Stokes curves.

Theorem 3.1 ([ATT2])

Set $\tilde{\psi}_+^{(r)} := (x - r)^{-\frac{1}{2}-\sigma\eta}\psi_+^{(r)}$. There is a neighborhood U of $x = r$ such that $\tilde{\psi}_+^{(r)}$ is Borel summable in $U - \{r\}$ and $x = r$ is a removable singularity of the Borel sum $\tilde{\Psi}_+^{(r)}$. Hence it is holomorphic in $U \times \{\eta;\ \mathrm{Re}\,\eta >> 0\}$. Moreover

$$\tilde{\Psi}_+^{(r)}(r,\eta) = \tilde{\psi}_+^{(r)}(r,\eta) = (\sigma\eta)^{-\frac{1}{2}}$$

holds.

- **Analytic solutions at the regular singular point r and WKB solutions**

⋄ **The characteristic exponents of our equation at $x = r$ are $\dfrac{1}{2} \pm \sigma\eta$.**

⋄ **There exist two independent analytic solutions $\Phi_\pm$ of the forms**

$$\Phi_\pm(x,\eta) = (x-r)^{\frac{1}{2}\pm\sigma\eta}\Phi_{\pm,0}(x,\eta)$$

of (3.1) **such that $\Phi_{\pm,0}(x,\eta)$ are holomorphic in a neighborhood of $x = r$ and $\Phi_{\pm,0}(r,\eta) = 1$.**

⋄ **By Theorem 3.1, the Borel sum $\Psi_+^{(r)}$ of $\psi_+^{(r)}$ near $x = r$ has the form**

$$\Psi_+^{(r)}(x,\eta) = (x-r)^{\frac{1}{2}+\sigma\eta}\tilde{\Psi}_+^{(r)}(x,\eta),$$

where $\tilde{\Psi}_+^{(r)}(x,\eta)$ is holomorphic near $x = r$ and $\tilde{\Psi}_+^{(r)}(r,\eta) = (\sigma\eta)^{-\frac{1}{2}}$.

Theorem 3.2 ([ATT2])

Under the assumptions and notation given above, we have the relation

$$\Phi_+(x,\eta) = (\sigma\eta)^{\frac{1}{2}}\Psi_+^{(r)}(x,\eta)$$

in a neighborhood of $x = r$.

Remark: If $\mathrm{Re}\,\rho_0 < 0$, we have to exchange "+" and "−".

4. Exact WKB analysis of the hypergeometric differential equation

- **We apply Theorem 3.2 to the hypergeometric differential equation**

$$x(1-x)\frac{d^2w}{dx^2} + (c-(a+b+1)x)\frac{dw}{dx} - abw = 0. \tag{4.1}$$

Introduce a large parameter η by setting

$$a = \alpha_0 + \alpha\eta, \quad b = \beta_0 + \beta\eta, \quad c = \gamma_0 + \gamma\eta.$$

Eliminate the first order term:

$$w = x^{-\frac{c}{2}}(1-x)^{-\frac{1}{2}(a+b-c+1)}\psi.$$

Equation for ψ: $\left(-\frac{d^2}{dx^2} + \eta^2 Q\right)\psi = 0.$ **Here $Q = Q_0 + \eta^{-1}Q_1 + \eta^{-2}Q_2$ with**

$$Q_0 = \frac{(\alpha-\beta)^2x^2 + 2(2\alpha\beta - \alpha\gamma - \beta\gamma)x + \gamma^2}{4x^2(x-1)^2},$$

$$Q_1 = \frac{(\alpha-\beta)(\alpha_0-\beta_0)x^2 + (2(\alpha\beta_0+\alpha_0\beta) - \beta\gamma_0 - \beta_0\gamma - \gamma\alpha_0 - \gamma_0\alpha + \gamma)x + \gamma(\gamma_0-1)}{2x^2(x-1)^2},$$

$$Q_2 = \frac{(\alpha_0-\beta_0+1)(\alpha_0-\beta_0-1)x^2 + 2(2\alpha_0\beta_0 - \beta_0\gamma_0 - \gamma_0\alpha_0 + \gamma_0)x + \gamma_0(\gamma_0-2)}{4x^2(x-1)^2}.$$

