On Voros coefficients in exact WKB analysis

Tatsuya KOIKE (Kobe Univ.)

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Outline

§1. Exact WKB analysis

§2. Voros coefficients for the Weber equation

§3. Transformation Theory to the Weber equation

§4. Conclusion

Exact WKB analysis

For an equation

$$\left(-\hbar^2\frac{d^2}{dx^2}+Q(x)\right)\psi=0 \quad (\hbar>0\text{: small},\quad x\in\mathbb{C}),$$

we can construct a formal solution called a WKB solution of the form

$$\psi(x,\hbar)=\exp\left[\int_{x_0}^{x}S(x,\hbar)dx\right],$$
$$S(x,\hbar)=\frac{1}{\hbar}S_{-1}(x)+S_0(x)+\hbar S_1(x)+\cdots$$

with an appropriate point x_0. By substitution, we find

$$S_{-1}(x)=\pm\sqrt{Q(x)},\quad S_0(x)=-\frac{Q'(x)}{4Q(x)},\quad S_1(x)=\pm\left(\frac{Q''(x)}{8Q(x)^{3/2}}-\frac{5}{32}\frac{Q'(x)^2}{Q(x)^{5/2}}\right),$$

and so on. Thus we have two formal solutions

$$\psi_\pm(x,\hbar)=\exp\left[\int_{x_0}^{x}S^{(\pm)}(x,\hbar)dx\right]=\exp\left[\sum_{n\geq -1}\hbar^n\int_{x_0}^{x}S_n^{(\pm)}(x)dx\right].$$

Following Voros ([V83]), we give an analytic meaning to a formal WKB solution by Borel resummation with respect to $\hbar$. In this talk, we assume $Q(x)$ is a polynomial or a rational function.

Theorem (Ko-Schäfke)

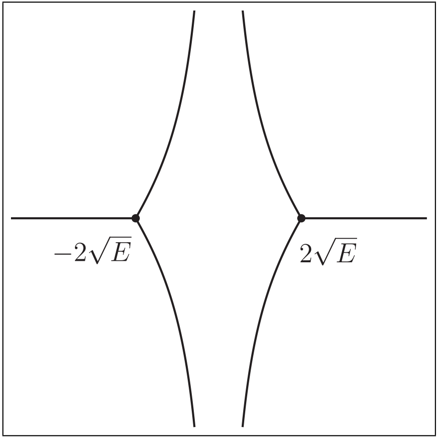
WKB solutions is Borel summable (in a direction 0) if a path of integration in WKB soluitons does not intersect any Stokes curves.

Here Stokes curves are curves defined by

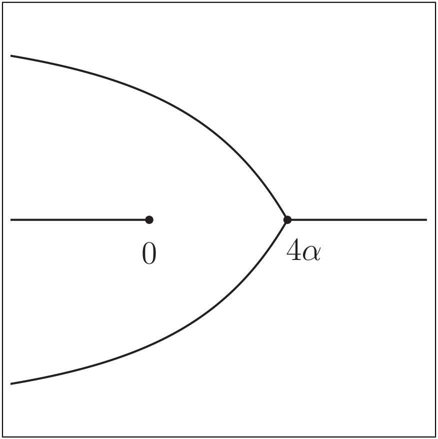
$$\mathrm{Im} \int_a^x \sqrt{Q(x)} dx = 0,$$

where a is a turning point (i.e., a zero of $Q(x)$) or a simple pole of $Q(x)$ (cf., e.g., [Ko00]).

Examples of Stokes curves



$$Q(x) = \frac{1}{4}x^2 - E \ (E > 0)$$



$$Q(x) = \frac{x - 4\alpha}{4x} \ (\alpha > 0)$$

Voros coefficients

For singular points b_0 and b_1 of the equation (precisely speaking, b_0 and b_1 are poles of $Q_0(x)$ of order ≥ 2), we define the Voros coefficient by

$$V \text{ ``$=$'' } \int_{b_0}^{b_1} S^{(+)}(x,\hbar)dx.$$

Here "$=$" means that the LHS should be properly regularized.

