iTHES mini-workshop "Exploration of hidden symmetries in atomic nuclei"
July 27, 2013, RIKEN, Japan

Pseudospin symmetry in supersymmetric quantum mechanics

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Outline

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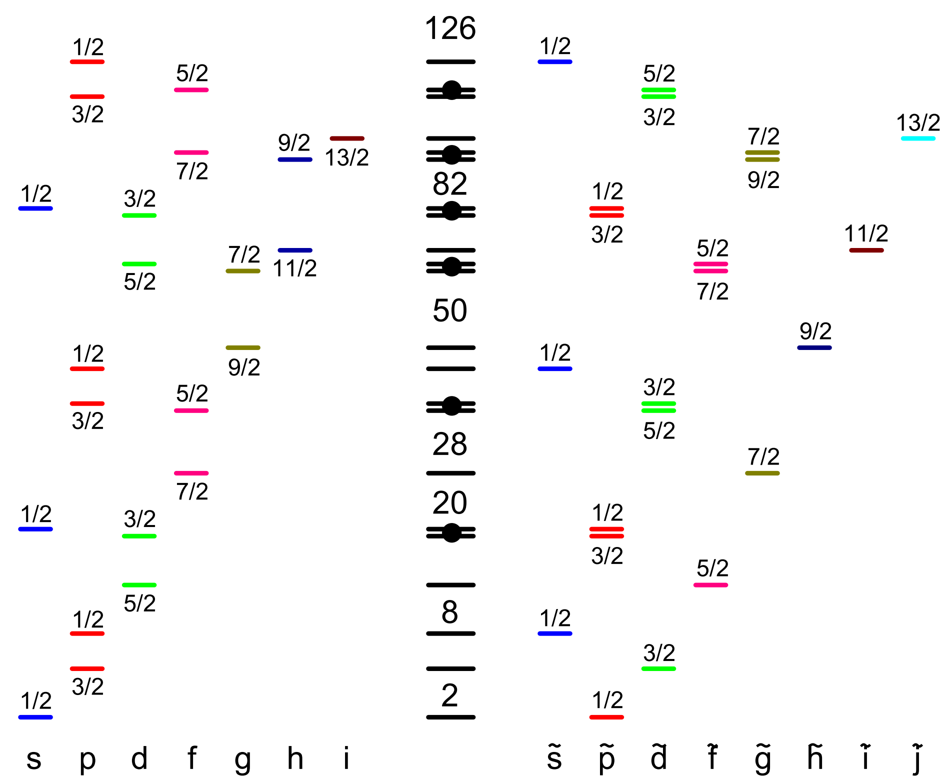
Spin and pseudospin symmetries

- Spin symmetry (SS) breaking, i.e., remarkable spin-orbit splitting in
  $(n, l, j = l \pm 1/2)$
  Haxel:1949, Mayer:1949

- Pseudospin symmetry (PSS), i.e., near degeneracy in
  $$\begin{cases} (n-1, l+2, j = l+3/2) \\ (n, l, j = l+1/2) \end{cases}$$
  by defining
  $(\tilde{n} = n-1, \tilde{l} = l+1, j = \tilde{l} \pm 1/2)$



In shell model scheme

- No spin-orbit coupling $\Rightarrow$ total spin S a good quantum number $\Rightarrow$ LS extreme $\times$
- No pseudo s.o. coupling $\Rightarrow$ total spin $\tilde{S}$ a good quantum number $\Rightarrow$ $\tilde{L}\tilde{S}$ extreme

Hecht & Adler, *NPA* **137**, 129 (1969); Arima, Harvey, Shimizu, *PLB* **30**, 517 (1969)

From spin scheme to pseudospin scheme

- From spin scheme to pseudospin scheme

$$H\psi = E\psi \quad \text{with} \quad H = \frac{\mathbf{p}^2}{2M} + V(r) + W(r)\mathbf{l}\cdot\mathbf{s}$$

$$(UHU^\dagger)U\psi = EU\psi \quad \text{with} \quad UHU^\dagger = \frac{\mathbf{p}^2}{2M} + \tilde{V}(r) + \tilde{W}(r)\tilde{\mathbf{l}}\cdot\tilde{\mathbf{s}}$$