We assume $(\alpha,\beta,\gamma) \notin E_0 \cup E_1 \cup E_2$, where

$$E_0 = \{(\alpha,\beta,\gamma) \in \mathbb{C}^3 \mid \alpha\beta\gamma(\alpha-\beta)(\alpha-\gamma)(\beta-\gamma)(\alpha+\beta-\gamma) = 0\},$$
$$E_1 = \{(\alpha,\beta,\gamma) \in \mathbb{C}^3 \mid \mathrm{Re}\,\alpha\,\mathrm{Re}\,\beta\,\mathrm{Re}(\gamma-\alpha)\mathrm{Re}(\gamma-\beta) = 0\},$$
$$E_2 = \{(\alpha,\beta,\gamma) \in \mathbb{C}^3 \mid \mathrm{Re}(\alpha-\beta)\mathrm{Re}(\alpha+\beta-\gamma)\mathrm{Re}\,\gamma = 0\}.$$

Then there are two distinct turning points $a_0,\ a_1$ and no Stokes curves connect turning point(s).

We take the branch of $\sqrt{Q_0}$ as

$$\operatorname*{Res}_{x=0} \sqrt{Q_0} = \frac{\gamma}{2}.$$

Then we have

$$\operatorname*{Res}_{x=0} S_{\mathrm{odd}} = \frac{\gamma_0 - 1 + \gamma\eta}{2} = \frac{c-1}{2}$$

and

$$\operatorname*{Res}_{x=1} S_{\mathrm{odd}} = -\frac{\alpha_0 + \beta_0 - \gamma_0 + (\alpha+\beta-\gamma)\eta}{2} = -\frac{a+b-c}{2}$$

if we take the branch cut for $\sqrt{Q_0}$ suitably.

Normalization of WKB solutions:

⋄ **WKB solutions normalized at a_0 (a simple turning point):**

$$\psi_{\pm} = \frac{1}{\sqrt{S_{\text{odd}}}} \exp\left(\pm \int_{a_0}^{x} S_{\text{odd}} dx\right),$$

⋄ **WKB solutions normalized at the origin:**

$$\psi_{\pm}^{(0)} = \frac{x^{\pm \frac{1}{2}(c-1)}}{\sqrt{S_{\text{odd}}}} \exp\left(\pm \int_{0}^{x} \left(S_{\text{odd}} - \frac{c-1}{2x}\right) dx\right).$$

⋄ **WKB solutions normalized at $x = 1$:**

$$\psi_{\pm}^{(1)} = \frac{(x-1)^{\pm(c-a-b)}}{\sqrt{S_{\text{odd}}}} \exp\left(\pm \int_{1}^{x} \left(S_{\text{odd}} - \frac{c-a-b}{2(x-1)}\right) dx\right).$$

◇ **Dominance:**

$\mathrm{Re}\,\gamma > 0 \implies \psi_+$ is recessive at $x = 0$ $(\longleftrightarrow u_1)$

$\mathrm{Re}\,\gamma < 0 \implies \psi_-$ is recessive at $x = 0$ $(\longleftrightarrow u_5)$

Theorem 4.1

(i) If $\mathrm{Re}\,\gamma > 0$, then

$$ {}_2F_1(a,b,c;x) = \sqrt{\frac{c-1}{2}}\,x^{-\frac{c}{2}}(1-x)^{-\frac{1}{2}(a+b-c+1)}\Psi_+^{(0)} $$

holds near the origin. Here $\Psi_+^{(0)}$ denotes the Borel sum of the WKB solution $\psi_+^{(0)}$ normalized at the origin.