- If both of b_j's are irregular singular points (i.e., poles of $Q(x)$ of order ≥ 3), we define
$$V = \int_{b_0}^{b_1} \left\{ S^{(+)}(x,\hbar) - \frac{1}{\hbar} S_{-1}^{(+)}(x) - S_0^{(+)}(x) \right\} dx.$$
- If one of b_j's is a double pole of $Q(x)$, this integral does not converge because $S_j(x)$ has a simple pole at a double pole of $Q(x)$. One way to regularize this integral is to substract more terms from the integrand. The other way is to add lower order terms w.r.t. $\hbar$ to $Q(x)$ in order to cancel simple poles of $S_j(x)$ out. (We will not discuss this case in this talk.)

- Some global quantities (such as secular equations for eigenvalues) are expressed by using them (and "periods").
- They are also useful to study singularity structures, especially the so-called fixed singular points, of the Borel transform of WKB solutions.
  - Parametric Stokes phenomenons (i.e., Stokes phenomenons with respect to parameters of the equation in question).

There are many results about Voros coefficients:

- Voros ([Vo83]) in his study of the anharmonic oscillator.
- General Properties:
  - Delabaere-Dillinger-Pham ([DDP93]): Voros symbols, Voros multipliers, Voros ring, $\cdots$
  - Iwaki-Nakanishi (e.g., [IN14]) : a link to the cluster algebra.

- Explicit forms for
  - the Weber equation ([Vo83], [KT94], [SS08], [T08]) ,
  - the Whittaker equation ([KoT11]),
  - the Legendre equation ([Ko]),
  - the Bessel equation ([Iwaki, 2013]),
  - Gauss' hypergeometric equation (Aoki-Tanda, [AKT13], $\cdots$),
  - Kummer's confluent hypergeometric equation, (Aoki-Takahashi-Tanda, [ATT13] and Takahashi [T17], $\cdots$).
- Transformation theory:
  - MTP (= Merging-Turning-Points) equation (Aoki-Kawai-Takei, [AKT10])
  - MPPT (= Mering Pair of a simple Pole and simple Turning point) equation (Kawai-Kamimoto-Ko-Takei, [KKKoT10])
  - M2P1T (= Merging triplet of Two simple Poles and One simple Turning point) equations (Kamimoto-Kawai-Takei, [KKT14]).

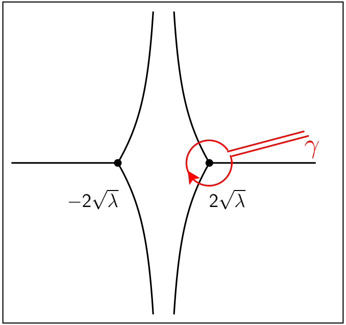
Voros coefficients for the Weber equation

$$\left(-\hbar^2\frac{d^2}{dx^2}+\frac{1}{4}x^2-\lambda\right)\psi=0$$

In this case

$$\psi_\pm(x,\lambda;\hbar)=\exp\left[\int^x S^{(\pm)}(x,\lambda;\hbar)dx\right],$$

$$S^{(\pm)}(x,\lambda;\hbar)=\pm\frac{1}{\hbar}\sqrt{\frac{1}{4}x^2-E}-\frac{x}{2(x^2-4\lambda)}\mp\hbar\frac{3x^2+8\lambda}{4(x^2-4\lambda)^{5/2}}+\cdots$$



$$\left(-\hbar^2\frac{d^2}{dx^2}+\frac{1}{4}x^2-\lambda\right)\psi=0$$

We then obtain

$$\begin{aligned} V(\lambda;\hbar) &= \int_\gamma \left\{ S^{(+)}(x,\lambda;\hbar)-\frac{1}{\hbar}S_{-1}(x,\lambda)-S_0(x,\lambda)\right\}dx \\ &= \sum_{n=1}^{\infty}\frac{2^{1-2n}-1}{2n(2n-1)}\cdot B_{2n}\cdot\left(\frac{\hbar}{\lambda}\right)^{2n-1} \end{aligned}$$

where B_{2n} is the $2n$-th Bernoulli number. This relation is proved by

- Voros (1983) (by using functional determinants),
- Shen and Silverstone (2008) (by comparing the asymptotic behavior of WKB solutions and the Weber function),
- Takei (2008) (by using the ladder operators).

Takei's proof can be applicable to equations whose ladder operaters are known. All the result of explicit forms of the Voros coefficients listed before are, more or less, based on his idea.