- Special ratio for v_{sl}/v_{ll}, e.g., $U_r = \mathbf{s}\cdot\hat{\mathbf{r}}$

$$\begin{aligned} H &= H_{\mathrm{HO}} + v_{ll}\mathbf{l}^2 + v_{ls}\mathbf{l}\cdot\mathbf{s} \\ \tilde{H} &= \tilde{H}_{\mathrm{HO}} + v_{ll}\tilde{\mathbf{l}}^2 + (4v_{ll} - v_{ls})\tilde{\mathbf{l}}\cdot\tilde{\mathbf{s}} \end{aligned}$$

⋆ Parameters for the modified oscillator potential.

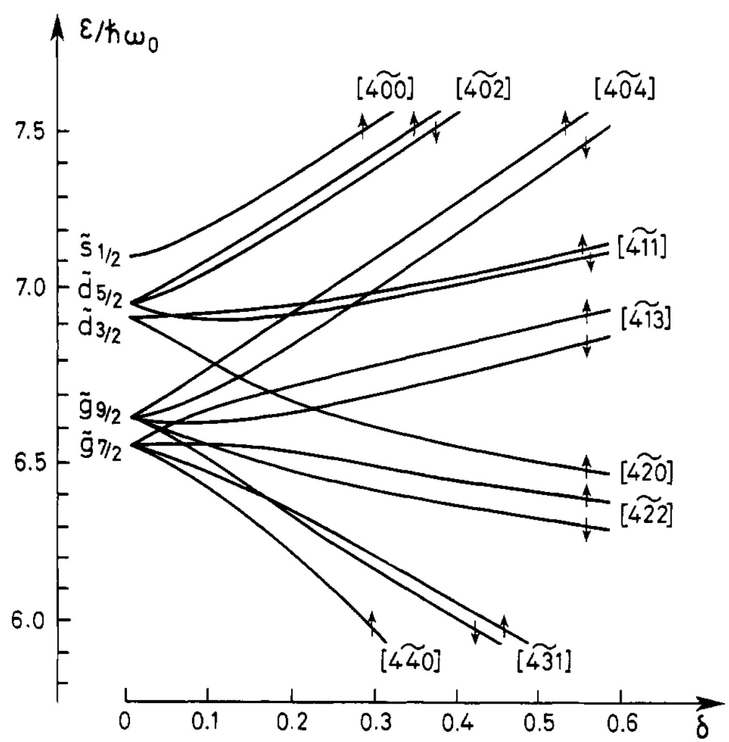
Bohr, Hamamoto, Mottelson, *Phys. Scr.* **26**, 267 (1982)

Region	$-v_{ls}$	$-v_{ll}$	$-\tilde{v}_{ls}$
$50 < Z < 82$	0.127	0.0382	0.026
$82 < N < 126$	0.127	0.0268	-0.019
$82 < Z < 126$	0.115	0.0375	0.035
$126 < N$	0.127	0.0206	-0.045