(ii) If $\mathrm{Re}\,\gamma < 0$, then

$$ x^{1-c}\,{}_2F_1(a-c+1, b-c+1, 2-c;x) = \sqrt{\frac{c-1}{2}}\,x^{-\frac{c}{2}}(1-x)^{-\frac{1}{2}(a+b-c+1)}\Psi_-^{(0)} $$

holds near the origin. Here $\Psi_-^{(0)}$ denotes the Borel sum of the WKB solution $\psi_-^{(0)}$ normalized at the origin.

$$\mathrm{Re}\,(\alpha+\beta-\gamma)>0 \implies \psi_- \text{ is recessive at } x=1 \quad (\longleftrightarrow u_2)$$
$$\mathrm{Re}\,(\alpha+\beta-\gamma)<0 \implies \psi_+ \text{ is recessive at } x=1 \quad (\longleftrightarrow u_6)$$

Theorem 4.2

(i) If $\mathrm{Re}\,(\alpha+\beta-\gamma)>0$, then

$$ {}_2F_1(a,b,a+b-c+1;1-x) = \sqrt{\frac{a+b-c}{2}}\,x^{-\frac{c}{2}}(1-x)^{-\frac{1}{2}(a+b-c+1)}\Psi_-^{(1)}$$

holds near $x=1$. Here $\Psi_-^{(1)}$ denotes the Borel sum of the WKB solution $\psi_-^{(1)}$ normalized at $x=1$.

(ii) If $\mathrm{Re}\,(\alpha+\beta-\gamma)<0$, then

$$(1-x)^{c-a-b}\,{}_2F_1(c-a,c-b,c-a-b+1;1-x) = \sqrt{\frac{c-a-b}{2}}\,x^{-\frac{c}{2}}(1-x)^{-\frac{1}{2}(a+b-c+1)}\Psi_+^{(1)}$$

holds near $x=1$. Here $\Psi_+^{(1)}$ denotes the Borel sum of the WKB solution $\psi_+^{(1)}$ normalized at $x=1$.

The first statement of Theorem 4.1 gives the relation between the hypergeometric function ${}_2F_1(a,b,c;x)$ and the WKB solution $\psi_+^{(0)}$ normalized at the origin when $\mathrm{Re}\,\gamma > 0$.

We consider the following two questions:

Q1: What is the relation between ${}_2F_1(a,b,c;x)$ and the WKB solutions $\psi_\pm$ normalized at the simple turning point a_0 when $\mathrm{Re}\,\gamma > 0$?

Q2: What happens when $\mathrm{Re}\,\gamma < 0$?

Answer to Q1:

Formally we have

$$\psi_{\pm}^{(0)} = \exp(\pm V_0)\,\psi_{\pm}$$

with

$$V_0 = \int_0^{a_0} \left(S_{\mathrm{odd}} - \frac{c-1}{2x} \right) dx + \frac{1}{2}(c-1)\log a_0$$

and

$$\psi_{\pm}^{(1)} = \exp(\pm V_1)\,\psi_{\pm}$$

with

$$V_1 = \int_1^{a_0} \left(S_{\mathrm{odd}} - \frac{c-a-b}{2(x-1)} \right) dx + \frac{1}{2}(c-a-b)\log(a_0-1).$$

We call V_0 (resp. V_1) the Voros coefficient of our equation of the origin (resp. of $x = 1$).

We may write

$$V_0 = V_{0,>0} + V_{0,\leq 0}$$

with

$$V_{0,>0} := \frac{1}{2}\int_{C_0} S_{\mathrm{odd},>0}\,dx,$$

$$V_{0,\leq 0} := \lim_{x\to 0}\frac{1}{2}\left(\int_{C_x} S_{\mathrm{odd},\leq 0}\,dx + (c-1)\log x\right),$$

where

$$S_{\mathrm{odd},>0} = \sum_{j>0}\eta^{-j}S_{\mathrm{odd},j}, \quad S_{\mathrm{odd},\leq 0} = \sum_{j\leq 0}\eta^{-j}S_{\mathrm{odd},j},$$

