SEMICLASSICAL ORIGINS OF NUCLEAR SHELL STRUCTURES

Ken-ichiro Arita (Nagoya Inst. Tech.)

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"Exploration of hidden symmetries in atomic nuclei"

1. Introduction
   – Semiclassical theory of shell structures
   – Periodic orbit bifurcations and local dynamical symmetries
2. The radial power-law potential model with spin-orbit coupling
   – Approximation to the Woods-Saxon model
   – Scaling properties and Fourier transformation technique
3. Semiclassical analysis of nuclear shell structures
   3-1. Spherical magic numbers, bifurcation and dynamical symmetry
   3-2. Semiclassical description of the prolate-shape dominance
   3-3. Anomalous shell effect at large tetrahedral deformation
4. Summary

1. Introduction

❏ *Level density and shell energy*

Nuclear deformations

⇔ Shell structures in single-particle spectra for deformed potentials

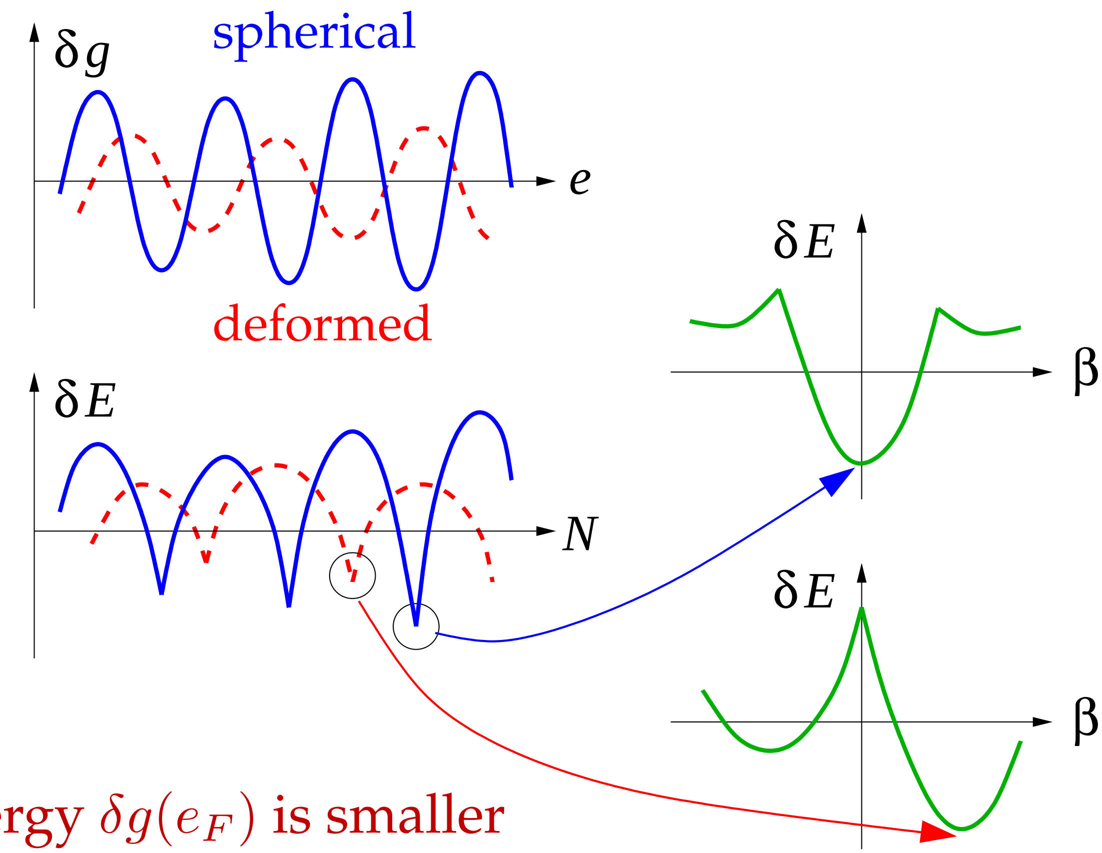
Spherical and deformed magic numbers

Single-particle level density

$$g(e) = \bar{g}(e) + \delta g(e)$$

Energy (mass) of nucleus

$$E(N) = \bar{E}(N) + \delta E(N)$$



System prefers shapes where

the level density at Fermi energy $\delta g(e_F)$ is smaller

❑ *Semiclassical theory of shell structures*

Level density (density of states)

$$g(e) = \mathrm{Tr}\,\delta(e-\hat{H}) = \frac{1}{2\pi\hbar}\int d\boldsymbol{r}\int_{-\infty}^{\infty} dt\, e^{iet/\hbar}\langle\boldsymbol{r}|e^{-i\hat{H}t/\hbar}|\boldsymbol{r}\rangle$$

Path integral representation of the transition amplitude

$$K(\boldsymbol{r}'',\boldsymbol{r}';t) = \langle\boldsymbol{r}''|e^{-i\hat{H}t/\hbar}|\boldsymbol{r}'\rangle = \int\mathcal{D}\boldsymbol{r}\exp\left[\frac{i}{\hbar}\int_0^t L(\boldsymbol{r},\dot{\boldsymbol{r}})dt'\right]$$

... integration over all the path connecting $\boldsymbol{r}(0)=\boldsymbol{r}'$ and $\boldsymbol{r}(t)=\boldsymbol{r}''$

Semiclassical evaluation of the integrals using stationary phase method

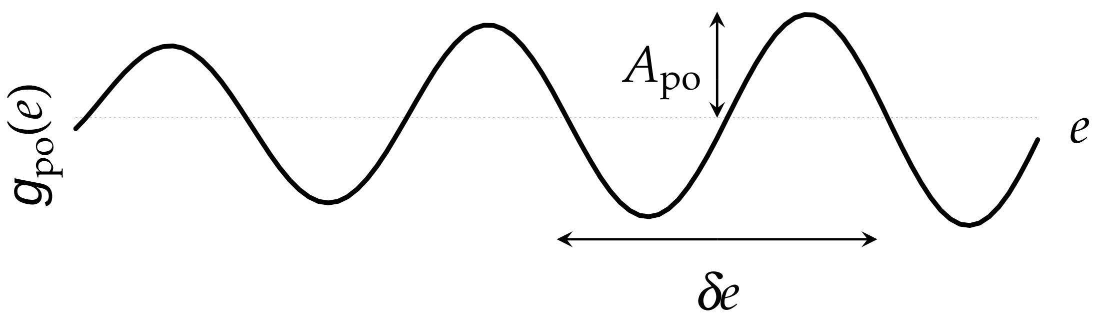
- path integral: arbitrary paths $\to$ classical trajectories
- Fourier transform: action $R=\int L dt \to S = R + e\,t = \int \boldsymbol{p}\cdot d\boldsymbol{r}$
- trace integral: closed trajectories $\to$ periodic orbits (po)

Finally one obtains the **Trace Formula**

$$\delta g(e) \sim \sum_{\mathrm{po}} A_{\mathrm{po}}(e)\cos\left[\frac{S_{\mathrm{po}}(e)}{\hbar}-\nu_{\mathrm{po}}\right], \qquad S_{\mathrm{po}}(e) = \oint_{\mathrm{po}}\boldsymbol{p}\cdot d\boldsymbol{r}$$

☞ M. C. Gutzwiller, J. Math. Phys. **8** (1967), 1979; **12** (1971), 343.

Oscillating part $\delta g \quad \Leftrightarrow \quad$ contribution of classical periodic orbits



Period δe of the level density oscillation:

$$\delta S_{\rm po} = \frac{\partial S_{\rm po}}{\partial e}\delta e = T_{\rm po}\delta e \approx 2\pi\hbar \;\Rightarrow\; \delta e \approx \frac{2\pi\hbar}{T_{\rm po}}$$

GROSS shell structure (large δe) $\Leftrightarrow$ SHORT periodic orbits (small $T_{\rm po}$)

Amplitude $A_{\rm po} \Leftrightarrow$ stability of the orbit

Gutzwiller's formula for isolated PO

$$A_{\rm po} = \frac{T_{\rm po}}{\pi\sqrt{|\det(I - \tilde{M}_{\rm po})|}}, \quad \tilde{M}_{\rm po} = \frac{\partial(\boldsymbol{p}'', \boldsymbol{r}'')_\perp}{\partial(\boldsymbol{p}', \boldsymbol{r}')_\perp} \quad \text{(monodromy matrix)}$$

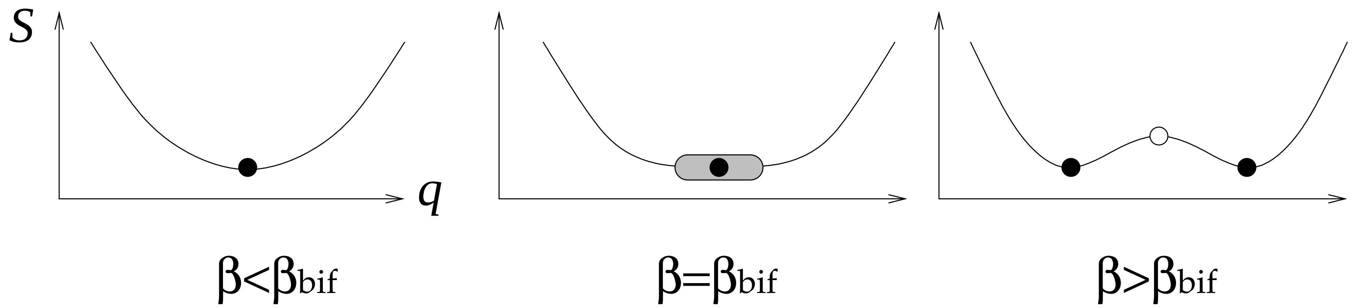
Enhanced around the bifurcation points $\Rightarrow$ strong shell effect

❑ *Periodic orbit bifurcation and local dynamical symmetry*

Effect of PO bifurcation to shell structure

$$\delta g_{\rm po}(e) \sim \int \sqrt{D} e^{iS(\boldsymbol{q},\boldsymbol{q})/\hbar} d\boldsymbol{q}, \quad S(\boldsymbol{q},\boldsymbol{q}') = \int_{\boldsymbol{q}}^{\boldsymbol{q}'} \boldsymbol{p}\cdot d\boldsymbol{q}$$

Illustration of bifurcation (Example: “pitchfork” bifurcation)



Periodic orbit: stationary point $\displaystyle \frac{dS}{d\boldsymbol{q}} = \left[\frac{\partial S}{\partial \boldsymbol{q}'} + \frac{\partial S}{\partial \boldsymbol{q}}\right]_{\boldsymbol{q}'=\boldsymbol{q}} = \boldsymbol{p}' - \boldsymbol{p} = 0$

Bifurcation ⇔ zero curvature $S''(q) = 0$

→ continuous family of quasi-stationary points (local family of PO)

local dynamical symmetry (q is the “negligible” coordinate)

coherent contribution to the trace integral

→ enhancement of $A_{\rm po}$ → growth of shell effect

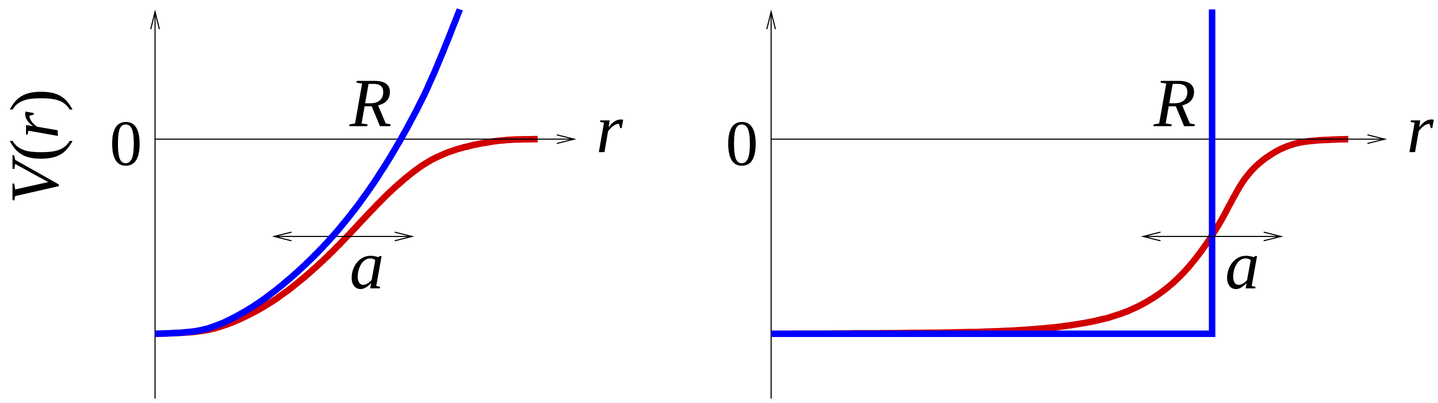
Deformed shell structures ⇔ Bifurcations of short PO

2. The radial power-law potential model

❑ *Nuclear mean-field potential*

Woods-Saxon potential

- Potential depth $W_0 \approx 50\mathrm{MeV}$
- Radius $R \approx r_0 A^{1/3}$ ($r_0 \approx 1.3$ fm, A: mass number)
- Surface diffuseness $a \approx 0.7$ fm



Light nuclei $(R \sim a)$ $\sim$ Harmonic oscillator

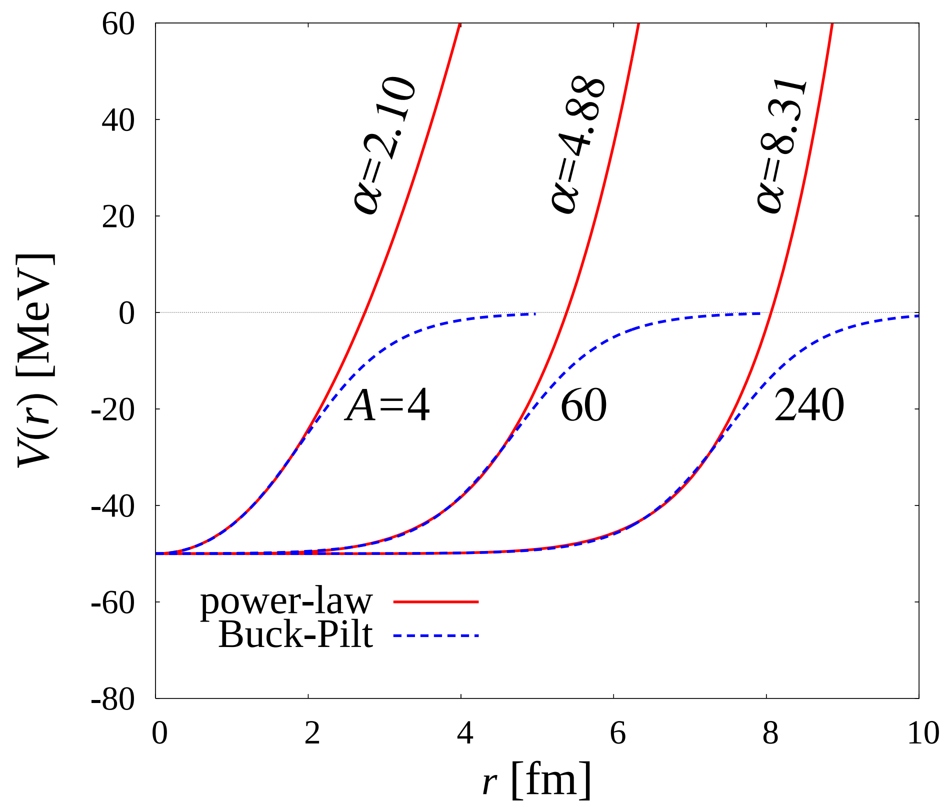
Heavy nuclei $(R \gg a)$ $\sim$ Square well $\sim$ Infinite well