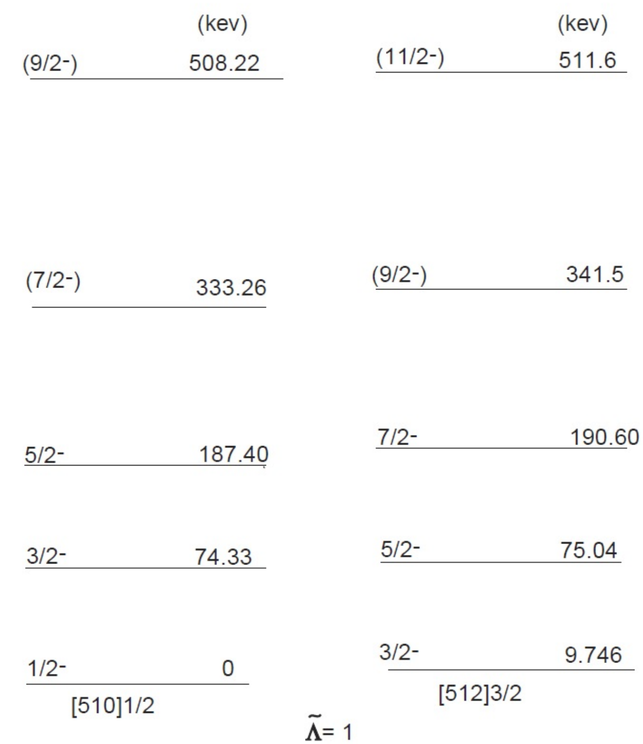
PSS in deformed nuclei

- Single-particle states: $[Nn_z\Lambda]\Omega$ & $[Nn_z\Lambda+2]\Omega+1 \Rightarrow [\widetilde{Nn_z\Lambda}]$ with $\tilde{N}=N-1, \tilde{\Lambda}=\Lambda+1, \Omega=\tilde{\Lambda}\pm 1/2$

- Rotational bands: from $\tilde{\Lambda}\Omega IM$ coupling to $\tilde{\Lambda}\tilde{R}IM$ coupling



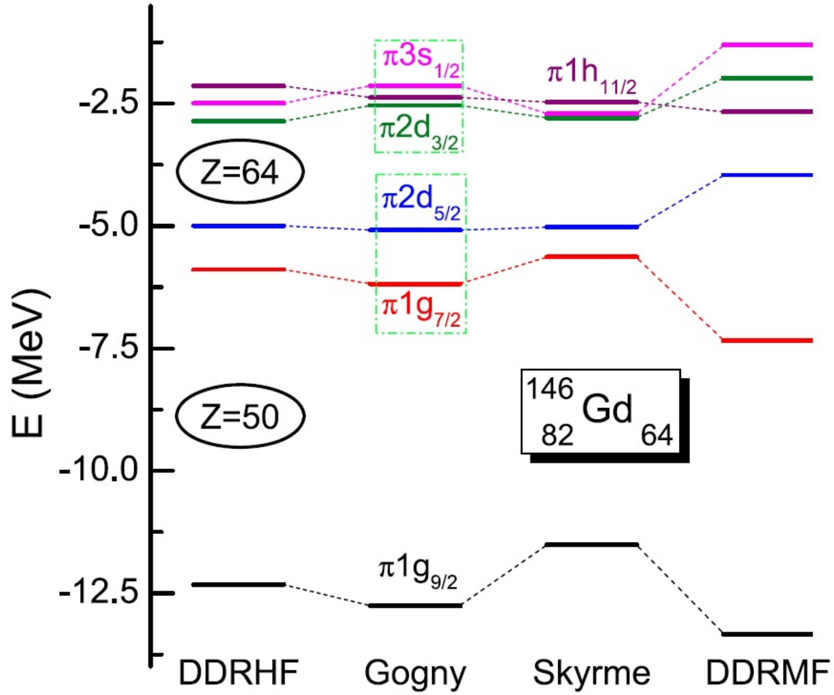
Bohr, Hamamoto, Mottelson, *Phys. Scr.* **26**, 267 (1982)