Sketch of a proof by Takei

- Because $\mathcal{A} = \hbar \frac{d}{dx} - \frac{1}{2}x$ is the rasing operator, we have

$$\begin{aligned}\psi_+(x, \lambda + \hbar; \hbar) &= (\text{const.}) \times \mathcal{A}\psi_+(x, \lambda; \hbar) \\ &= (\text{const.}) \times \left\{ \hbar S(x, \lambda; \hbar) - \frac{1}{2}x \right\} \psi_+(x, \lambda; \hbar)\end{aligned}$$

- The logarighmic derivative of both hand sides gives

$$S^{(+)}(x, \lambda + \hbar; \hbar) = S^{(+)}(x, \lambda; \hbar) - \frac{d}{dx} \log \left[S^{(+)}(x, \lambda; \hbar) - \frac{1}{2}x \right]$$

- By integration,

$$V(\lambda + \hbar; \hbar) = V(\lambda; \hbar) + 1 + \log(1 + \frac{\hbar}{2\lambda}) - \left(\frac{\lambda}{\hbar} + 1 \right) \log(1 + \frac{\hbar}{\lambda}).$$

- Solving this difference equation, we obtain the result.

Remark Voros coefficients for other equaitons:

- The Whittaker equation: $Q(x,\hbar) = \dfrac{x-4\alpha}{4x} + \hbar^2 \dfrac{\gamma(\gamma+1)}{x^2}$

$$V = 2\sum_{n\geq 1} \frac{B_{2n}(-\gamma)}{2n(2n-1)} \left(\frac{\hbar}{\alpha}\right)^{2n-1}.$$

  Here $B_m(X)$ is the m-th Bernoulli polynomial.

- Kummer' equaiton ([Aoki-Takahashi-Tanda, [ATT13] and Takahashi [T17]):

$$Q(x,\hbar) = \frac{x^2 + 2(2\alpha-\gamma)x + \gamma^2}{4x^2} + \hbar\frac{(2\alpha_0-\gamma_0)x + \gamma(\gamma_0-1)}{2x^2} + \hbar^2\frac{\gamma_0^2 - 2\gamma_0}{4x^2}$$

$$V_0 = \sum_{n\geq 1} \frac{(-1)^n\hbar^n}{n(n+1)} \left\{ \frac{B_{n+1}(\alpha_0)}{\alpha^n} + \frac{B_{n+1}(\gamma_0-\alpha_0)}{(\gamma-\alpha)^n} - \frac{B_{n+1}(\gamma_0) + B_n(\gamma_0-1)}{\gamma^n} \right\}$$

- Higher order differential equations are also discussed. For a parameter brought by middle convolution, we can find the ladder operators. We can find how Voros coefficients depend on such parameters (Iwaki-Ko, [IKo14]).

A consequence of the explicit formula: Stokes phenomenon w.r.t. λ

By the Borel transformation ($\hbar^m \mapsto \dfrac{y^{m-1}}{(m-1)!}$), we obtain

$$V_B(\lambda; y) = \frac{1}{2y}\left(-\frac{2\lambda}{y} + \frac{1}{e^{y/(2\lambda)} - 1} + \frac{1}{e^{y/(2\lambda)} + 1}\right).$$

Therefore V_B has singular ponits at

$$y = 2m\pi i\lambda = m \cdot 2\int_{-2\sqrt{\lambda}}^{2\sqrt{\lambda}} S_{-1}(x, \lambda)dx \qquad (m \in \mathbb{Z} \setminus \{0\}).$$

All of them are simple poles, and

$$\operatorname*{Res}_{y=2m\pi i\lambda} V_B(\lambda, \hbar) = \frac{(-1)^m}{2im}$$

(1) From these formula, we obtain

$$\begin{aligned}\mathcal{S}[V](\lambda e^{(\frac{\pi}{2}+\varepsilon)i}) &= \mathcal{S}[V](\lambda e^{(\frac{\pi}{2}-\varepsilon)i}) - \sum_{m=1}^{\infty} 2\pi i \operatorname*{Res}_{y=2m\pi i\lambda}[V_B(\lambda; y)]e^{-2mi\pi\lambda} \\ &= \mathcal{S}[V](\lambda e^{(\frac{\pi}{2}-\varepsilon)i}) + \log(1 + e^{-2\pi i\lambda}).\end{aligned}$$