❏ *Approximation of Woods-Saxon potential*

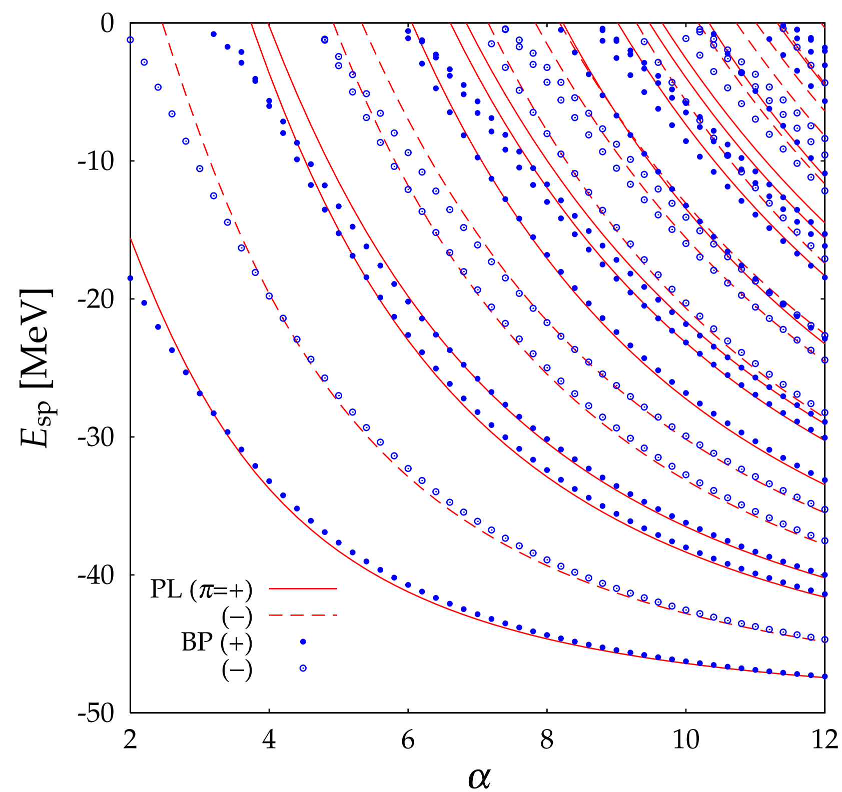
Woods-Saxon potential ≈ simpler potential with r^α radial dependence:

$$V(\boldsymbol{r}) = -\frac{W_0}{1+\exp\left(\frac{r-R(\theta,\varphi)}{a}\right)} \approx -W_0 + U_0\left(\frac{r}{R(\theta,\varphi)}\right)^\alpha$$

fitting of the potential



quantum spectra



❑ *Spin-orbit coupling*

Introduction of spin-orbit coupling in the power-law potential

$$H = \frac{p^2}{2m} + U_0\left(\frac{r}{R}\right)^{\alpha} + 2\kappa\boldsymbol{\nabla}U_{\rm so}\cdot(\boldsymbol{s}\times\boldsymbol{p}), \quad U_{\rm so}(\boldsymbol{r}) = \frac{1}{m}\left(\frac{r}{R(\theta,\varphi)}\right)^{\alpha_{\rm so}}$$
$$= \frac{p^2}{2m} + U_0\left(\frac{r}{R}\right)^{\alpha} - 2\kappa\boldsymbol{B}\cdot\boldsymbol{s}, \quad \boldsymbol{B} = \boldsymbol{\nabla}U_{\rm so}\times\boldsymbol{p}$$

Classical spin canonical variable $\Leftarrow$ SU(2) coherent state path integral

$$\boldsymbol{s} = (s\sin\vartheta\cos\varphi, s\sin\vartheta\sin\varphi, s\cos\vartheta), \quad (q_s, p_s) = (\varphi, s_z = s\cos\vartheta)$$

EOM in the spin part

$$\{s_i, s_j\}_{\rm P.B.} = \frac{\partial s_i}{\partial q_s}\frac{\partial s_j}{\partial p_s} - \frac{\partial s_i}{\partial p_s}\frac{\partial s_j}{\partial q_s} = \varepsilon_{ijk}s_k$$
$$\dot{s}_i = \{s_i, H\}_{\rm P.B.} = -2\kappa B_j\{s_i, s_j\}_{\rm P.B.} = -2\kappa\varepsilon_{ijk}B_j s_k = -2\kappa(\boldsymbol{B}\times\boldsymbol{s})_i$$

spin perpendicular to symmetry plane

$$\boldsymbol{B} \parallel \boldsymbol{s} \quad \Rightarrow \quad \dot{\boldsymbol{s}} = 0 \quad \text{(frozen-spin orbits)}$$

❑ *Scaling properties*

Taking the spin-orbit radial parameter as $\alpha_{\rm so} = 1 + \frac{\alpha}{2}$,

$$H(\boldsymbol{p}, \boldsymbol{r}, \boldsymbol{s}) = \frac{\boldsymbol{p}^2}{2m} + U_0 \left(\frac{r}{R(\theta, \varphi)} \right)^{\alpha} - \frac{2\kappa}{m} \boldsymbol{\nabla} \left(\frac{r}{R(\theta, \varphi)} \right)^{1+\frac{\alpha}{2}} (\boldsymbol{s} \times \boldsymbol{p})$$

classical Hamiltonian obeys the scaling relation

$$H\left(c^{1/2}\boldsymbol{p}, c^{1/\alpha}\boldsymbol{r}, \boldsymbol{s}\right) = cH(\boldsymbol{p}, \boldsymbol{r}, \boldsymbol{s})$$

For frozen-spin orbits, classical EOM are invariant under the scaling transformation

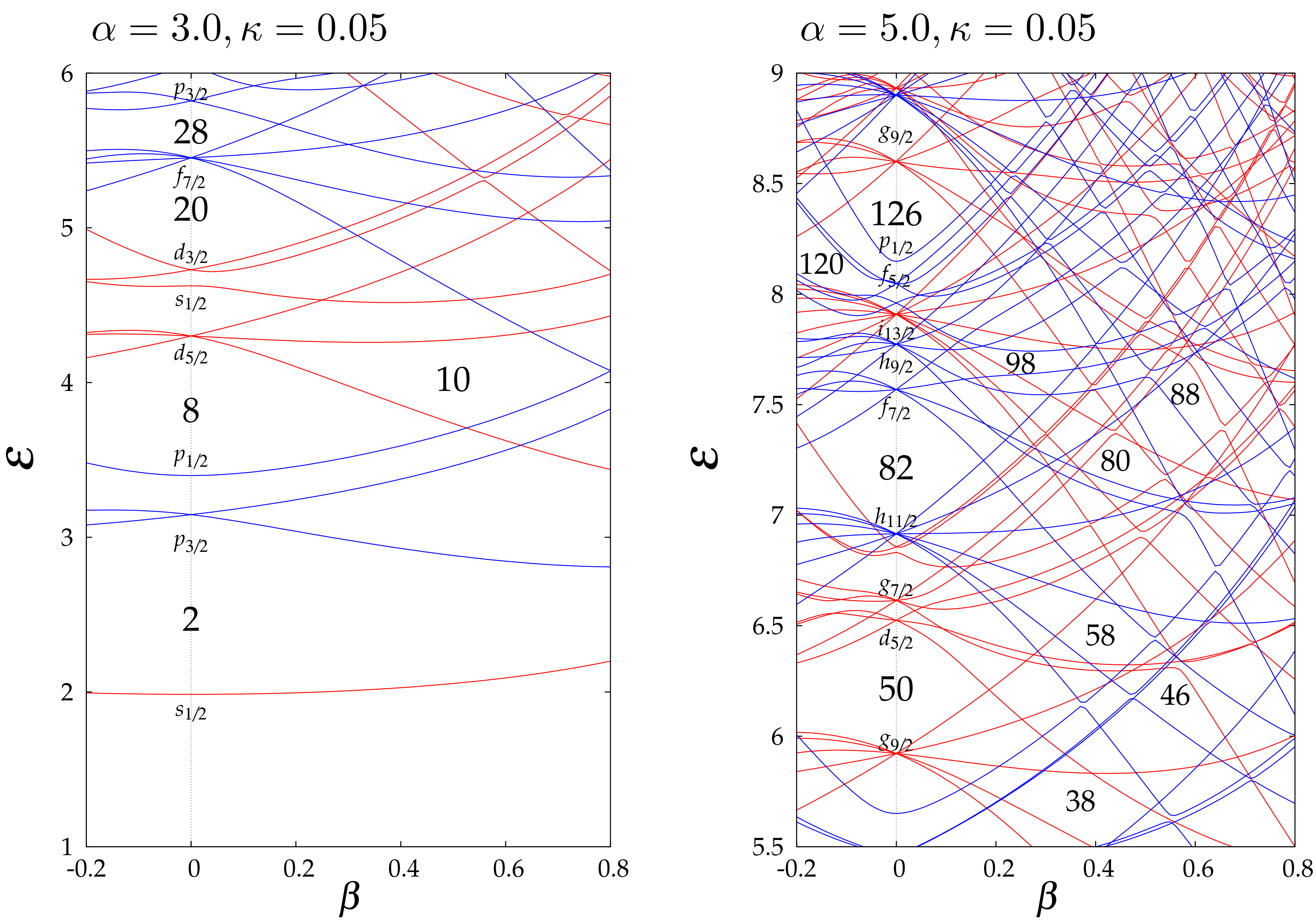
$$\boldsymbol{p} \to c^{1/2}\boldsymbol{p}, \quad \boldsymbol{r} \to c^{1/\alpha}\boldsymbol{r}, \quad t \to c^{-(\frac{1}{2}-\frac{1}{\alpha})}t \quad \text{as} \quad e \to c\, e$$

$\Rightarrow$ Same frozen-spin periodic orbits in any energy

Action integral along the PO

$$S_{\rm po}(e = cU_0) = \oint_{{\rm po}(cU_0)} \boldsymbol{p} \cdot d\boldsymbol{r} = c^{\frac{1}{2}+\frac{1}{\alpha}} \oint_{{\rm po}(U_0)} \boldsymbol{p} \cdot d\boldsymbol{r}$$
$$= (e/U_0)^{\frac{1}{2}+\frac{1}{\alpha}} S_{\rm po}(U_0)$$

Single-particle level diagram



❑ *Fourier Transformation technique for scaling system*

Action integral along PO

$$S_{\mathrm{po}}(e) = \oint_{\mathrm{po}(e)} \boldsymbol{p}\cdot d\boldsymbol{r} = \left(\frac{e}{U_0}\right)^{\frac{1}{2}+\frac{1}{\alpha}} \oint_{\mathrm{po}(U_0)} \boldsymbol{p}\cdot d\boldsymbol{r} \equiv \mathcal{E}\hbar\tau_{\mathrm{po}}$$

scaled energy $\mathcal{E} = (e/U_0)^{\frac{1}{2}+\frac{1}{\alpha}}$, scaled period $\tau_{\mathrm{po}} = \frac{1}{\hbar}\oint_{\mathrm{po}(e=U_0)} \boldsymbol{p}\cdot d\boldsymbol{r}$

Trace Formula for scaled energy level density

$$g(\mathcal{E}) = g(e)\frac{de}{d\mathcal{E}} = g_0(\mathcal{E}) + \sum_{\mathrm{po}} A_{\mathrm{po}}(\mathcal{E})\cos\left(\mathcal{E}\tau_{\mathrm{po}} - \nu_{\mathrm{po}}\right)$$

Fourier Transform of level density

$$F(\tau) = \int d\mathcal{E}\, e^{i\tau\mathcal{E}} g(\mathcal{E}) = \sum_n e^{i\tau\mathcal{E}_n}, \quad \mathcal{E}_n = (e_n/U_0)^{\frac{1}{2}+\frac{1}{\alpha}} \qquad \text{(quantum)}$$

$$\sim F_0(\tau) + \pi\hbar \sum_{\mathrm{po}} e^{i\nu_{\mathrm{po}}} \widetilde{A}_{\mathrm{po}}\delta(\tau - \tau_{\mathrm{po}}) \qquad \text{(semiclassical)}$$

$F(\tau)$ ... peaks at periodic orbits $\tau = \tau_{\mathrm{po}}$ with height proportional to A_{po}
Information on PO out of quantum energy spectrum
⇒ Explanation of quantum shell effect in terms of classical PO

3. Semiclassical analysis of nuclear shell structures

3.1. Spherical magic numbers, bifurcation and dynamical symmetry

❏ *Pseudospin symmetry*

Nilsson model:

$$h_{\text{Nilsson}} = h_{\text{HO}} - v_{ls}\boldsymbol{l}\cdot\boldsymbol{s} - v_{ll}\boldsymbol{l}^2,$$

$$v_{ls} \approx 4v_{ll}$$

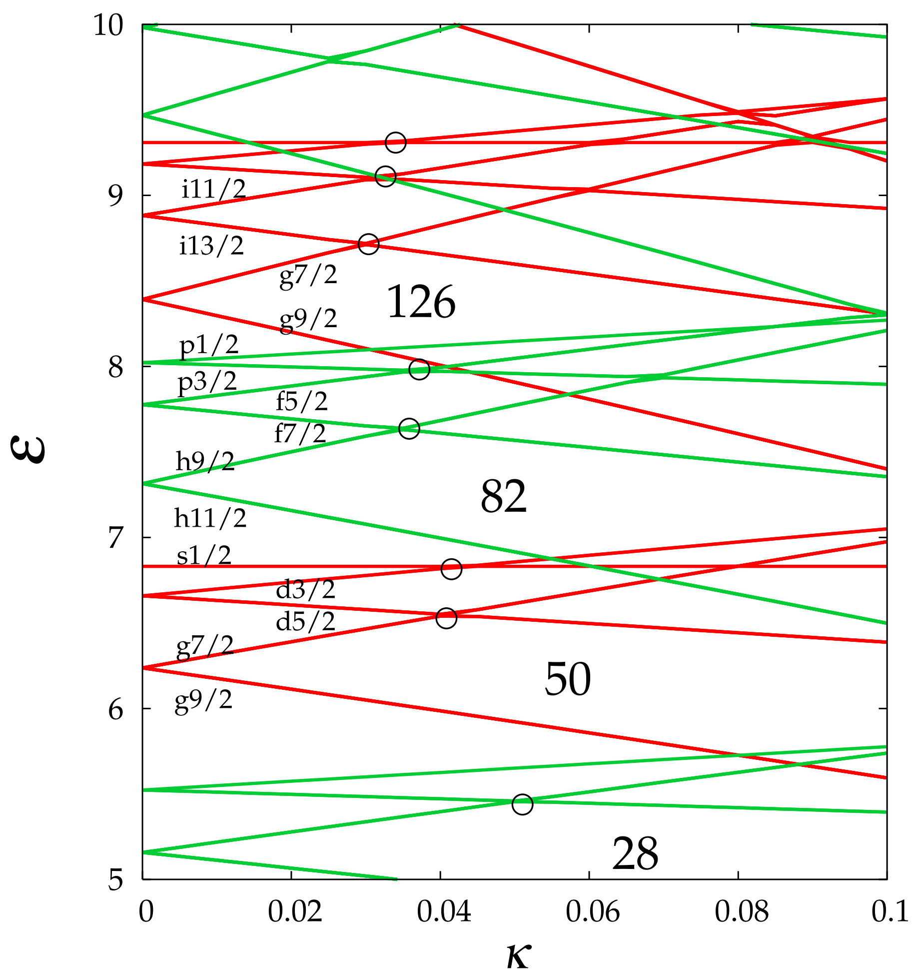
$$= \tilde{h}_{\text{HO}} - (4v_{ll} - v_{ls})\tilde{\boldsymbol{l}}\cdot\tilde{\boldsymbol{s}} - v_{ll}\tilde{\boldsymbol{l}}^2$$

$$\approx \tilde{h}_{\text{HO}} - v_{ll}\tilde{\boldsymbol{l}}^2$$

In power-law potential model:

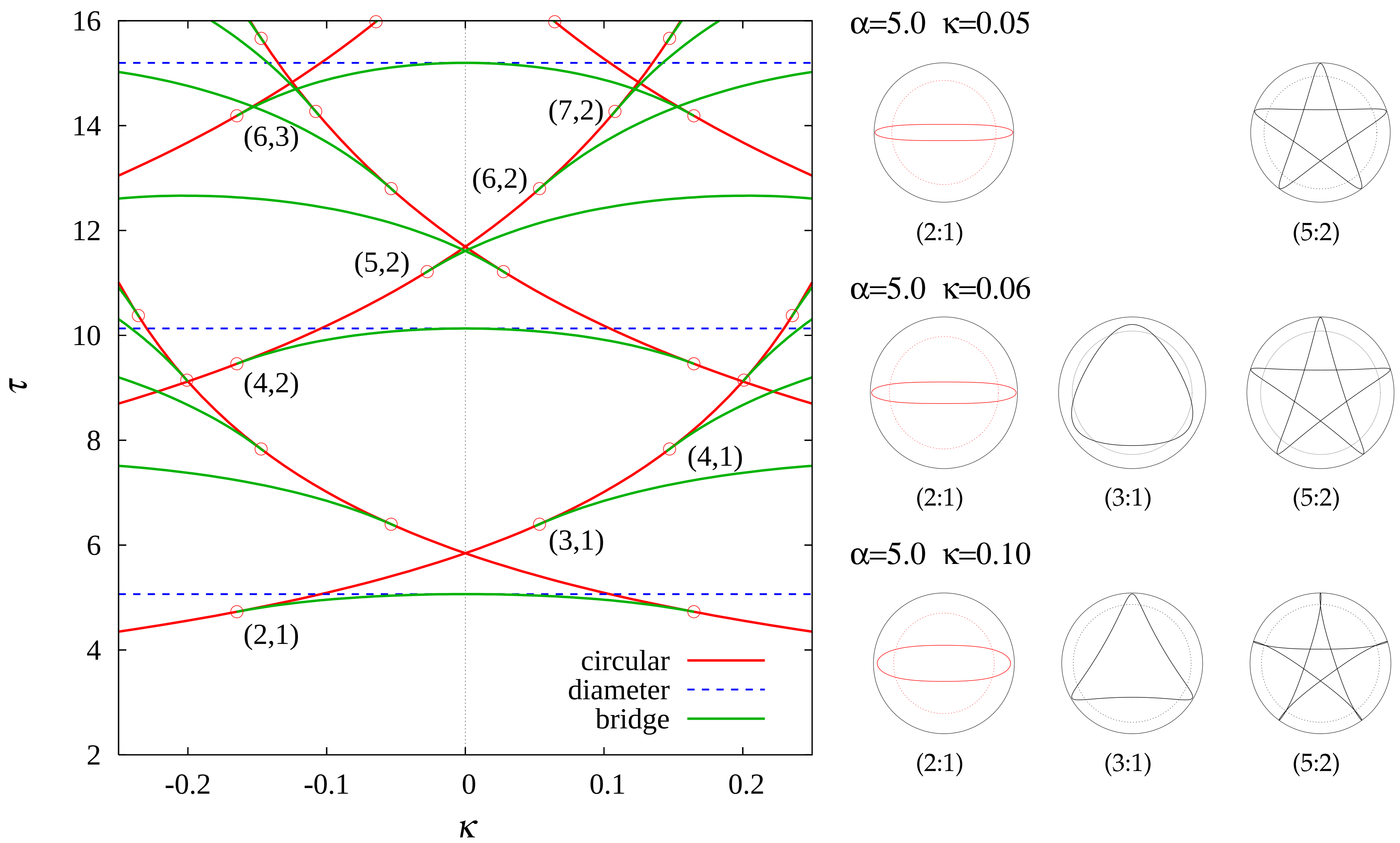
Level crossings of $\tilde{L}\tilde{S}$ partners around $\kappa \approx 0.05$

$\Rightarrow$ large energy gaps distinct magic numbers

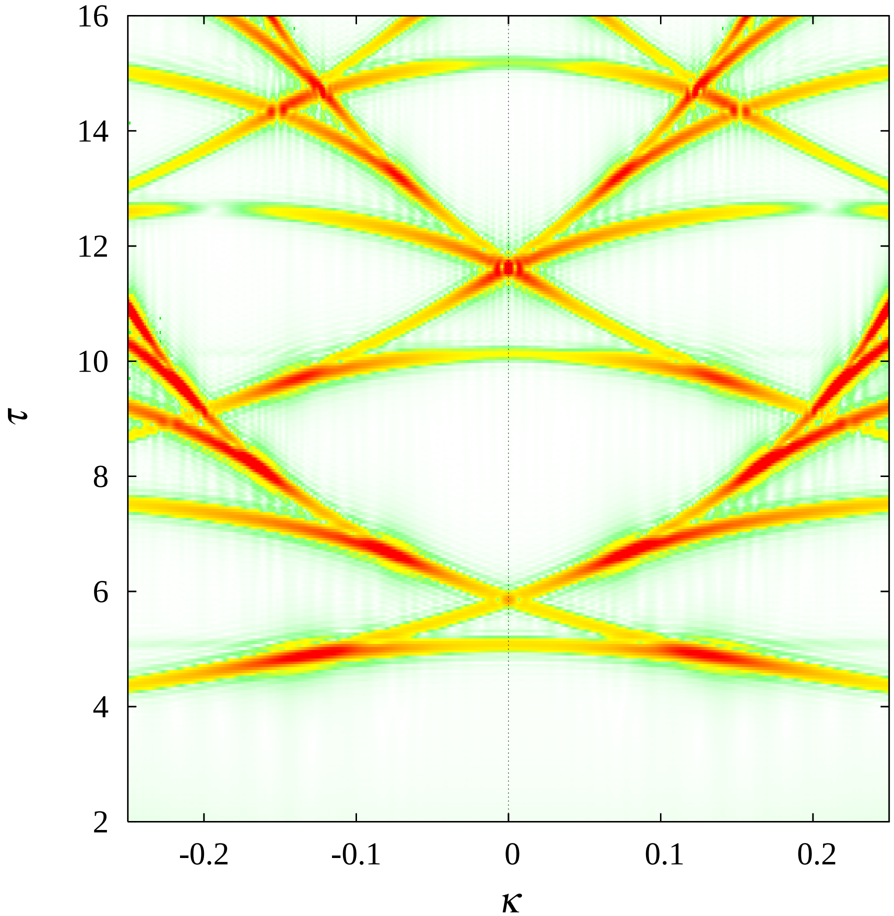


❑ *Semiclassical origin of the gross shell structures around* $\kappa \approx 0.05$

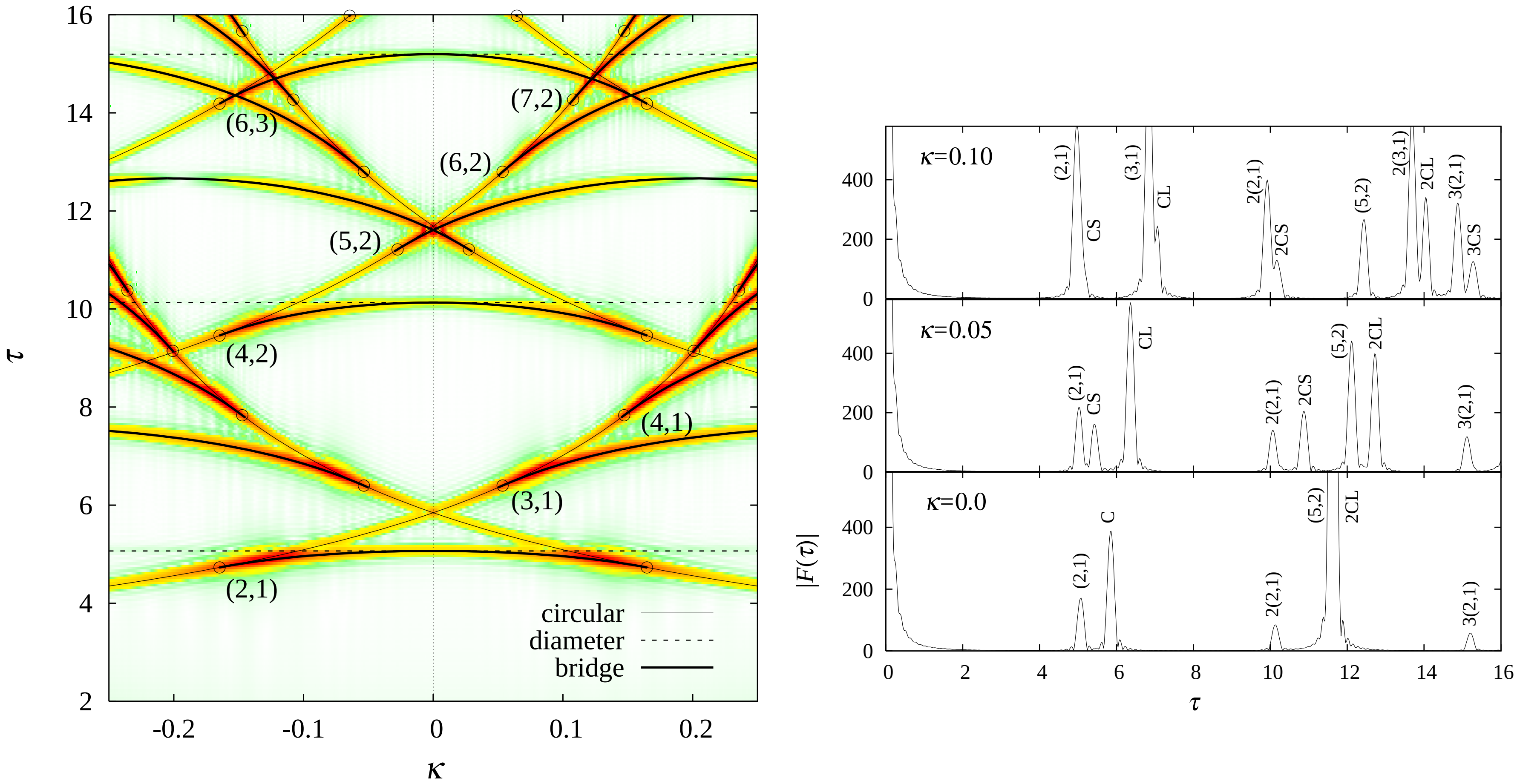
Classical periodic orbits (frozen spin) for radial parameter $\alpha = 5.0$



Fourier transforms of quantum level density

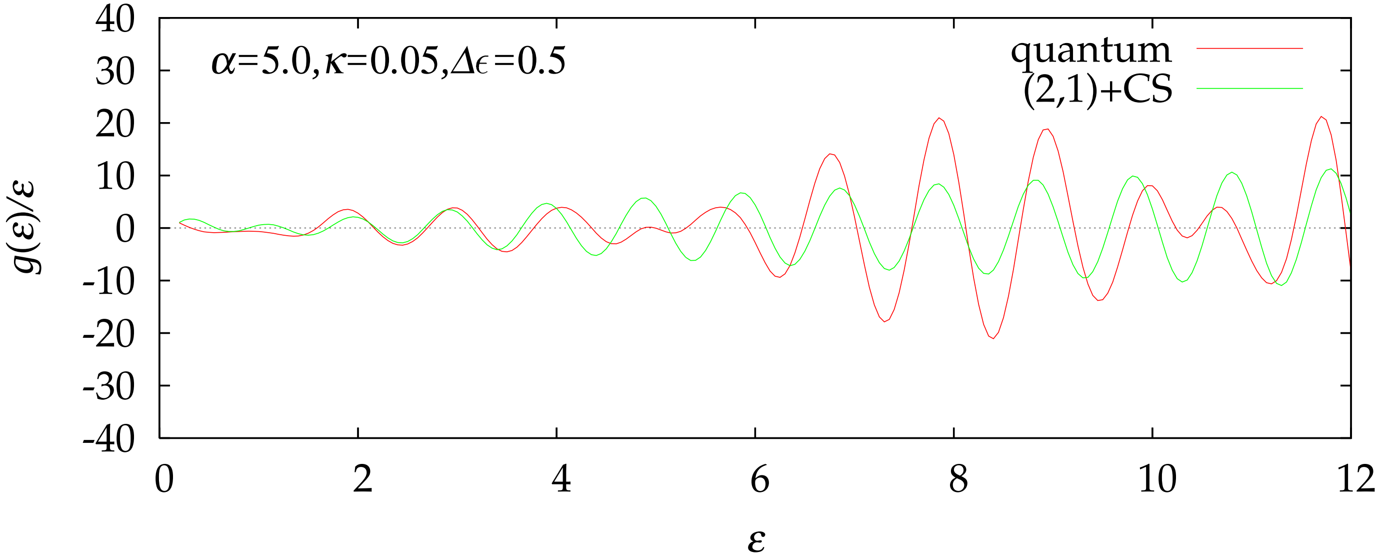


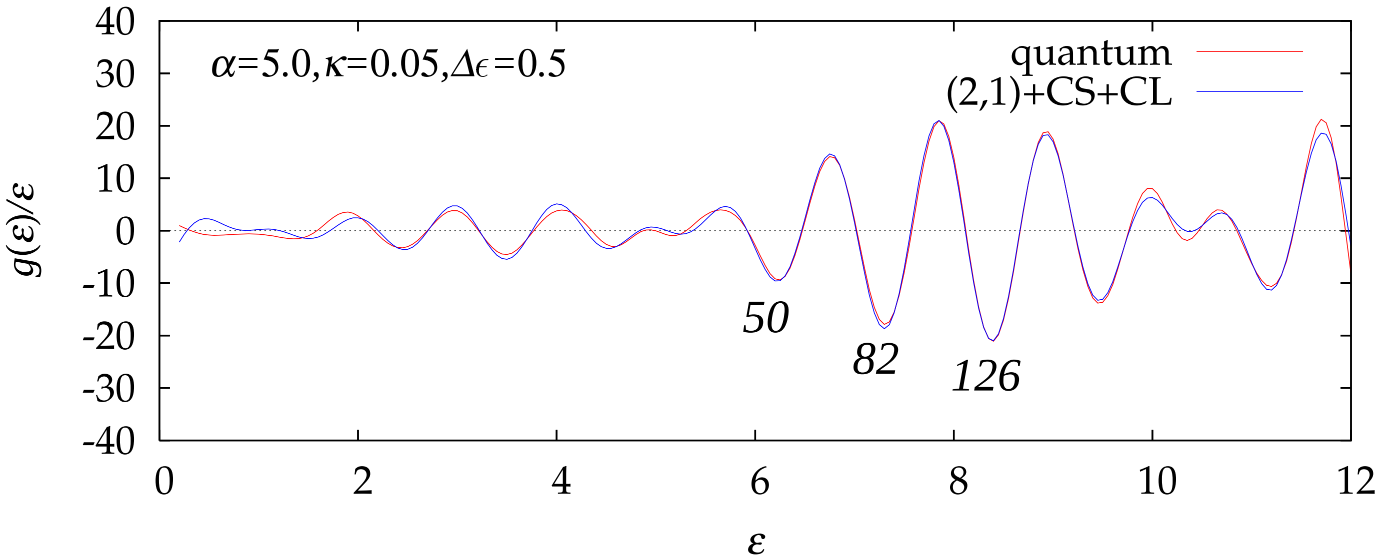
Fourier transforms of quantum level density



Enhancement of peak corresponding to CL near the bifurcation point (3,1)

Semiclassical level density





3.2 Semiclassical description of the prolate-shape dominance

– Most of the nuclear ground state deformations are prolate
– Stronger preference to prolate shapes in heavier nuclei
– Related with prolate-oblate asymmetry of deformed shell structures which becomes more pronounced for heavier nuclei

Explanations using models without spin-orbit coupling

◆ Semiclassical periodic orbit theory (POT)
H. Frisk, Nucl. Phys. A511 (1990), 309.
K. A., Phys. Rev. C86 (2012), 034317.