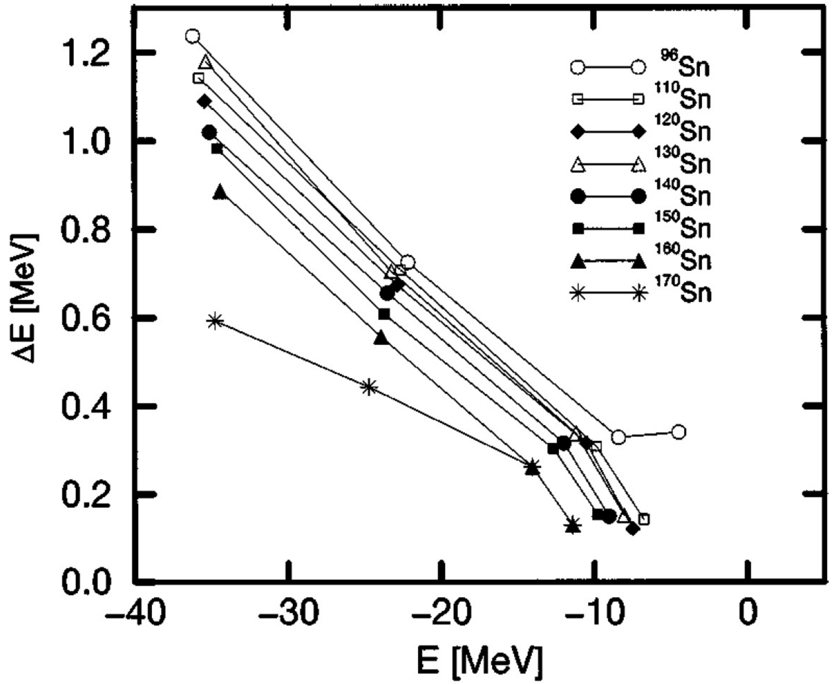
⋆ g.s. & neighboring bands in ^{187}Os

Data: Bruce *et al.*, *PRC* **56**, 1438 (1997)

PSS in shell structure evolutions



⋆ Proton single-particle energies for ^{146}Gd
Long, Nakatsukasa, Sagawa, Meng, Nakada, Zhang, *PLB* **680**, 428 (2009)



⋆ Pseudospin-orbit splitting in Sn isotopes
Meng, Sugawara-Tanabe, Yamaji, Arima *PRC* **59**, 154 (1999)

- Splitting of both spin and pseudospin doublets play important roles in the shell structure evolutions.
- It is a fundamental task to explore the origin of SS and PSS, as well as the mechanism of their breaking.