C_x**: a contour starting from** x**, going around** a_0 **and back to** x**.**

Similarly,

$$V_1 = V_{1,>0} + V_{1,\leq 0},$$

with

$$V_{1,>0} := \frac{1}{2}\int_{C_1} S_{\mathrm{odd},>0}\,dx,$$

$$V_{1,\leq 0} := \lim_{x\to 1}\frac{1}{2}\left(\int_{C_x} S_{\mathrm{odd},\leq 0}\,dx + (c-a-b)\log(x-1)\right).$$

Explicit forms of the Voros coefficients:

Theorem 4.3

$$V_{0,>0} = \frac{1}{2}\sum_{n=2}^{\infty}\frac{(-1)^{n-1}\eta^{1-n}}{n(n-1)}\left(\frac{B_n(\alpha_0)}{\alpha^{n-1}} + \frac{B_n(\beta_0)}{\beta^{n-1}} + \frac{B_n(\gamma_0-\alpha_0)}{(\gamma-\alpha)^{n-1}} + \frac{B_n(\gamma_0-\beta_0)}{(\gamma-\beta)^{n-1}} - \frac{B_n(\gamma_0)+B_n(\gamma_0-1)}{\gamma^{n-1}}\right),$$

$$V_{1,>0} = -\frac{1}{2}\sum_{n=2}^{\infty}\frac{(-1)^{n-1}\eta^{1-n}}{n(n-1)}\left(\frac{B_n(\alpha_0)}{\alpha^{n-1}} + \frac{B_n(\beta_0)}{\beta^{n-1}} - \frac{B_n(\gamma_0-\alpha_0)}{(\gamma-\alpha)^{n-1}} - \frac{B_n(\gamma_0-\beta_0)}{(\gamma-\beta)^{n-1}} - \frac{B_n(\alpha_0+\beta_0-\gamma_0)+B_n(\alpha_0+\beta_0-\gamma_0+1)}{(\alpha+\beta-\gamma)^{n-1}}\right).$$

Here $B_n(x)$ denotes the n-th Bernoulli polynomial :

$$\frac{te^{xt}}{e^t-1} = \sum_{n=0}^{\infty}\frac{B_n(x)}{n!}t^n.$$

Explicit forms of $V_{0,\leq 0}$ and $V_{1,\leq 0}$ depend on the choice of the simple turning point a_0 and of the branch of logarithms. Under suitable choice, we may write, for example,

$$V_{0,\leq 0} = \frac{1}{4}\left\{(a+b-c)\log\frac{(\alpha-\gamma)(\beta-\gamma)}{\alpha\beta} + (a-c)\log\frac{\beta(\alpha-\gamma)}{\alpha(\beta-\gamma)} - (c-1)\log\frac{\alpha\beta(\alpha-\gamma)(\beta-\gamma)}{\gamma^4}\right\},$$

$$V_{1,\leq 0} = \frac{1}{4}\left\{(c-1)\log\frac{\alpha\beta}{(\alpha-\gamma)(\beta-\gamma)} + (a-c)\log\frac{\alpha(\alpha-\gamma)}{\beta(\beta-\gamma)} + (a+b-c)\log\frac{\alpha\beta(\alpha-\gamma)(\beta-\gamma)}{(\alpha+\beta-\gamma)^4}\right\}.$$

The explicit forms of V_0 and V_1 are obtained by solving the following system of difference equations:

Lemma 4.4

$$\Delta_\alpha \partial_\alpha V_0 = \frac{1}{2}\left(\frac{1}{\alpha - \gamma + (\alpha_0 - \gamma_0 + 1)\eta^{-1}} - \frac{1}{\alpha + \alpha_0 \eta^{-1}}\right),$$

$$\Delta_\beta \partial_\beta V_0 = \frac{1}{2}\left(\frac{1}{\beta - \gamma + (\beta_0 - \gamma_0 + 1)\eta^{-1}} - \frac{1}{\beta + \beta_0 \eta^{-1}}\right),$$

$$\Delta_\gamma \partial_\gamma V_0 = \frac{1}{2}\left(\frac{1}{\gamma + \gamma_0 \eta^{-1}} + \frac{1}{\gamma + (\gamma_0 - 1)\eta^{-1}} + \frac{1}{\alpha - \gamma + (\alpha_0 - \gamma_0)\eta^{-1}} + \frac{1}{\beta - \gamma + (\beta_0 - \gamma_0)\eta^{-1}}\right).$$

Here we set $\Delta_\alpha := \exp(\eta^{-1}\partial_\alpha) - 1$, $\partial_\alpha := \partial/\partial\alpha$, etc.