◆ Asymmetric ways of level spreading (fanning)
I. Hamamoto and B. R. Mottelson, Phys. Rev. C79 (2009), 034317.

Effect of spin-orbit coupling

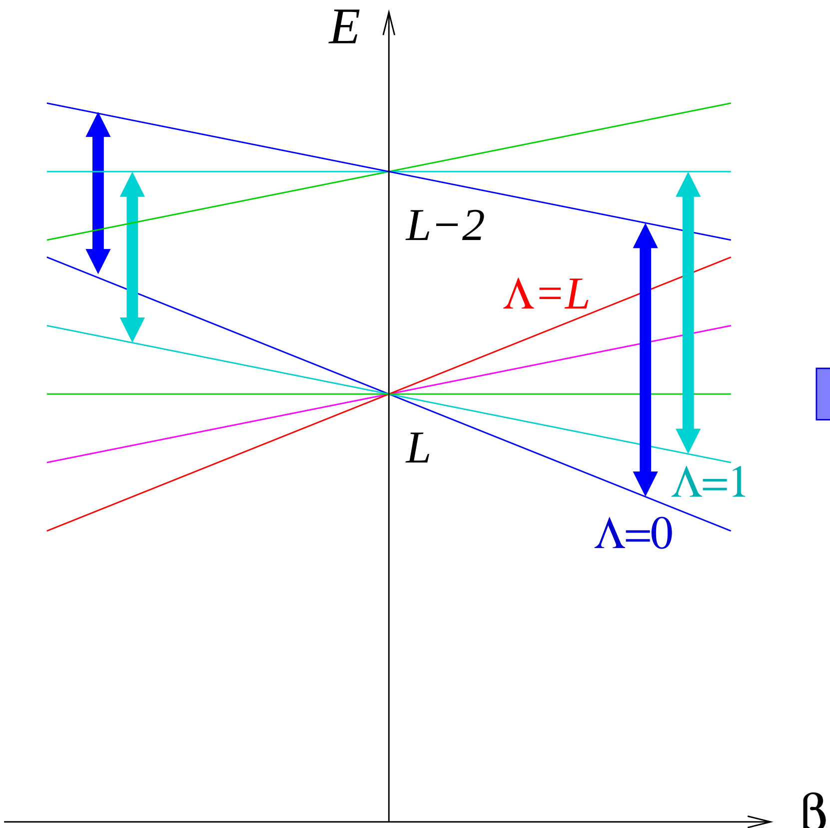
◆ Interplay between spin-orbit coupling and surface diffuseness
N. Tajima and N. Suzuki, Phnys. Rev. C64 (2001), 037301.
S. Takahara et al., Phys. Rev. C86 (2012) 064323.

$\Rightarrow$ Semiclassical explanation using POT

➤ Origin of asymmetric level splittings

– Higher l levels in HO shell have lower energies as the surface becomes sharp

– Interaction between levels with same Λ play different roles in prolate and oblate sides



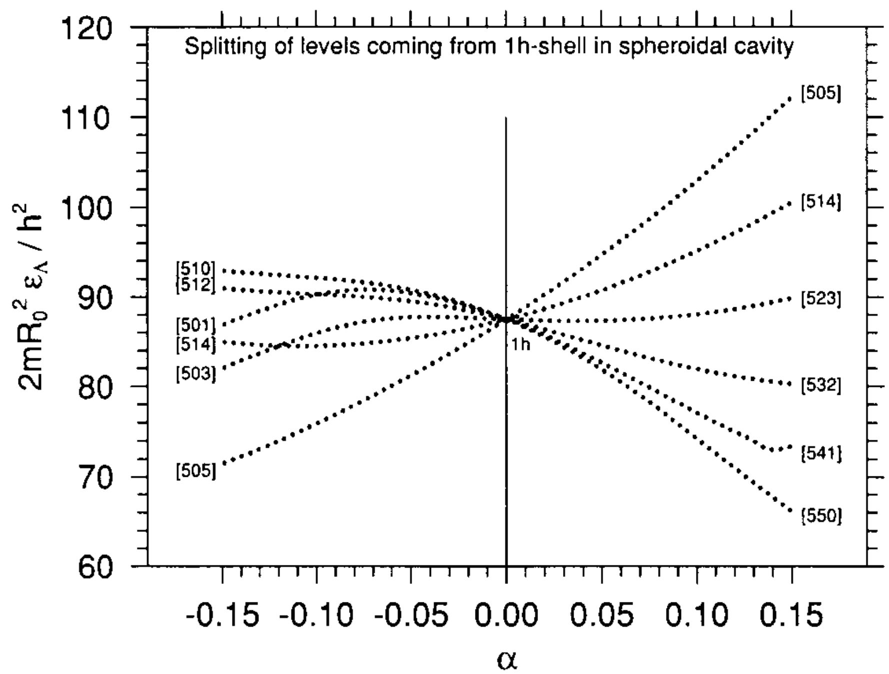


FIG. 5. Splitting of levels originating from the $1h$ shell in spheroidal cavity. The asymptotic quantum numbers $[N\ n_z\ \Lambda]$ are assigned to the levels on both prolate and oblate sides. See the text for details.

I. Hamamoto and B.R. Mottelson, Phys. Rev. C79 (2009), 034317.

❑ *Periodic orbits in power-law potential model w/o spin-orbit coupling*

power-law potential with spheroidal deformation

$$H_\beta = \frac{\boldsymbol{p}^2}{2m} + U_0 \left(\frac{r}{R(\theta,\beta)} \right)^\alpha, \quad R(\theta,\beta) = \frac{R_0}{\sqrt{e^{-\frac{4}{3}\beta}\cos^2\theta + e^{\frac{2}{3}\beta}\sin^2\theta}}$$

Spheroidal deformation parameter β: $R_z/R_\perp = e^\beta$

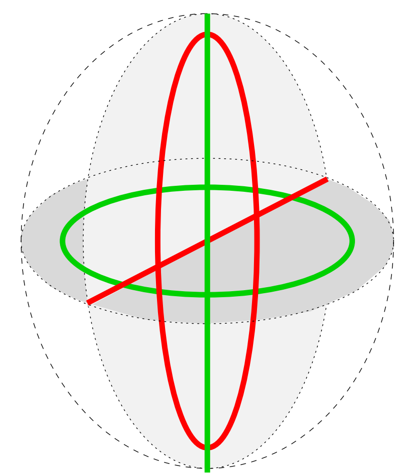
spherical: $\beta = 0$, superdeformed(axis ratio 2:1): $\beta = \pm\log 2 \approx \pm 0.7$

Non-integrable except for the cases $\alpha = 2$ (HO) and $\alpha = \infty$ (cavity)

Simple periodic orbits at normal deformations ($\beta \lesssim 0.3$)

- isolated diametric orbit along symmetry axis
- degenerate diameter orbit in equatorial plane
- isolated circular orbit in equatorial plane
- degenerate oval orbit in meridian plane

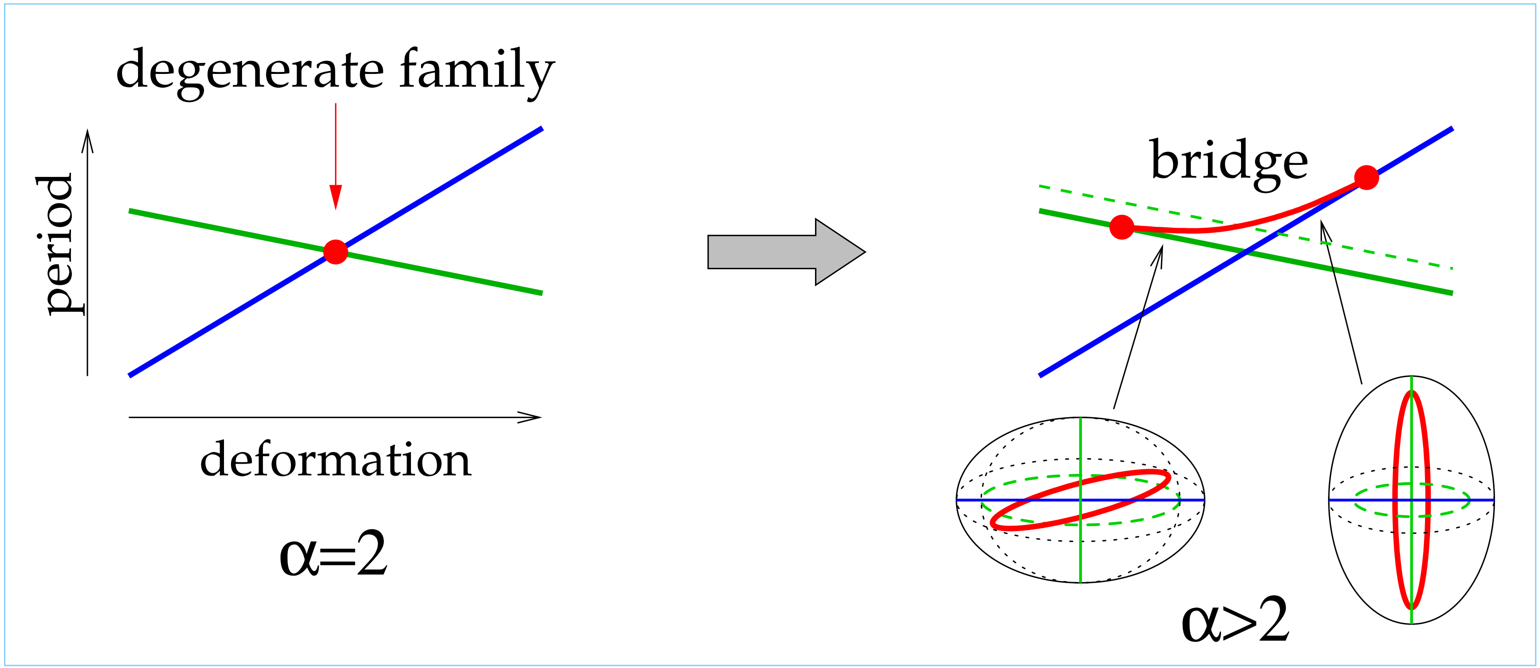
... **Bridge orbit** between two diametric orbits



❑ *Bridge orbit bifurcations*

$\alpha = 2$ (HO) ... degenerate family at $\beta = 0$

$\alpha > 2$ ... appearance of bridge orbits over two diameters



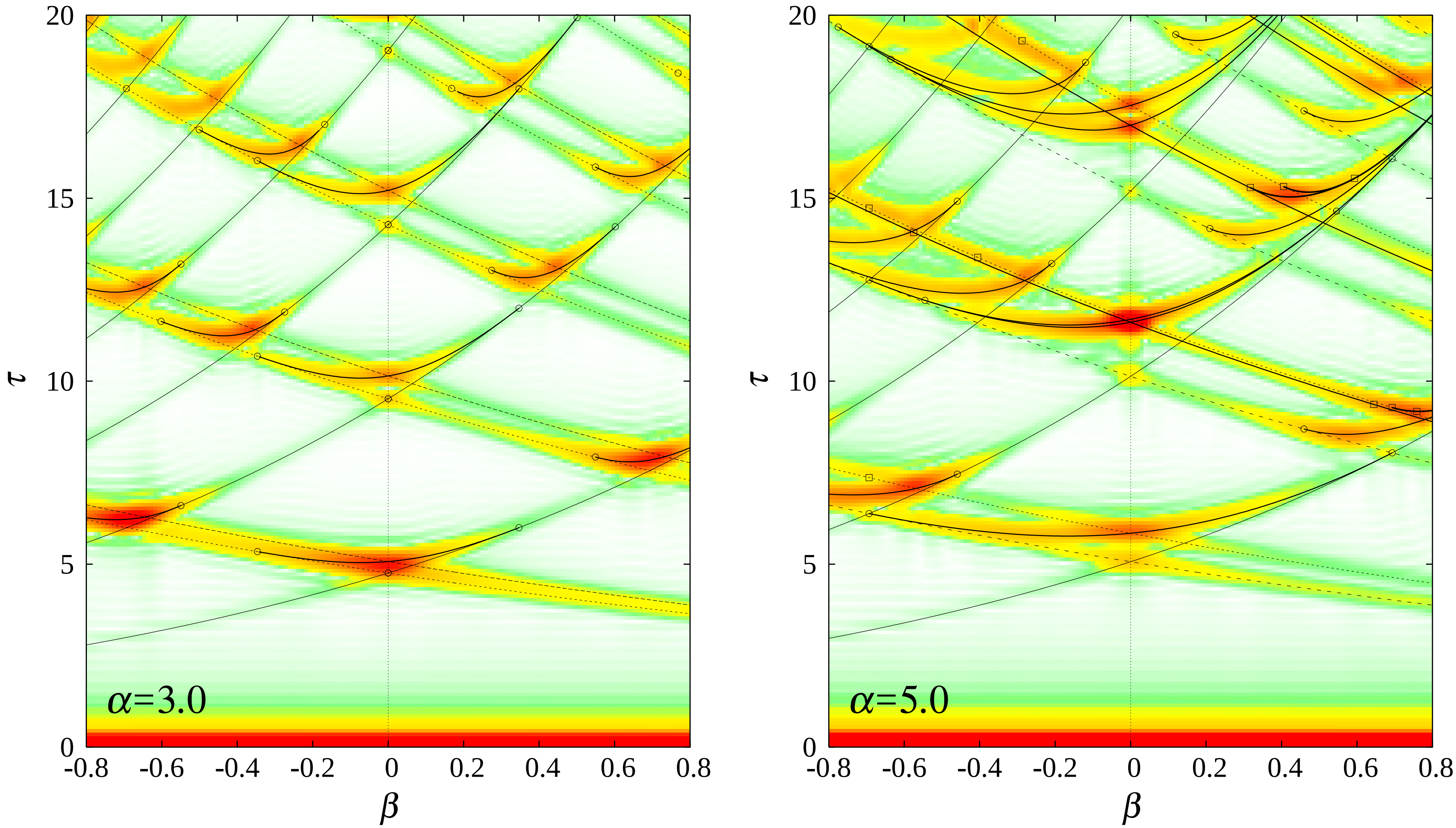
Classical and semiclassical theory of bridge orbit bifurcation

☞ K. A. and M. Brack, J. of Phys. **A41** (2008), 385207.