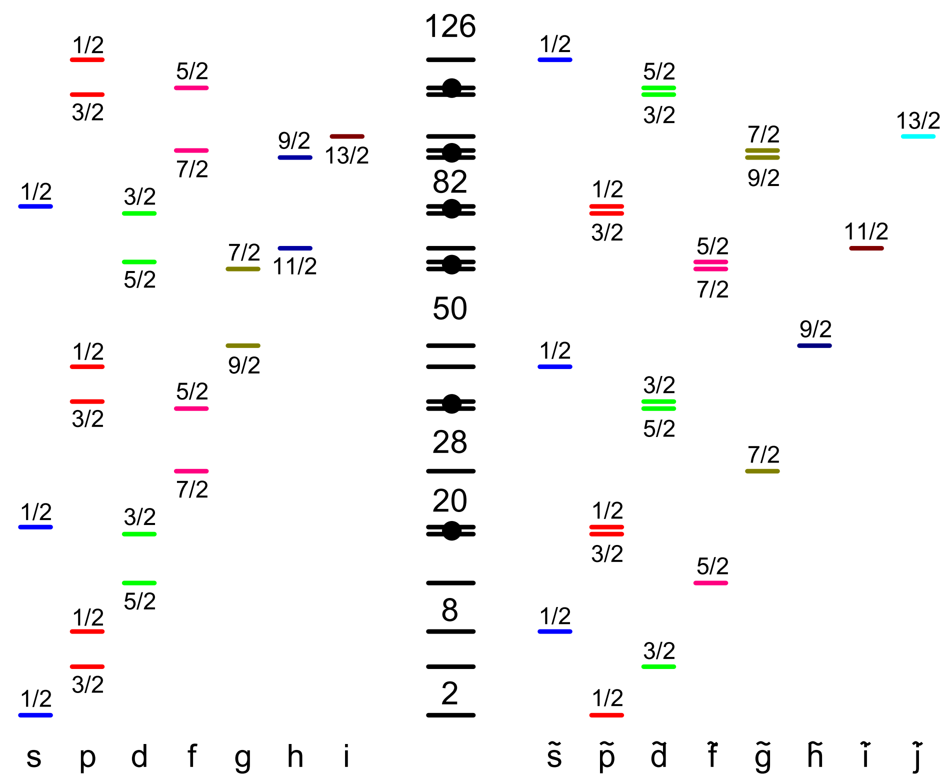
Intruder states in PSS

- Spin symmetry (SS) breaking, i.e., remarkable spin-orbit splitting in $(n, l, j = l \pm 1/2)$

  Haxel:1949, Mayer:1949

- Pseudospin symmetry (PSS), i.e., near degeneracy in
  $$\begin{cases} (n-1, l+2, j = l+3/2) \\ (n, l, j = l+1/2) \end{cases}$$
  by defining
  $$(\tilde{n} = n-1, \tilde{l} = l+1, j = \tilde{l} \pm 1/2)$$

  Arima:1969, Hecht:1969



- The intruder states do not have their own pseudospin partners.

Intruder states in PSS

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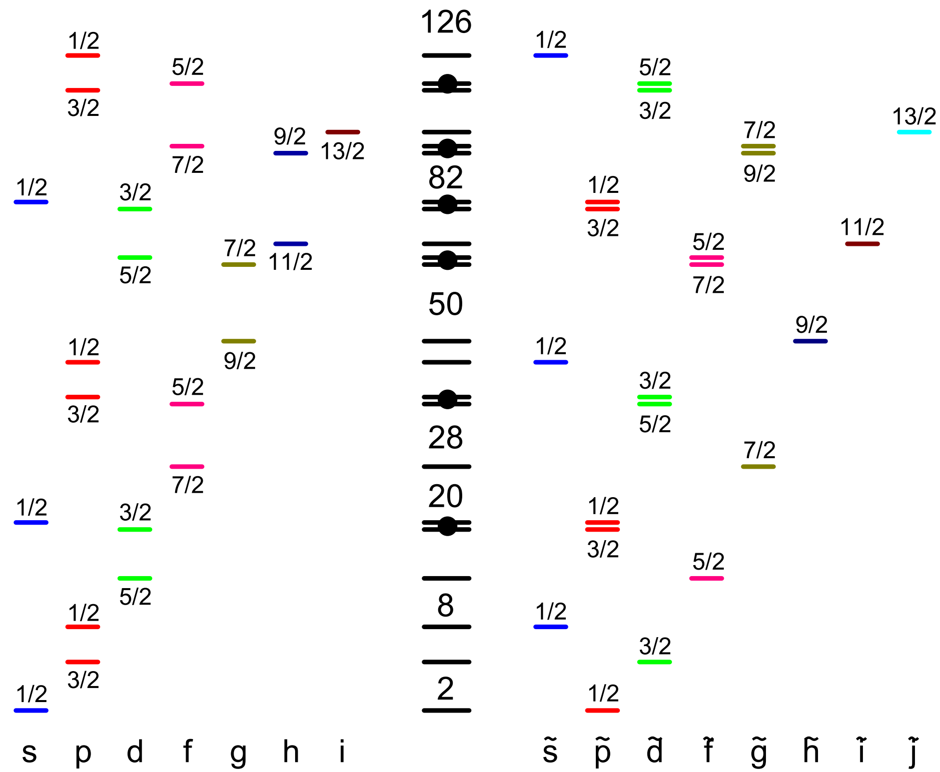

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⇒ Supersymmetric (SUSY) quantum mechanics

Leviatan, *PRL* **92**, 202501 (2004); Typel, *NPA* **806**, 156 (2008)

Outline

SUSY quantum mechanics (I)