Lemma 4.5

$$\Delta_\alpha \partial_\alpha V_1 = \frac{1}{2}\left(\frac{1}{\alpha + \alpha_0\eta^{-1}} + \frac{1}{\alpha - \gamma + (\alpha_0 - \gamma_0 + 1)\eta^{-1}} - \frac{1}{\alpha + \beta - \gamma + (\alpha_0 + \beta_0 - \gamma_0)\eta^{-1}} - \frac{1}{\alpha + \beta - \gamma + (\alpha_0 + \beta_0 - \gamma_0 + 1)\eta^{-1}}\right),$$

$$\Delta_\beta \partial_\beta V_1 = \frac{1}{2}\left(\frac{1}{\beta + \beta_0\eta^{-1}} + \frac{1}{\beta - \gamma + (\beta_0 - \gamma_0 + 1)\eta^{-1}} - \frac{1}{\alpha + \beta - \gamma + (\alpha_0 + \beta_0 - \gamma_0)\eta^{-1}} - \frac{1}{\alpha + \beta - \gamma + (\alpha_0 + \beta_0 - \gamma_0 + 1)\eta^{-1}}\right),$$

$$\Delta_\gamma \partial_\gamma V_1 = \frac{1}{2}\left(\frac{1}{\gamma - \alpha - \beta + (\gamma_0 - \alpha_0 - \beta_0 + 1)\eta^{-1}} + \frac{1}{\gamma - \alpha - \beta + (\gamma_0 - \alpha_0 - \beta_0)\eta^{-1}} - \frac{1}{\gamma - \alpha + (\gamma_0 - \alpha_0)\eta^{-1}} - \frac{1}{\gamma - \beta + (\gamma_0 - \beta_0)\eta^{-1}}\right).$$

These systems can be solved by using formal differential operators of infinite order of the form

$$(\exp(\eta^{-1}\partial_\alpha) - 1)^{-1}\eta^{-1}\partial_\alpha \exp(\alpha_0\eta^{-1}\partial_\alpha) = \sum_{n=0}^{\infty} \frac{B_n(\alpha_0)}{n!}(\eta^{-1}\partial_\alpha)^n.$$

• Borel sums of the Voros coefficients

Divergent parts of the Voros coefficients consist of sums of formal series of the form

$$U(\tau, s, \eta) := \frac{1}{2}\sum_{n=2}^{\infty} \frac{(-1)^{n-1}B_n(s)\eta^{1-n}}{n(n-1)\tau^{n-1}}.$$

This is Borel summable if $\mathrm{Re}\,\tau \neq 0$ and the Borel sum $\mathcal{U}_{\pm}$ of U with respect to η^{-1} depends on the signature of $\mathrm{Re}\,\tau$:

$$\mathrm{Re}\,\tau > 0 \quad \Longrightarrow \quad \mathcal{U}_{+} = \frac{1}{2}\log\frac{(\tau\eta)^{\tau\eta+s-\frac{1}{2}}\sqrt{2\pi}}{\Gamma(s+\tau\eta)e^{\tau\eta}},$$

$$\mathrm{Re}\,\tau < 0 \quad \Longrightarrow \quad \mathcal{U}_{-} = \frac{1}{2}\log\frac{\Gamma(1-s-\tau\eta)(-\tau\eta)^{\tau\eta+s-\frac{1}{2}}}{e^{\tau\eta}\sqrt{2\pi}}.$$

Thus the explicit forms of the Borel sums of V_0 and V_1 depend on the signatures of

$$\mathrm{Re}\,\alpha, \quad \mathrm{Re}\,\beta, \quad \mathrm{Re}\,(\alpha-\gamma), \quad \mathrm{Re}\,(\beta-\gamma), \quad \mathrm{Re}\,(\alpha+\beta-\gamma).$$

These signatures determine the type of Stokes geometry of the hypergeometric differential equation.