"Length" of the bridge grows as increasing α

$\Rightarrow$ Increasing significance of bridge orbit for larger α (larger A)

❏ *Fourier transformation of quantum level density*



- large Fourier amplitude along **bridge orbits**

  ... significant contribution to the level density

- growth of bridge as increasing α

  $\Rightarrow$ Increasing prolate-oblate asymmetry for larger α (larger A)

❏ *Deformed shell structures and constant-action curves*

Dominant contribution of a certain orbit "po" in PO sum

$$\delta g(e,\beta) \approx A_{\mathrm{po}} \cos\left[S_{\mathrm{po}}(e,\beta)/\hbar - \nu_{\mathrm{po}}\right]$$

$$\delta E(N,\beta) \approx \frac{A_{\mathrm{po}}}{(T_{\mathrm{po}}/\hbar)^2} \cos\left[S_{\mathrm{po}}(e_F(N),\beta)/\hbar - \nu_{\mathrm{po}}\right]$$

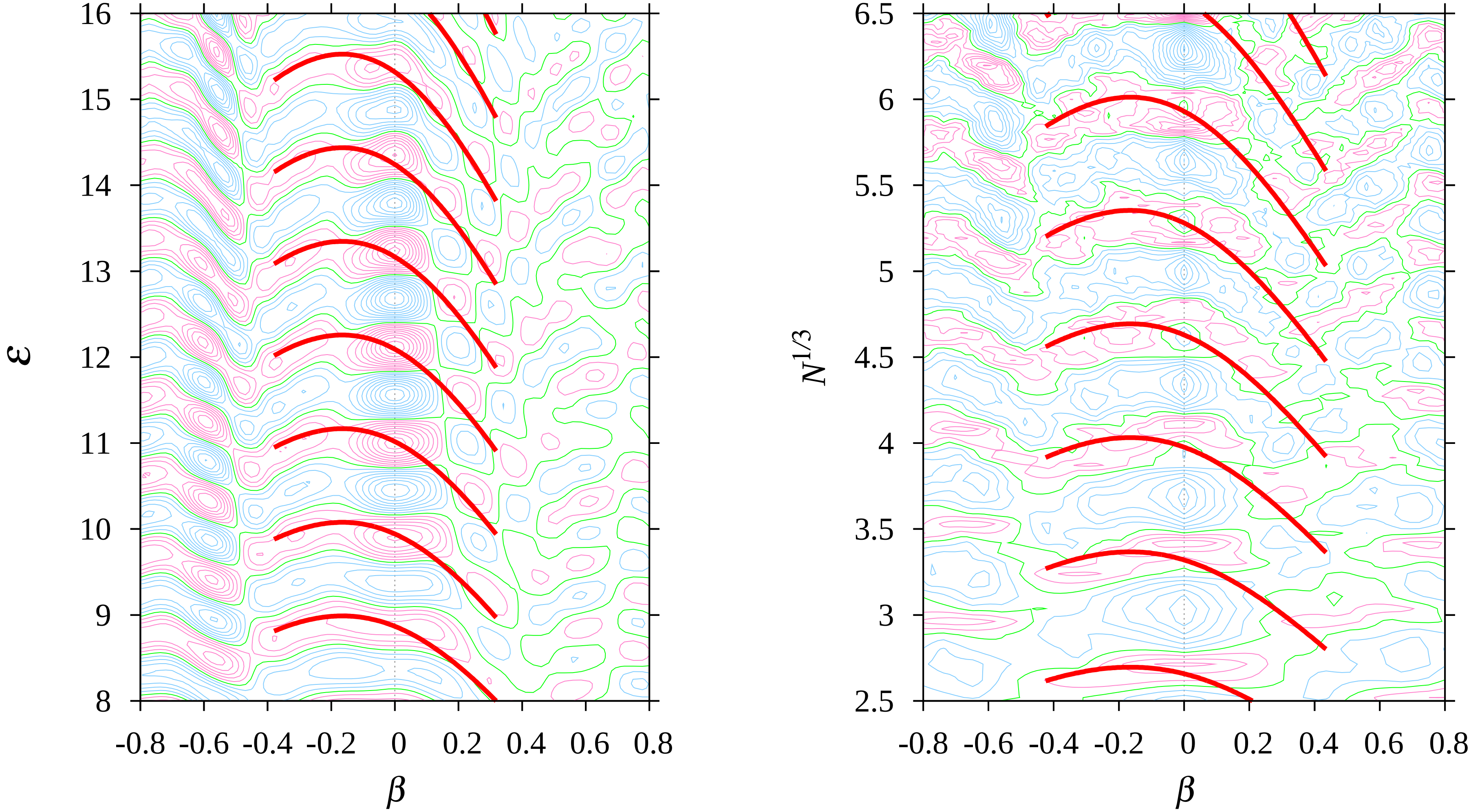
$$\Downarrow$$

Low level density (shell energy)

along the constant-action curves in (e,β) plane:

$$S_{\mathrm{po}}(e,\beta)/\hbar - \nu_{\mathrm{po}} = \pi(2n+1), \quad n = 0, 1, 2, \cdots$$

Oscillating level density and shell energy ($\alpha = 5.0$)



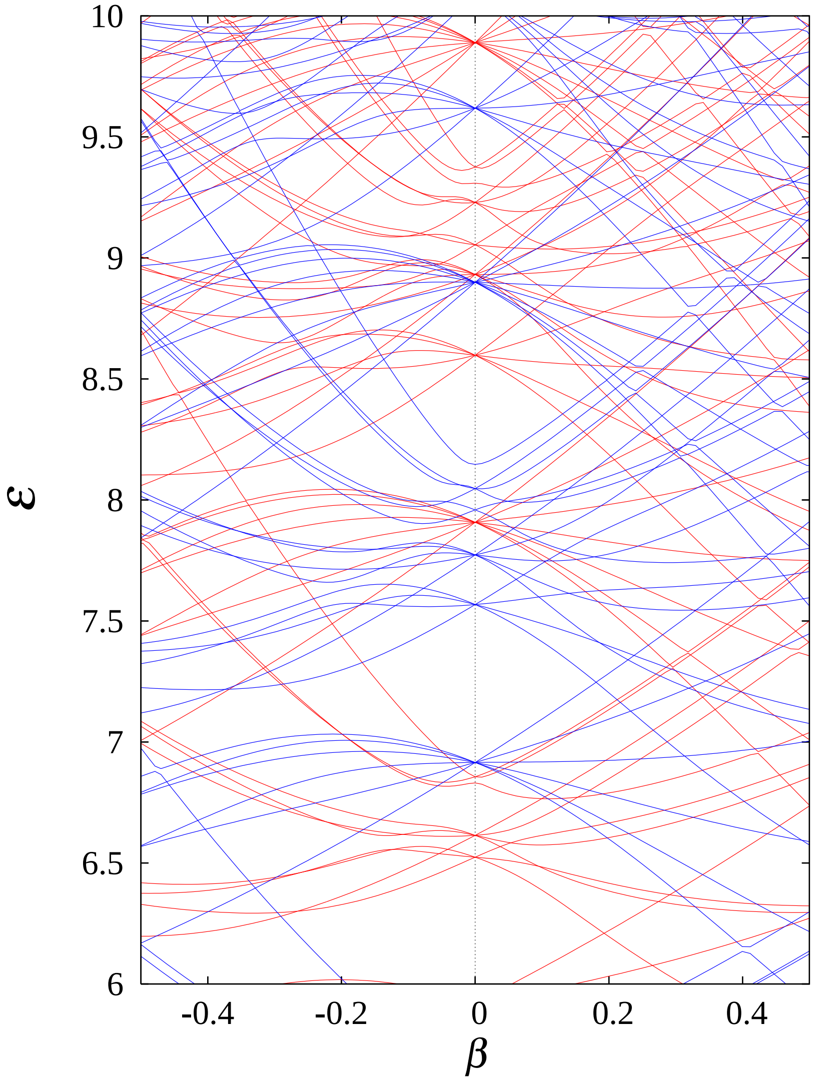
- valley lines of level density and shell energy ⇔ constant-action curves of bridge orbit

Prolate-shape dominance ⇔ prolate-oblate asymmetry of bridge orbit

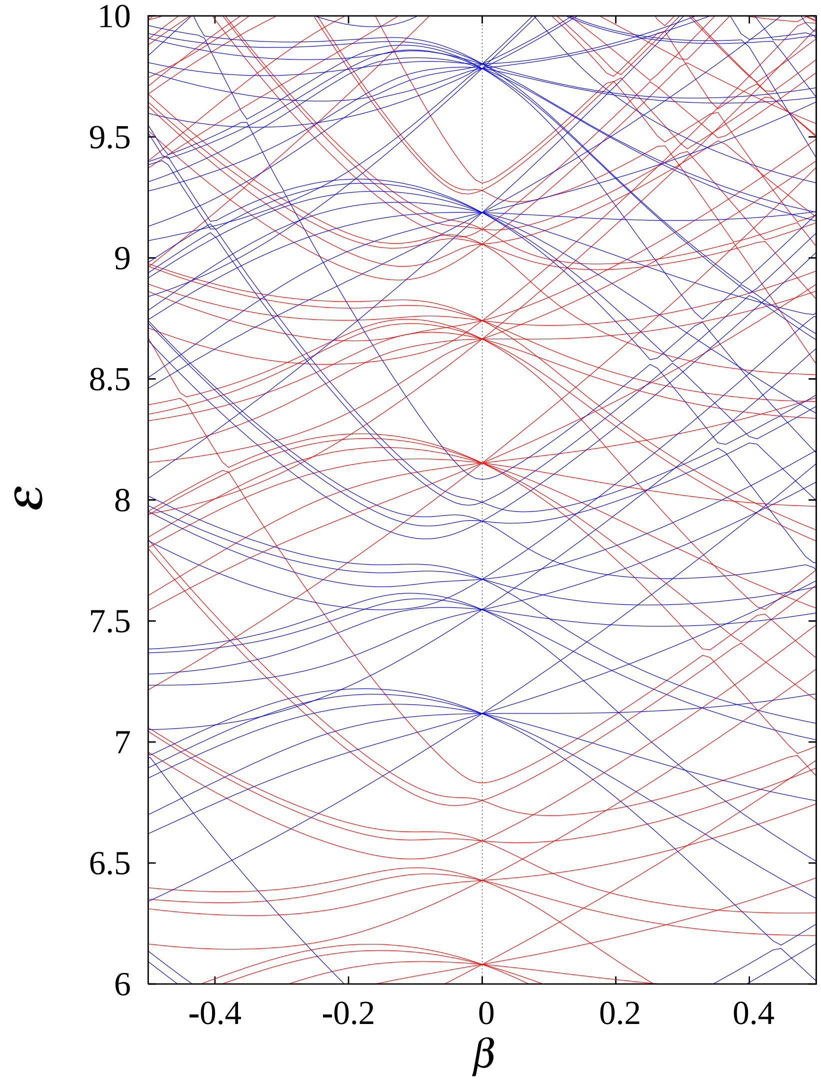
❑ *Effect of spin-orbit coupling*

Single-particle level diagram (power-law model with $\alpha = 5.0$)

$\kappa = 0.05$ (realistic)

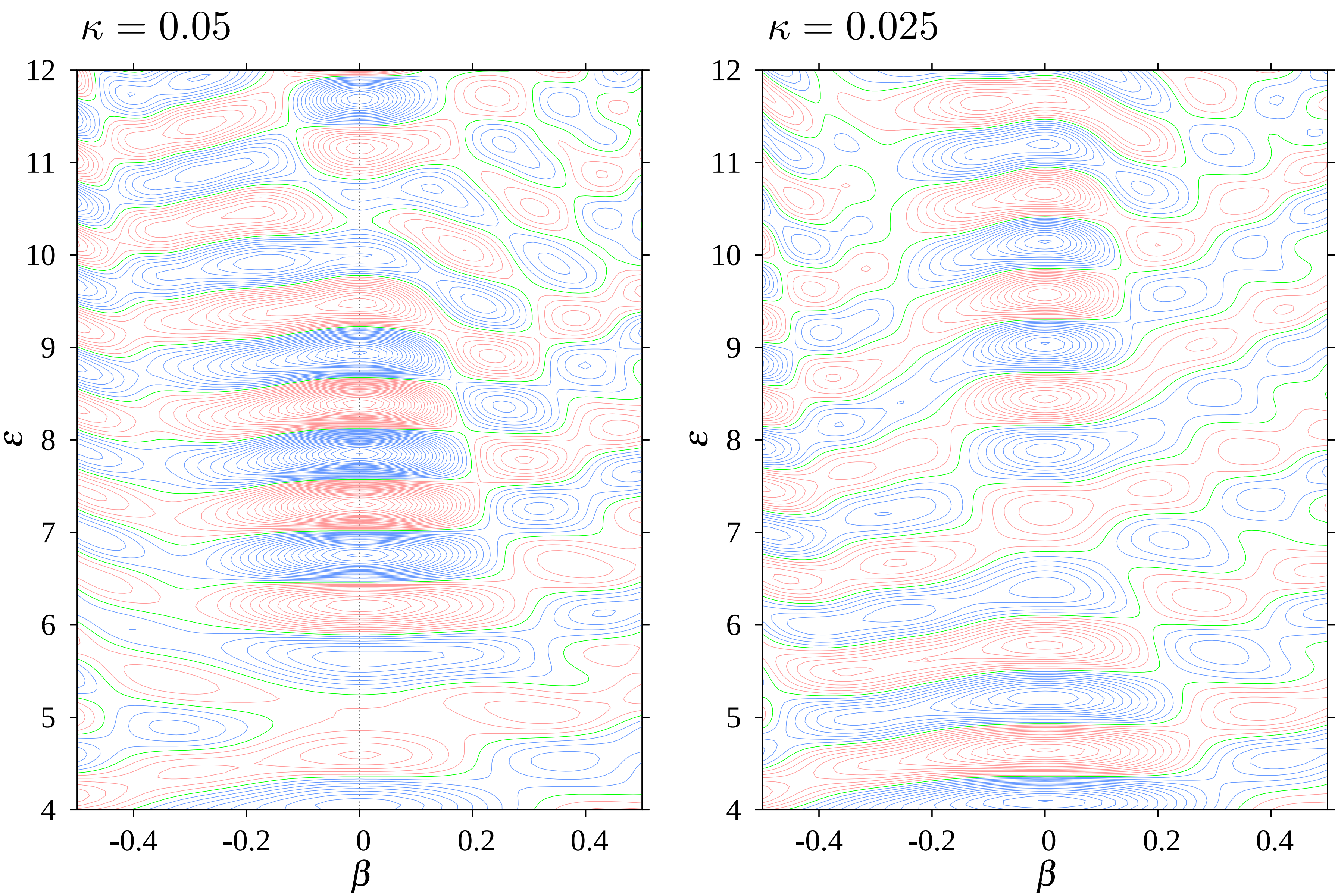


$\kappa = 0.025$ (reduced)



No manifest differences in level fannings

Oscillating level density



Remarkable differences in ridge-valley structures!