(2) Typical normalizations of WKB solutions are

$$\psi_{\pm} = \frac{1}{\sqrt{S_{\mathrm{odd}}(x,\lambda;\hbar)}} \exp\left(\pm \int_{2\sqrt{\lambda}}^{x} S_{\mathrm{odd}}(x,\lambda;\hbar)dx\right),$$

which is called WKB solutions normalized at a turning point, and

$$\begin{aligned}\varphi_{\pm}^{(\infty)} = \frac{1}{\sqrt{S_{\mathrm{odd}}(x,\lambda;\hbar)}} &\exp\left(\pm \frac{1}{\hbar}\int_{2\sqrt{\lambda}}^{x} S_{-1}(x,\lambda)dx\right) \\ &\times \exp\left(\pm \int_{\infty}^{x} \left\{S_{\mathrm{odd}}(x,\lambda;\hbar) - \frac{1}{\hbar}S_{-1}(x,\lambda)\right\}dx\right),\end{aligned}$$

which is called WKB solutions normalized at the infinity. Here we set

$$S_{\mathrm{odd}}(x,\lambda;\hbar) := \sum_{n\geq -1,\mathrm{odd}} \hbar^n S_n(x,\lambda), \quad S_{\mathrm{even}}(x,\lambda;\hbar) := \sum_{n\geq -1,\mathrm{even}} \hbar^n S_n(x,\lambda),$$

and use the relation

$$\begin{aligned} S^{(\pm)}(x,\lambda;\hbar) &= \pm S_{\mathrm{odd}}(x,\lambda;\hbar) + S_{\mathrm{even}}(x,\lambda;\hbar), \\ S_{\mathrm{even}}(x,\lambda;\hbar) &= -\frac{1}{2}\frac{d}{dx}\log S_{\mathrm{odd}}(x,\lambda;\hbar). \end{aligned}$$

- WKB solution normalized at a turning point is a suitable form to describe connection formula of (Borel sum of) WKB solutions:

$$\psi_+(x,\hbar) \longmapsto \psi_+(x,\hbar) + i\psi_-(x,\hbar)$$

  holds when we cross a Stokes curves in a counter-clockwise manner and ψ_+ is dominant there.

- WKB solutions normalized at the infinity has a good asymptotic behavior when $x \to \infty$:

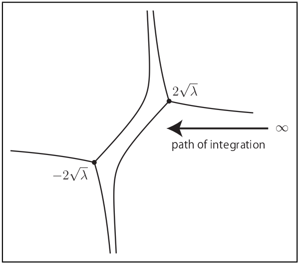
$$\varphi_+^{(\infty)}(x,\hbar) \sim \hbar^{-1/2} \frac{1}{\sqrt[4]{Q(x)}} \exp\left[\frac{1}{\hbar}\int_{2\lambda}^{x} Q(x)dx\right]\left(1 + O\left(\frac{1}{x}\right)\right),$$

  and hence it is easy to compare this and classical Weber functions.

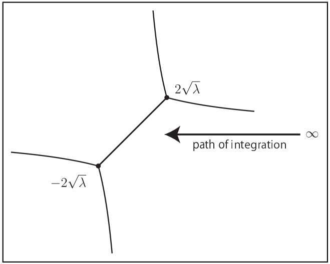
- The Voros coefficient connects these two solutions:

$$\begin{aligned}\psi_\pm(x,\lambda;\hbar) &= \exp\left[\pm\int_{2\sqrt{\lambda}}^{\infty}\left(S_{\mathrm{odd}}(x,\lambda;\hbar) - \frac{1}{\hbar}S_{-1}(x,\lambda)\right)dx\right]\varphi_\pm^{(\infty)}(x,\lambda;\hbar)\\ &= \exp\left[\pm\frac{1}{2}V(\lambda;\hbar)\right]\varphi_\pm^{(\infty)}(x,\lambda;\hbar).\end{aligned}$$