Characterization of the Stokes geometry in terms of parameters

$$\omega_1 = \{(\alpha,\beta,\gamma) \in \mathbb{C}^3 \mid 0 < \mathrm{Re}\,\alpha < \mathrm{Re}\,\gamma < \mathrm{Re}\,\beta\},$$
$$\omega_2 = \{(\alpha,\beta,\gamma) \in \mathbb{C}^3 \mid 0 < \mathrm{Re}\,\alpha < \mathrm{Re}\,\beta < \mathrm{Re}\,\gamma < \mathrm{Re}\,\alpha + \mathrm{Re}\,\beta\},$$
$$\omega_3 = \{(\alpha,\beta,\gamma) \in \mathbb{C}^3 \mid 0 < \mathrm{Re}\,\gamma < \mathrm{Re}\,\alpha < \mathrm{Re}\,\beta\},$$
$$\omega_4 = \{(\alpha,\beta,\gamma) \in \mathbb{C}^3 \mid 0 < \mathrm{Re}\,\gamma < \mathrm{Re}\,\alpha + \mathrm{Re}\,\beta < \mathrm{Re}\,\beta\},$$
$$G = \text{ group generated by } \iota_m \ (m = 0, 1, 2),$$

where ι_m $(m = 0, 1, 2)$ are involutions in the parameter space defined by $\iota_0 : (\alpha,\beta,\gamma) \mapsto (\beta,\alpha,\gamma)$, $\iota_1 : (\alpha,\beta,\gamma) \mapsto (\gamma-\alpha,\gamma-\beta,\gamma)$ and $\iota_2 : (\alpha,\beta,\gamma) \mapsto (-\alpha,-\beta,-\gamma)$.
Moreover, we set $\Pi_k = \bigcup_{r\in G} r(\omega_k) \quad (k = 1, \ldots, 4)$.

Theorem 4.6 [AT1].

Let n_* $(* = 0, 1, \infty)$ denote the number of Stokes curves flowing into the singular point $*$ and set $\hat{n} = (n_0, n_1, n_\infty)$.
(1) $(\alpha,\beta,\gamma) \in \Pi_1 \Longrightarrow \hat{n} = (2, 2, 2)$. (2) $(\alpha,\beta,\gamma) \in \Pi_2 \Longrightarrow \hat{n} = (4, 1, 1)$.
(3) $(\alpha,\beta,\gamma) \in \Pi_3 \Longrightarrow \hat{n} = (1, 4, 1)$. (4) $(\alpha,\beta,\gamma) \in \Pi_4 \Longrightarrow \hat{n} = (1, 1, 4)$.

By our assumption, $(\alpha, \beta, \gamma) \in \Pi_k$ for some k and then V_0 and V_1 are Borel summable. To specify the explicit forms of the Borel sums, we assume α, β, γ to be real.

Theorem 4.7

If $(\alpha, \beta, \gamma) \in \omega_1$, the Borel sum V_0^1 of V_0 has the following form:

$$V_0^1 = \frac{1}{2} \log \frac{\Gamma(b-c)\Gamma(c)\Gamma(c-1)e^{\frac{\pi i}{2}(a-c)}}{\Gamma(a)\Gamma(b)\Gamma(c-a)}.$$

Here we set

$$a = \alpha_0 + \alpha\eta, \quad b = \beta_0 + \beta\eta, \quad c = \gamma_0 + \gamma\eta.$$

Other cases can be managed similarly.
Taking the Borel sums of the formal relation

$$\psi_+^{(0)} = \exp(V_0)\psi_+,$$

we have the following analytic relation:

$$\Psi_+^{(0)} = \exp(V_0^1)\Psi_+.$$

Combining this and Theorem 4.1, (i), we have

Theorem 4.8

Suppose that $\gamma > 0$. Let Ψ_+ be the Borel sum of the recessive WKB solution ψ_+ at the origin normalized at the simple turning point a_0.