Evolutions of meridian frozen-spin orbits

- bridge orbits B $\rightarrow$ BL, BS
- diameters X, Z $\rightarrow$ BX, BZ (oval)

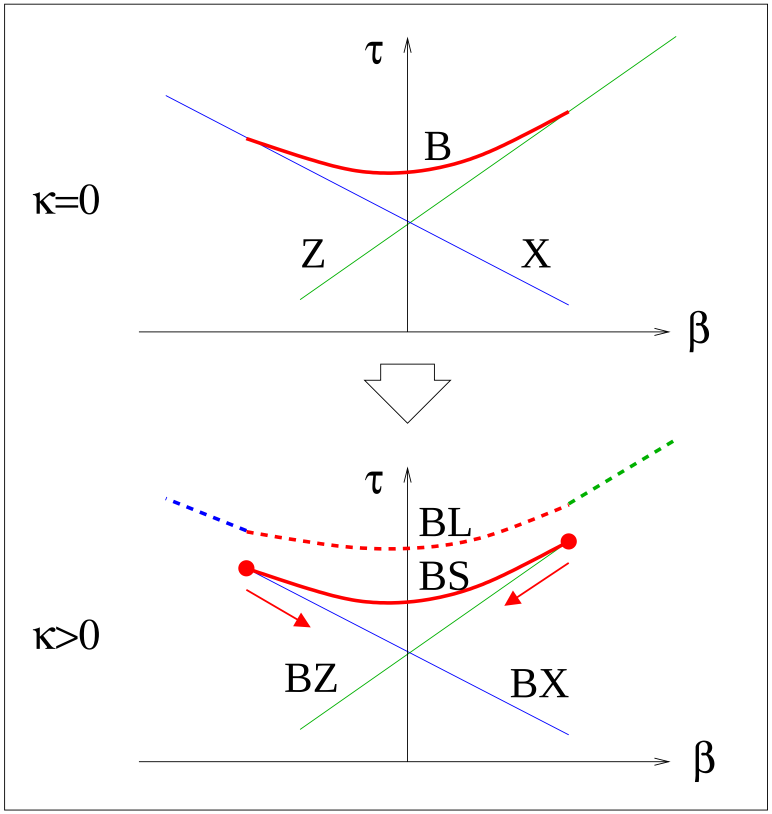
  at large deformation

  ... connect to BL

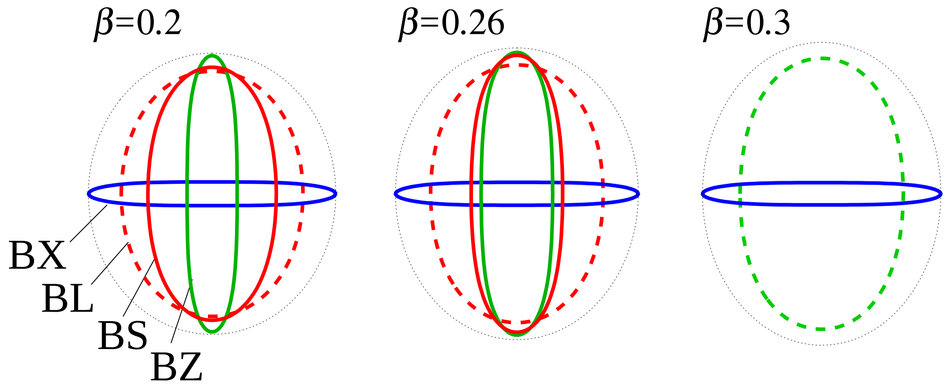
  at small deformation

  ... pair annihilate with BS

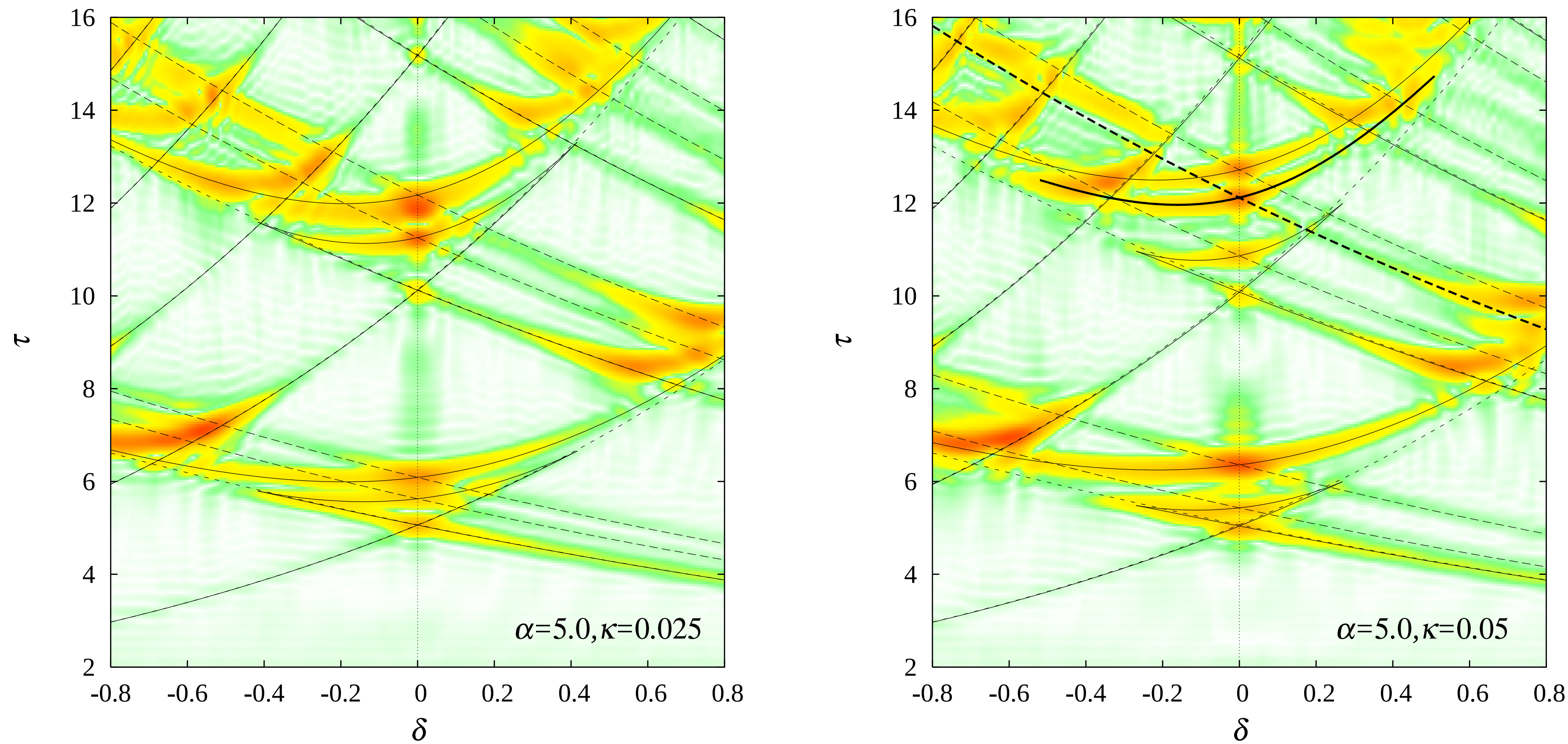
  BS shrinks with increasing κ



meridian frozen-spin orbit for $\kappa = 0.05$



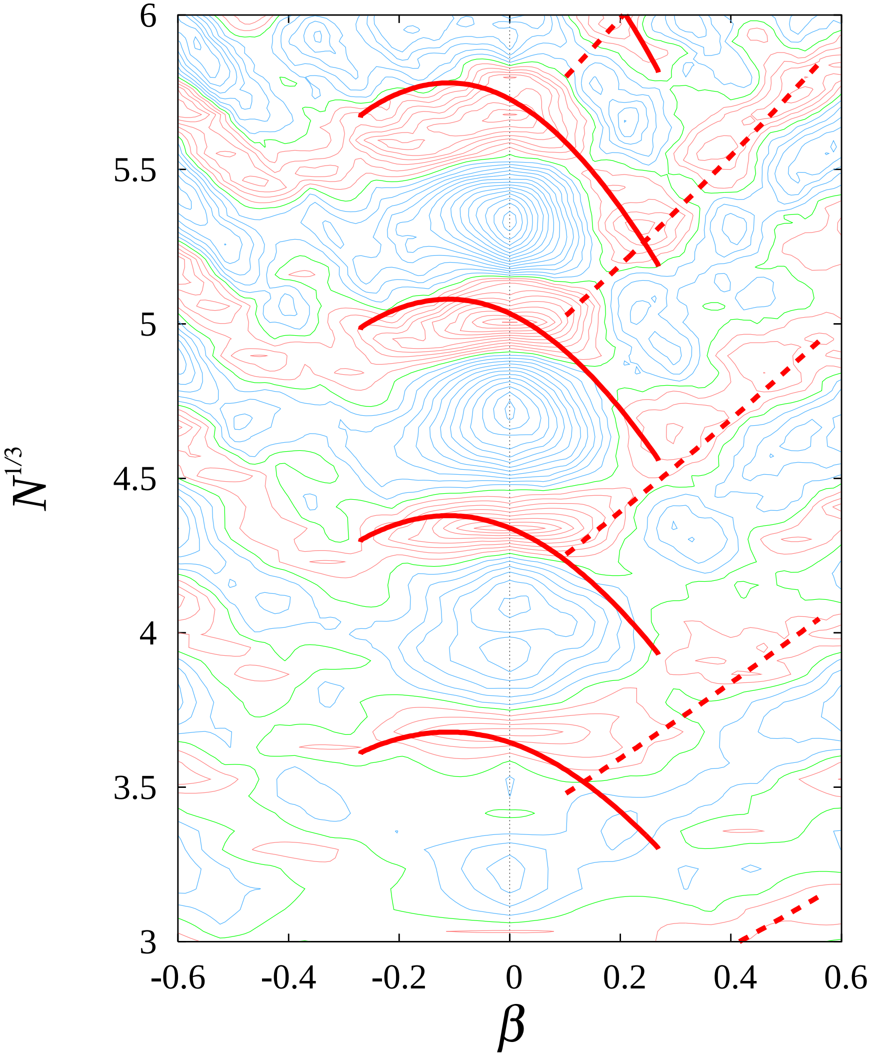
Fourier transforms of level density



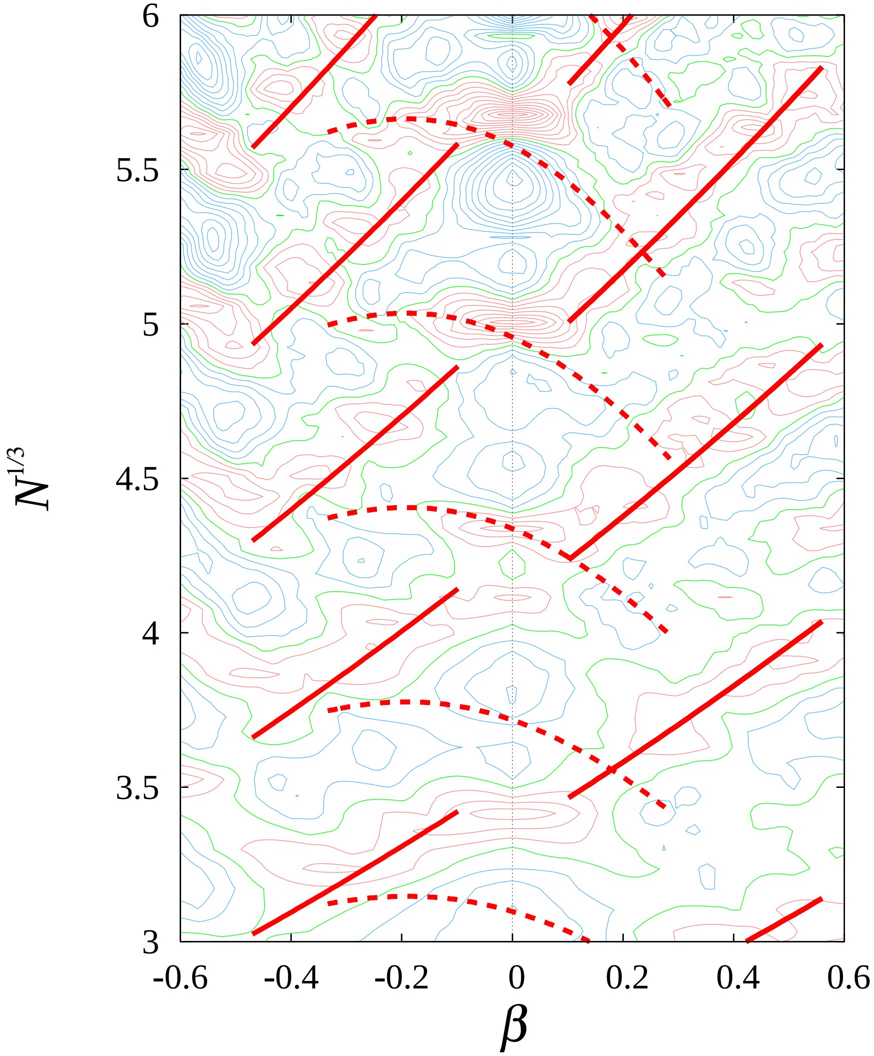
- New shell structures in oblate side
  due to the contributions of lower branches (BX with BS) at $\kappa \approx 0.025$
  $\Rightarrow$ *disappearance* of prolate dominance
- Shrink of the lower bridge at $\kappa \gtrsim 0.05$
  $\Rightarrow$ *revival* of prolate dominance

Shell energy and constant-action curves

$\kappa = 0.05$ (realistic)