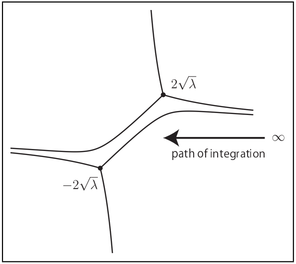
- When we change $\arg\lambda$ from $\pi/2-\varepsilon$ to $\pi/2+\varepsilon$, the path of integration in $\varphi_{\pm}^{(\infty)}$ does not cross any Stokes curves (Hence $\varphi_{\pm}^{(\infty)}$ is Borel summable).



arg $\lambda=\pi/2-\varepsilon$



arg $\lambda=\pi/2$



arg $\lambda=\pi/2+\varepsilon$

- Hence
$$\mathcal{S}[\varphi_{\pm}^{(\infty)}]\Big|_{\arg\lambda=\pi/2+\varepsilon} = \mathcal{S}[\varphi_{\pm}^{(\infty)}]\Big|_{\arg\lambda=\pi/2-\varepsilon}.$$
Therefore relations
$$\psi_{\pm} = \exp\left[\pm\frac{1}{2}V(\lambda;\hbar)\right]\varphi_{\pm}^{(\infty)}$$
and
$$\mathcal{S}[V]\Big|_{\arg\lambda=\pi/2+\varepsilon} = \mathcal{S}[V]\Big|_{\arg\lambda=\pi/2-\varepsilon} + \log(1+e^{-2\pi i\lambda}).$$
gives
$$\mathcal{S}[\psi_{\pm}]\Big|_{\arg\lambda=\pi/2+\varepsilon} = (1+e^{-2\pi i\lambda})^{\pm 1/2}\mathcal{S}[\psi_{\pm}]\Big|_{\arg\lambda=\pi/2-\varepsilon}.$$
- Here we study Stokes phenomenon of WKB solutions $\psi_{\pm}$. By the Borel transformation, we can also study the analytic structure (e.g., locations of singularities, their orders, etc.) of $\psi_{\pm,B}$, the Borel transform of WKB solutions.

Transformation Theory to the Weber equation

An MTP operator is introduced by Aoki-Kawai-Takei [AKT09]:

$$P = -\hbar^2 \frac{d^2}{dx^2} + Q(x, t),$$

where

- $Q(x, t)$ is holomorphic near $(x, t) = (0, 0)$,
- $Q(x, t) = cx^2 + O(x^3)$ with a constant c,
- for each t small enough, the equation $Q(x, t) = 0$ in x has two simple ditinct roots which merge together at $t = 0$, whereas other roots stay uniformly away from 0.
- We also assume some condition about the merging speed of two simple turing points.

In [AKT09], the formal change of variable

$$z = z(x,\hbar) = z_0(x) + z_1(x)\hbar + z_2(x)\hbar^2 + \cdots$$

is constructed for an MTP operator P such that $P\psi = 0$ transforms uniformly to

$$\left(-\hbar^2 \frac{d^2}{dz^2} - \frac{1}{4}z^2 + E(t,\hbar)\right)\varphi = 0$$

with

$$E(t,\hbar) = \frac{1}{2\pi i}\int_\gamma S_{\mathrm{odd}}(x,t,\hbar)dx.$$

Here γ is a closed path which encircles two merging turning points in a counter-clockwise manner. In this transformation, $\psi_\pm$ relates to $\varphi_\pm$ by

$$\psi_\pm(x,t;\hbar) = \left(\frac{\partial z}{\partial x}\right)^{-1/2} \varphi(z(x,\hbar), E(\hbar); \hbar)$$

After the Borel transformation, this relation becomes

$$\psi_{\pm,B}(x,t,y) = P(x,\partial_x,\partial_y)\varphi(z_0(x), E_0, y)$$

with an apropriate microdifferential operator P, and we can find the alien derivative of $\psi_{\pm,B}(x,t,y)$.

Conclusion

Voros coefficients are used to describe the analytic continuation of Borel sum of WKB solutions, and also used to study the analytic properties of Borel transform of WKB solution.

- Explicit computations of Voros coefficients.
- Stokes phenomenon w.r.t parameters:

  By using explicit forms of its Voros coefficient and WKB solutions normalized at infinity, we can describe Stokes phenomenons for WKB solutions normalized at a turning point.
- Transformation theory:

  Even more general equations, such as an MTP equation, we can study several properties by using transformation theory.

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