- Every second-order Hamiltonian can be factorized in a product of two Hermitian conjugate first-order operators Infeld:1951, Cooper:1995

$$H_1 = B^+ B^- .$$

- The Hermitian operators Q_1 and Q_2 called supercharges read

$$Q_1 = \begin{pmatrix} 0 & B^+ \\ B^- & 0 \end{pmatrix}, \qquad Q_2 = iQ_1\tau = \begin{pmatrix} 0 & -iB^+ \\ iB^- & 0 \end{pmatrix}.$$

- The supersymmetric Hamiltonian

$$H_S = Q_1^2 = Q_2^2 = \begin{pmatrix} H_1 & 0 \\ 0 & H_2 \end{pmatrix}$$

is obtained with the supersymmetric partners

$$H_1 = B^+ B^- \qquad \text{and} \qquad H_2 = B^- B^+ .$$

SUSY quantum mechanics (II)

- Since H_S is the square of the Hermitian operators Q_i, all eigenvalues $E_S(n)$ of the eigenvalue equation are non-negative

$$H_S \Psi_S(n) = E_S(n) \Psi_S(n)$$

with the two-component wave function

$$\Psi_S(n) = \begin{pmatrix} \psi_1(n) \\ \psi_2(n) \end{pmatrix}.$$

- H_1 and H_2 have the **same spectrum** of positive energies $E_S(n) > 0$.
- Operators B^+ and B^- connect the components of the wave function by

$$\psi_2(n) = \frac{B^-}{\sqrt{E_S(n)}} \psi_1(n), \qquad \psi_1(n) = \frac{B^+}{\sqrt{E_S(n)}} \psi_2(n).$$

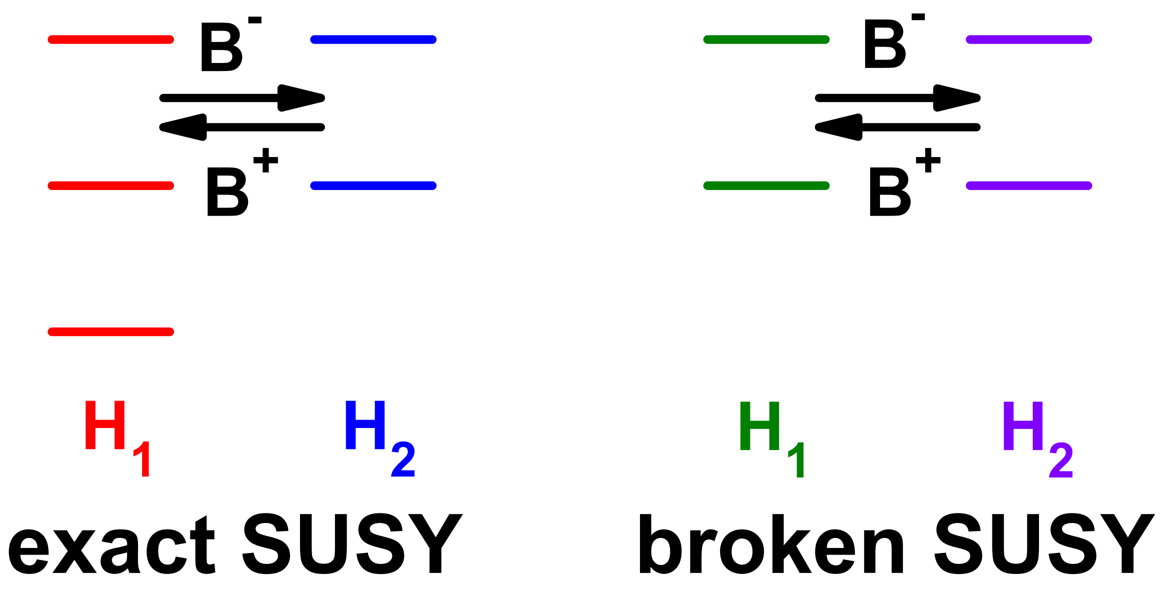
SUSY quantum mechanics (III)

- The supersymmetry is called exact if there is an eigenstate $\Psi_S(0)$ with energy $E_S(0) = 0$.