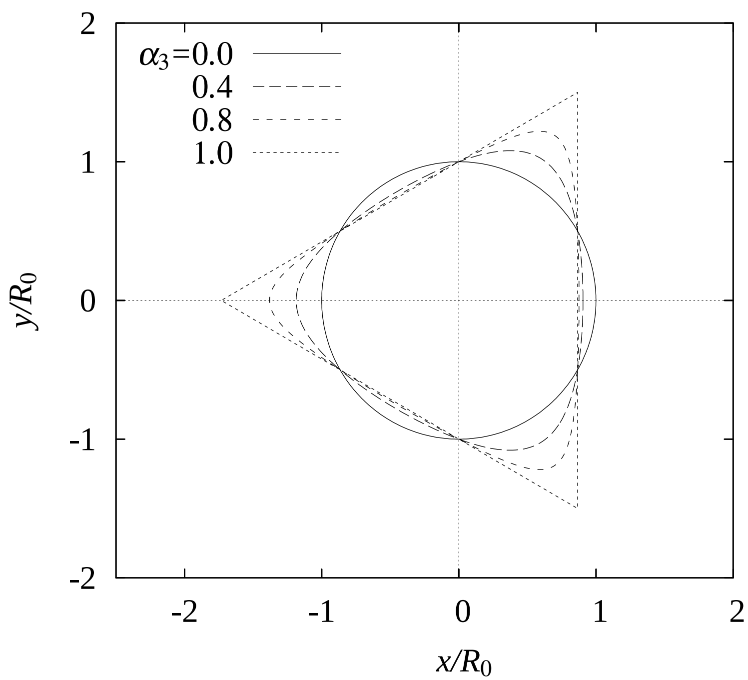


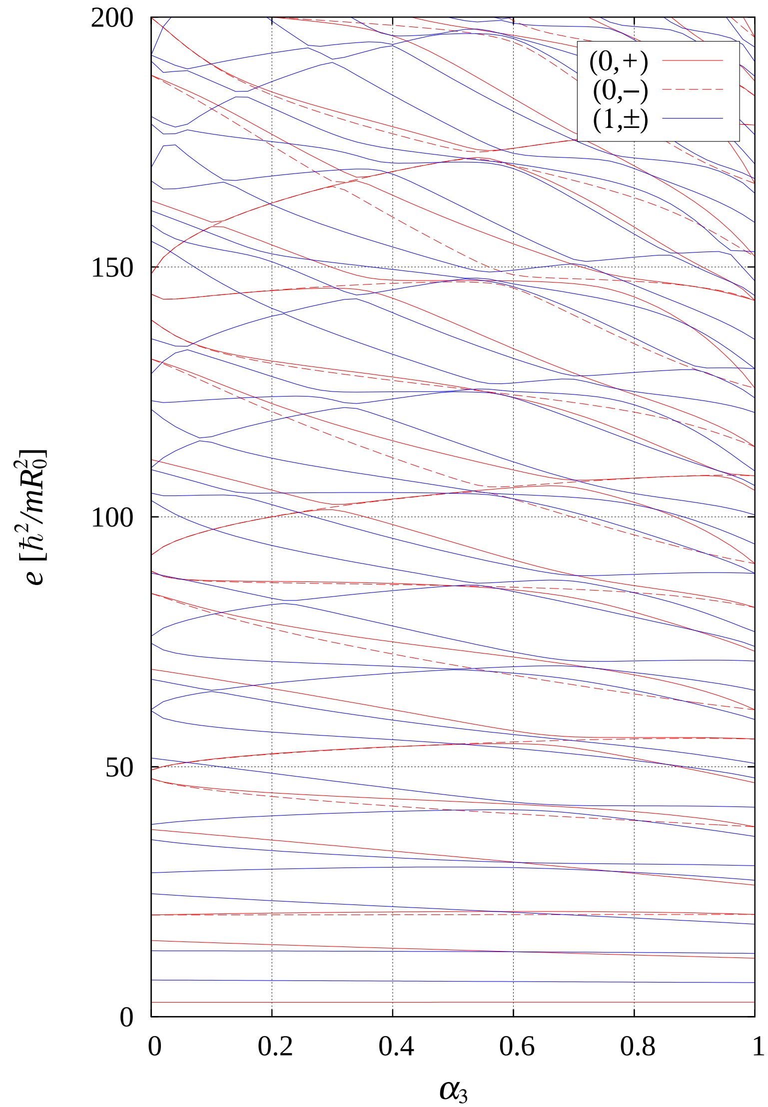
$\kappa = 0.025$ (reduced)



3.3. Anomalous shell effect at large tetrahedral deformation

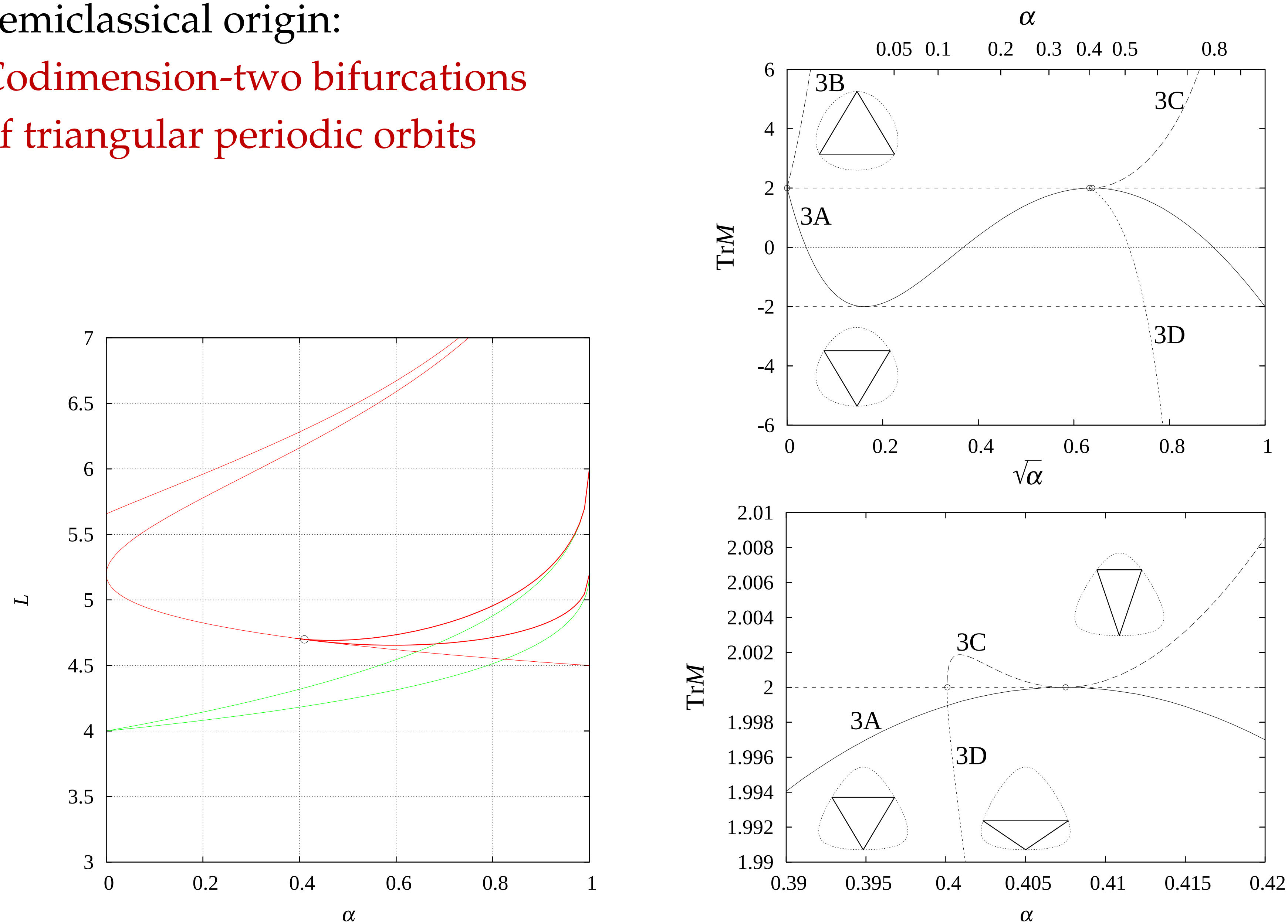
- *Anomalous shell effect in triangular billiard*
  K. A. and M. Brack,
  Phys. Rev. E77 (2008), 056211





Semiclassical origin:

Codimension-two bifurcations of triangular periodic orbits



❏ *Shell structures in nuclei with tetrahedral deformation*

Tetrahedral deformation (high point-group symmetry)

⇒ candidate for nuclear exotic-shape around $N = Z$ region

Power-law potential model with tetrahedral deformation

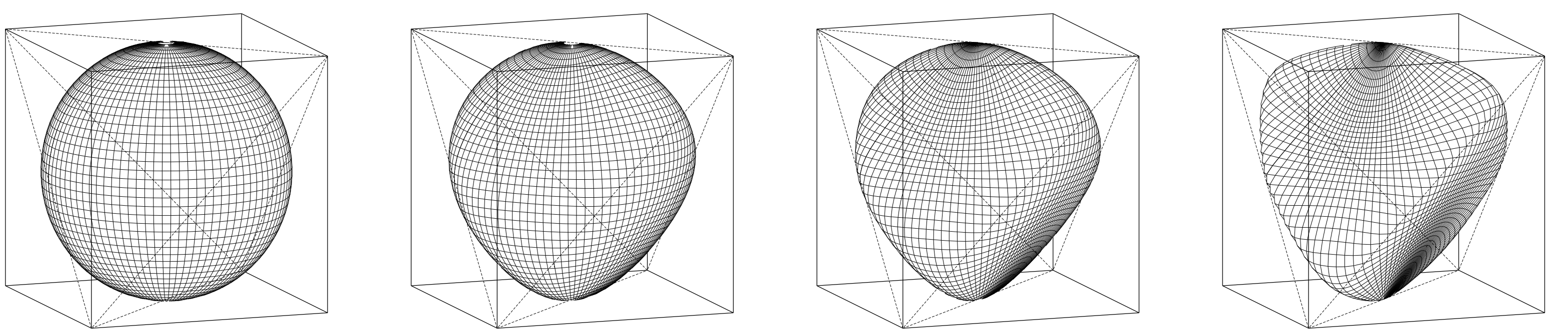
$$V(\boldsymbol{r}) = U_0 \left(\frac{r}{R(\theta, \varphi; c)} \right)^{\alpha}$$

Shape parametrization:

$$R^2 + u_0 + u_3(\theta, \varphi)R^3 + u_4(\theta, \varphi)R^4 = R_0^2 \quad \text{(tetrahedron)}$$

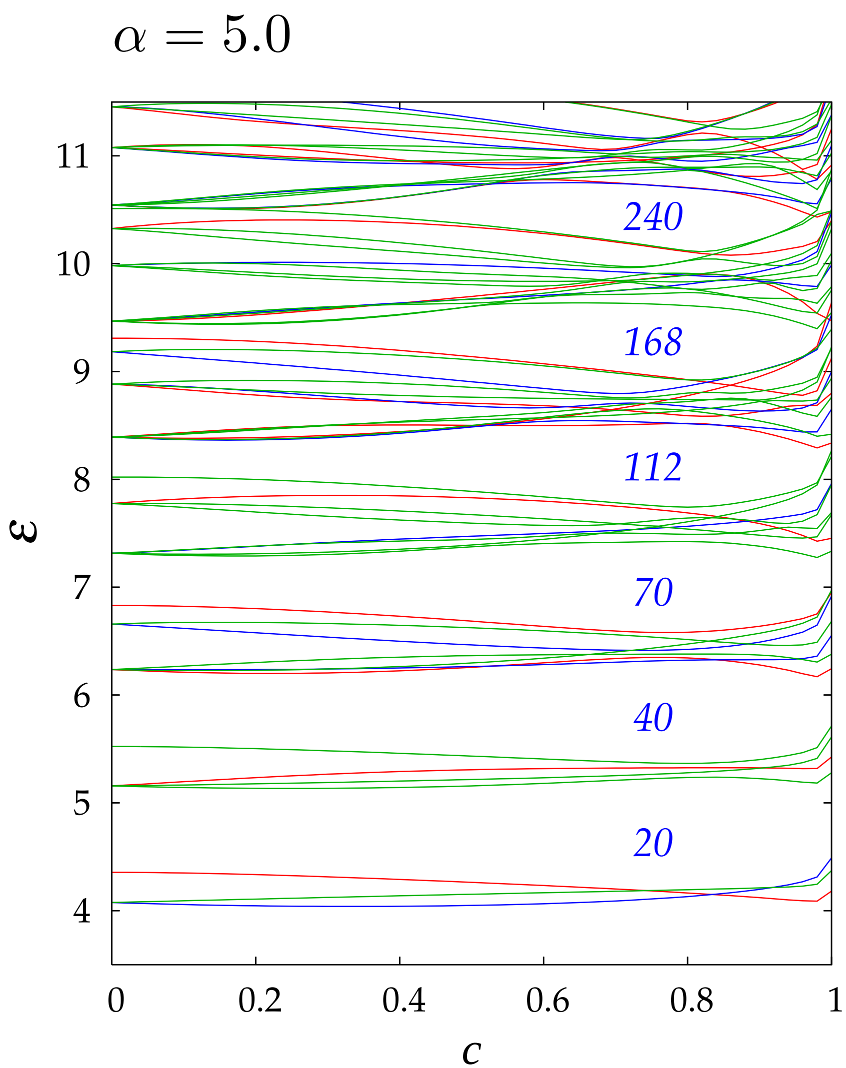
$$R_c^2 + \boldsymbol{c}\{u_0 + u_3(\theta, \varphi)R_c^3 + u_4(\theta, \varphi)R_c^4\} = R_0^2$$

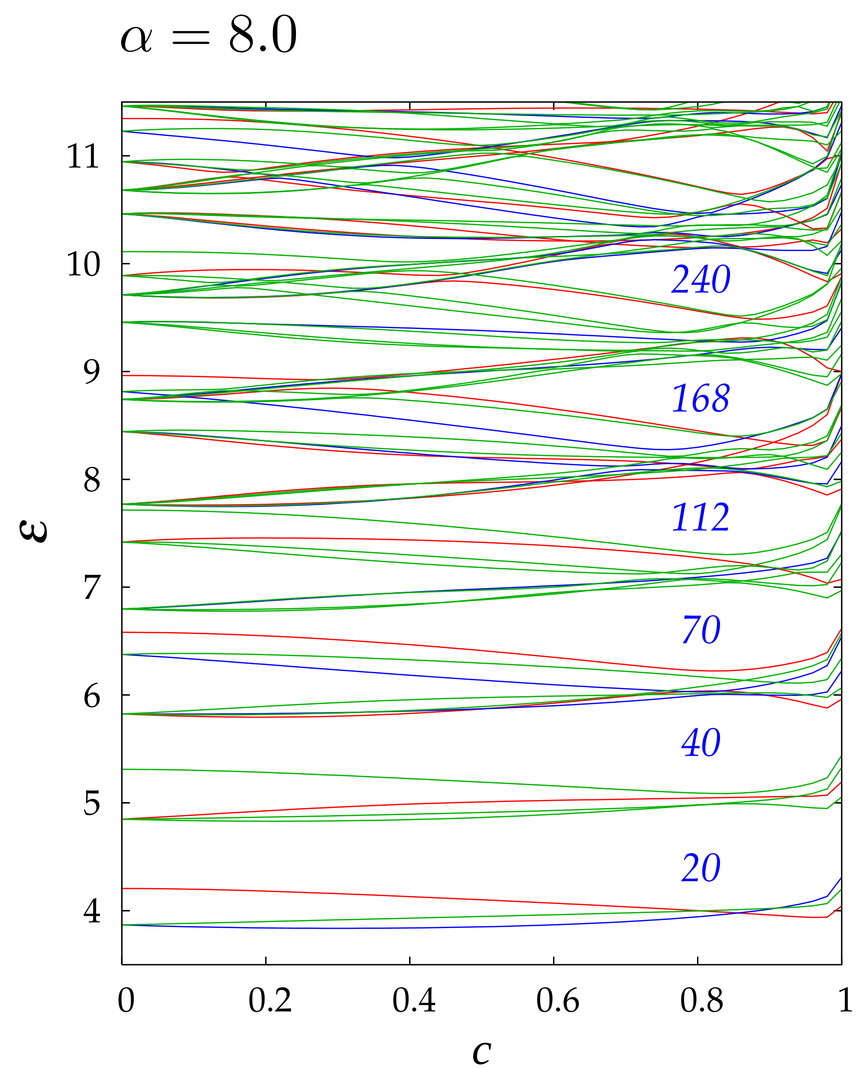
$c = 0$ ... sphere $(R = R_0) \longrightarrow c = 1$ ... tetrahedron