For $(\alpha, \beta, \gamma) \in \omega_j$ $(j = 1, 2, 3, 4)$, we have the relation:

$$_2F_1(a, b, c; x) = C_j\, x^{-\frac{c}{2}} (1 - x)^{-\frac{1}{2}(a+b-c+1)} \Psi_+.$$

with a constant C_j given by

$$C_1 = \frac{e^{-\frac{\pi i}{2}(c-a-\frac{1}{2})}\Gamma(c)\Gamma(b-c+1)^{\frac{1}{2}}}{\sqrt{2}\left\{\Gamma(a)\Gamma(b)\Gamma(c-a)\right\}^{\frac{1}{2}}}, \qquad C_2 = \frac{e^{-\frac{\pi i}{2}(2c-a-b-1)}\sqrt{\pi}\Gamma(c)}{\left\{\Gamma(a)\Gamma(b)\Gamma(c-a)\Gamma(c-b)\right\}^{\frac{1}{2}}},$$

$$C_3 = \frac{\Gamma(c)\left\{\Gamma(a-c+1)\Gamma(b-c+1)\right\}^{\frac{1}{2}}}{2\sqrt{\pi}\left\{\Gamma(a)\Gamma(b)\right\}^{\frac{1}{2}}}, \quad C_4 = \frac{e^{-\frac{\pi i}{2}(c-1)}\Gamma(c)\left\{\Gamma(1-a)\Gamma(b-c+1)\right\}^{\frac{1}{2}}}{2\sqrt{\pi}\left\{\Gamma(b)\Gamma(c-a)\right\}^{\frac{1}{2}}}.$$

Answer to Q2: **What happens when $\mathrm{Re}\,\gamma < 0$?**

We give an answer for the case where $(\alpha, \beta, \gamma) \in \iota_0(\omega_1)$. Other cases can be treated similarly.

Theorem 4.9

If $\beta < \gamma < \alpha < 0$, we have the relation

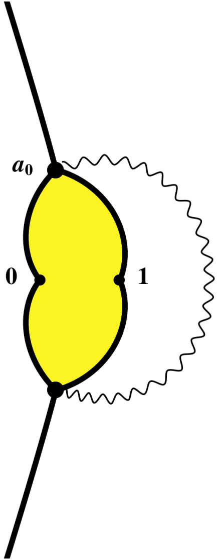
$$ {}_2F_1(a, b, c; x) = x^{-\frac{c}{2}}(1-x)^{-\frac{1}{2}(a+b-c+1)}(C_{11}\Psi_+ + C_{21}\Psi_-) $$

with

$$ C_{11} = \frac{e^{\frac{\pi i}{2}(b-c+\frac{1}{2})}\left\{\Gamma(1-a)\Gamma(1-b)\Gamma(a-c+1)\right\}^{\frac{1}{2}}}{\sqrt{2}\Gamma(1-c)\Gamma(c-b)^{\frac{1}{2}}}, $$

$$ C_{21} = \frac{e^{\frac{\pi i}{2}(c-b+\frac{1}{2})}\Gamma(c)\left\{\Gamma(1-a)\Gamma(1-b)\right\}^{\frac{1}{2}}}{\sqrt{2}\Gamma(1-a)\left\{\Gamma(c-b)\Gamma(a-c+1)\right\}^{\frac{1}{2}}}. $$

Here $\Psi_{\pm}$ are Borel sums of $\psi_{\pm}$ in the yellow-colored region:



• Applications

If we replace $\Psi_\pm$ by $\psi_\pm$ in our relations, we have asymptotic expansion formulas of ${}_2F_1(a, b, c; x)$ with respect to η^{-1} (Watson's Lemma).