$c = 0$ $c = 0.3$ $c = 0.6$ $c = 0.9$

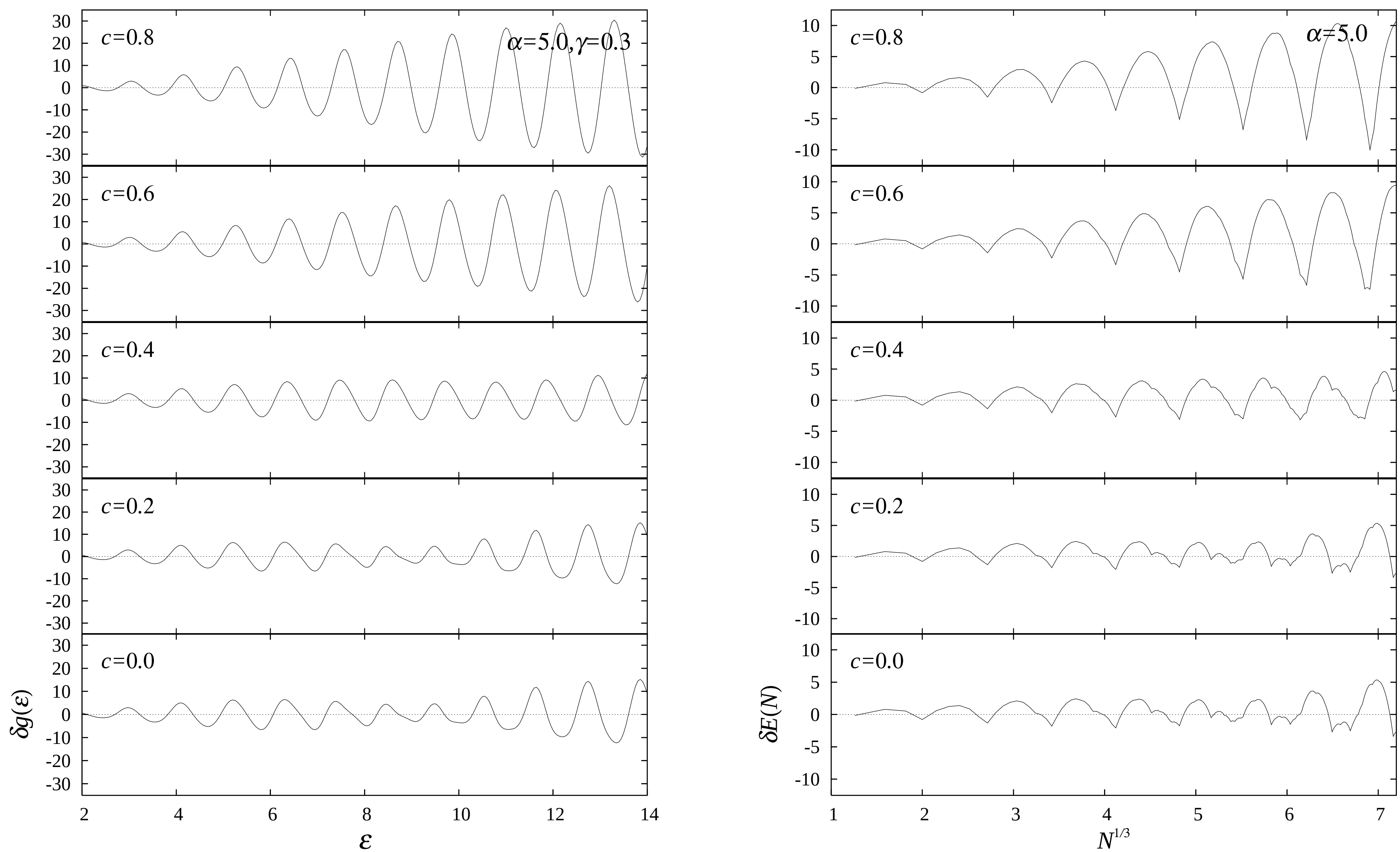
Single-particle level diagram





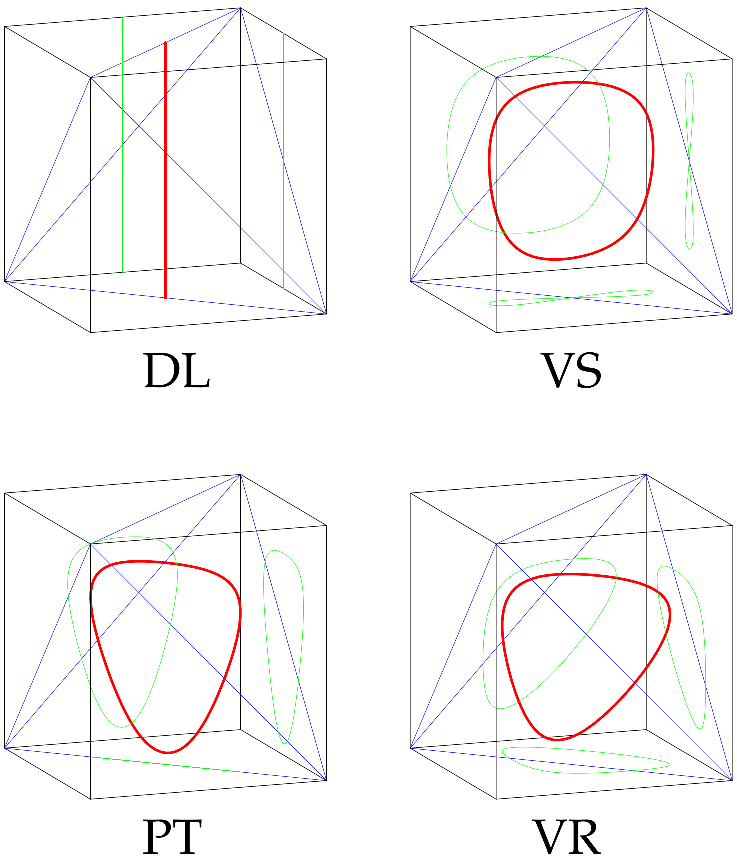
- very strong shell effect at $c = 0.7 \sim 0.8$
- magic numbers are exactly same as spherical HO ! SU(3) symmetry?

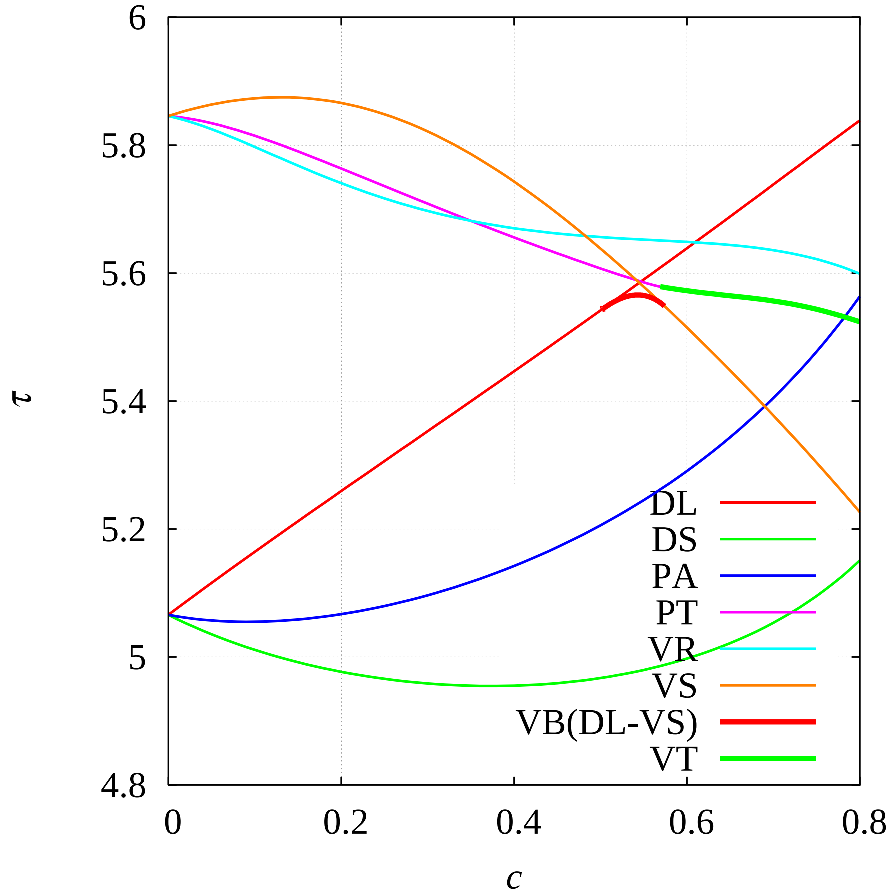
Oscillating level density and shell energy



Classical periodic orbits in tetrahedral power-law potential

$\alpha = 5.0, c = 0.3$



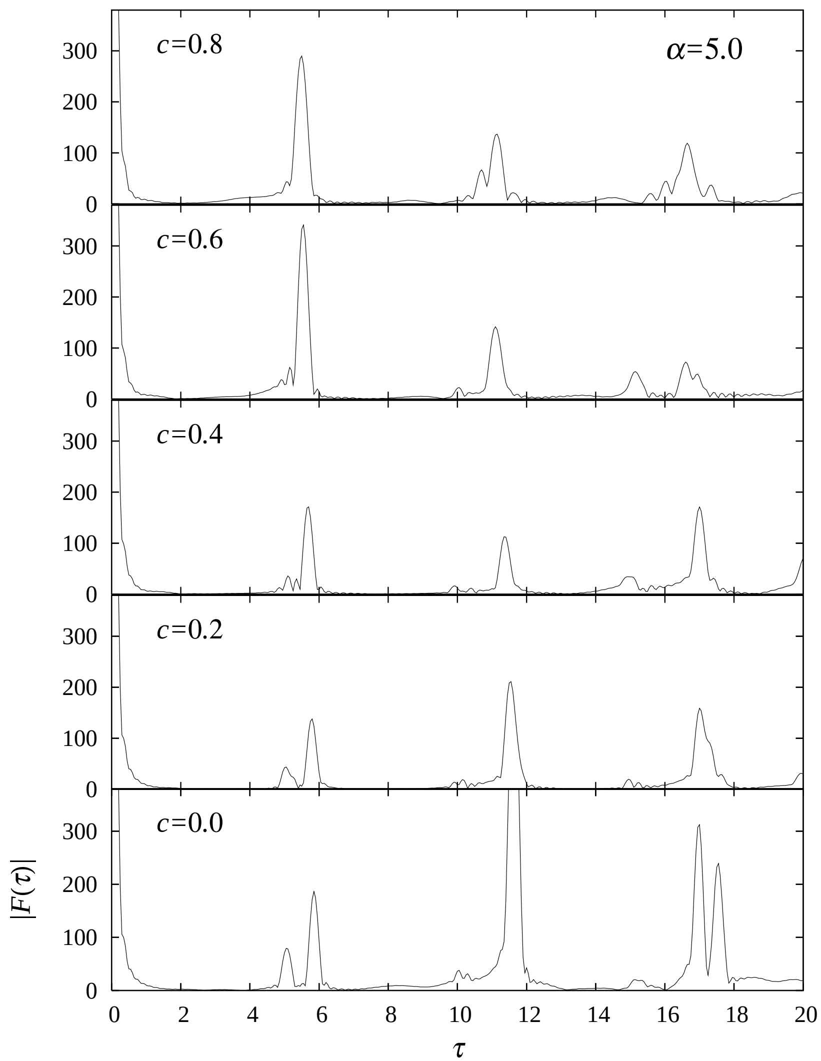


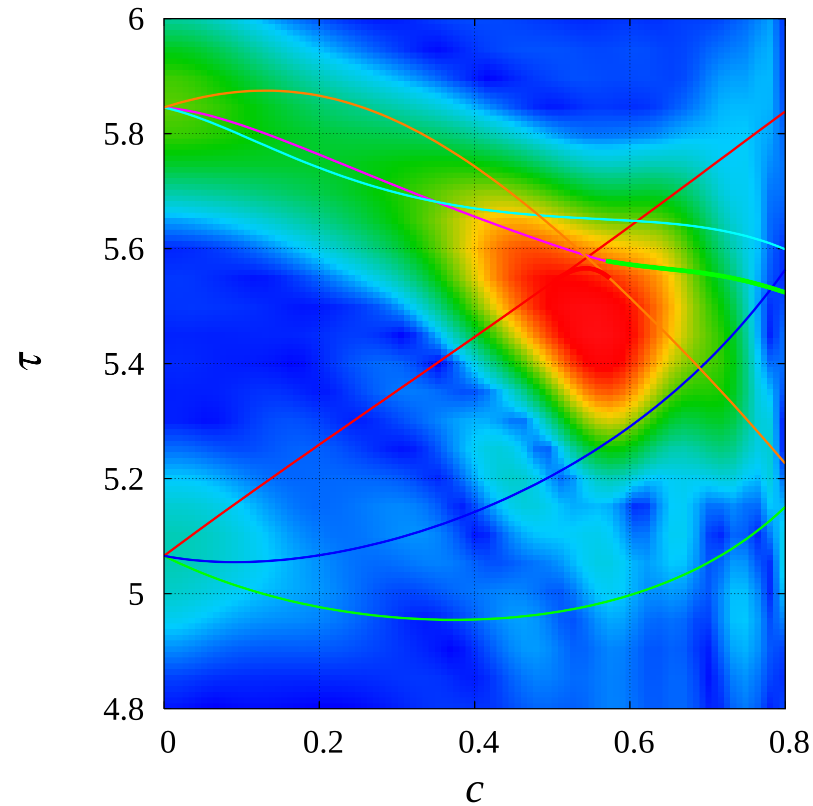
Simultaneous bifurcations of two PO's having almost the same period $\tau_{\rm po}$

local dynamical symmetries in different regions of the phase space

... existence of global dynamical symmetry ?

Fourier transforms of level density





Strong enhancement of Fourier amplitude around the bifurcation point
strong shell effect ⇔ PO bifurcations

4. Summary

Semiclassical analysis of spherical and deformed power-law potential with(w/o) spin-orbit coupling

Periodic orbit bifurcations play significant roles in emergence of strong shell effect

- ❑ Role of (3,1) orbit bifurcation for a certain combination of spin-orbit strength and surface diffuseness
  and possible relation to pseudospin symmetry
- ❑ Role of bridge orbit bifurcations for prolate-oblate asymmetry in deformed shell structures
- ❑ Role of simultaneous PO bifurcations at large tetrahedral deformation for the anomalous quantum shell effect.