The leading term of WKB solution ψ_+ is

$$\frac{\sqrt{2x(x-1)}}{G^{\frac{1}{4}}}\left\{\frac{(\alpha-\beta)^2x+2\alpha\beta-\beta\gamma-\gamma\alpha+(\alpha-\beta)\sqrt{G}}{(\alpha-\beta)^2x+2\alpha\beta-\beta\gamma-\gamma\alpha-(\alpha-\beta)\sqrt{G}}\right\}^{\frac{\alpha-\beta}{4}}$$

$$\times\left\{\frac{(\alpha^2+\beta^2+(\beta-\alpha)\gamma)x+2\alpha\beta-\beta\gamma-\gamma\alpha+\gamma^2-(\alpha+\beta-\gamma)\sqrt{G}}{(\alpha^2+\beta^2+(\beta-\alpha)\gamma)x+2\alpha\beta-\beta\gamma-\gamma\alpha+\gamma^2+(\alpha+\beta-\gamma)\sqrt{G}}\right\}^{\frac{\alpha+\beta-\gamma}{4}}$$

$$\times\left\{\frac{(2\alpha\beta-\beta\gamma-\gamma\alpha)x+\gamma^2+\gamma\sqrt{G}}{(2\alpha\beta-\beta\gamma-\gamma\alpha)x+\gamma^2-\gamma\sqrt{G}}\right\}^{\frac{\gamma}{4}}.$$

Here we set

$$G=(\alpha-\beta)^2x^2+2(2\alpha\beta-\beta\gamma-\gamma\alpha)x+\gamma^2.$$

For example, we have not only an alternative proof of the asymptotic formula for the monic Jacobi polynomial

$$\hat{P}_n^{(nA,nB)}(x) = 2^n \frac{\Gamma(n(A+1)+1)\Gamma(n(A+B+1)+1)}{\Gamma(n(A+B+2)+1)\Gamma(nA+1)} \times {}_2F_1(-n, n(A+B+1)+1, nA+1; \frac{1-x}{2}).$$

$$(-1 < A < 0,\ -1 < B < 0,\ -2 < A+B < -1)$$

as $n \to \infty$ obtained by Kuijlaars and Martínez-Finkelshtein but also an asymptotic expansion formula for all orders:

$$\hat{P}_n^{(nA,nB)}(1-2z) \sim$$

$$2^n z^{-\frac{1}{2}(1+nA)}(1-z)^{-\frac{1}{2}(1+nB)} \frac{\Gamma(n(1+A+B)+1)\Gamma(-(1+A+B)n)\Gamma(n+1)}{\Gamma(1+n(2+A+B))\Gamma(-(1+B)n)\Gamma(1-An)}$$

$$\times \Big[\sqrt{\frac{-An}{2}} e^{\lambda_1} \Big(\frac{\Gamma(1+n(1+A))\Gamma(1-An)\Gamma(-An)}{\Gamma(n+1)\Gamma(-(1+B+A)n)\Gamma(1+(B+1)n)} \Big)^{\frac{1}{2}} \psi_+$$

$$+ \frac{1}{i} \sqrt{\frac{-Bn}{2}} e^{\lambda_2} \Big(\frac{\Gamma(1+n(1+B))\Gamma(1-Bn)\Gamma(-Bn)}{\Gamma(n+1)\Gamma(-(1+B+A)n)\Gamma(1+(A+1)n)} \Big)^{\frac{1}{2}}$$

$$\times \frac{\Gamma(1-An)\Gamma(Bn)}{\Gamma(-(1+A)n)\Gamma(1+(1+B)n)} \psi_- \Big]$$

Summary and concluding remarks

- **A large parameter is introduced in the 3 parameters in the Gauss hypergeometric differential equation.**
- **One can construct WKB solutions of the equation. Taking the Borel sum, we have analytic solutions of the Gauss hypergeometric equation.**
- **The Gauss hypergeometric function $_2F_1(a, b, c; x)$ can be written explicitly in terms of those Borel resummed WKB solutions under the condition that the Stokes geometry is non-degenerate.**
- **As an application, one can obtain asymptotic expansion formulas for the hypergeometric function with respect to the parameter.**

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Thank you for your attention.