- As usual convention, this ground-state obeys

$$B^- \psi_1(0) = 0, \quad \psi_2(0) = 0,$$

i.e., H_1 has an additional state at zero energy that is not appearing in H_2.



Schrödinger equations without spin-orbit term

- Starting point: Schrödinger equations without spin-orbit term

$$\left[-\frac{1}{2M}\nabla^2 + V(\mathbf{r})\right]\psi(\mathbf{r}) = E\psi(\mathbf{r}).$$

- For the spherical symmetry,

$$HR_a(r) = E_a R_a(r)$$

with the Hamiltonian and wave functions

$$H = -\frac{d^2}{2Mdr^2} + \frac{\kappa(\kappa+1)}{2Mr^2} + V(r), \qquad \psi_\alpha(\mathbf{r}) = \frac{R_a(r)}{r}\mathscr{Y}^l_{jm}(\hat{\mathbf{r}}),$$

where $\kappa = \mp(j+1/2)$ for $j = l \pm 1/2$ as adopted in the relativistic framework.

- H has an explicit spin symmetry (SS).
- To investigate the pseudospin symmetry (PSS) and its breaking, the critical point is to identify the $\tilde{l}(\tilde{l}+1) = \kappa(\kappa-1)$ term.
- One of the promising tricks is the SUSY quantum mechanics. Typel, *NPA* **806**, 156 (2008)

SUSY for Schrödinger equations (I)

- SUSY for Schrödinger equations without spin-orbit term

$$H = -\frac{d^2}{2Mdr^2} + \frac{\kappa(\kappa+1)}{2Mr^2} + V(r)$$

- Two Hermitian conjugate first-order operators

$$B_\kappa^+ = \left[Q_\kappa(r) - \frac{d}{dr}\right]\frac{1}{\sqrt{2M}}, \qquad B_\kappa^- = \frac{1}{\sqrt{2M}}\left[Q_\kappa(r) + \frac{d}{dr}\right],$$

- SUSY partner Hamiltonians

$$\begin{aligned} H_1 &= B_\kappa^+ B_\kappa^- = \frac{1}{2M}\left[-\frac{d^2}{dr^2} + Q_\kappa^2 - Q_\kappa'\right], \\ H_2 &= B_\kappa^- B_\kappa^+ = \frac{1}{2M}\left[-\frac{d^2}{dr^2} + Q_\kappa^2 + Q_\kappa'\right]. \end{aligned}$$

SUSY for Schrödinger equations (II)

- Furthermore, setting the reduced supermomenta

$$q_\kappa(r) = Q_\kappa(r) - \frac{\kappa}{r},$$

so that the SUSY partner Hamiltonians read

$$\begin{aligned} H_1 &= B_\kappa^+ B_\kappa^- = \frac{1}{2M}\left[-\frac{d^2}{dr^2} + \frac{\kappa(\kappa+1)}{r^2} + q_\kappa^2 + \frac{2\kappa}{r} q_\kappa - q_\kappa'\right], \\ H_2 &= B_\kappa^- B_\kappa^+ = \frac{1}{2M}\left[-\frac{d^2}{dr^2} + \frac{\kappa(\kappa-1)}{r^2} + q_\kappa^2 + \frac{2\kappa}{r} q_\kappa + q_\kappa'\right]. \end{aligned}$$

- The centrifugal barrier term $\kappa(\kappa+1)$ leading to SS appears in H_1.
- The pseudo-centrifugal barrier term $\kappa(\kappa-1)$ leading to PSS appears in H_2.

Energy shifts

- H and H_1 are connected by

$$H_1(\kappa) + e(\kappa) = H$$

with the **energy shifts** $e(\kappa)$ to be determined.

- It is equivalent that

$$\frac{1}{2M}\left[q_\kappa^2(r) + \frac{2\kappa}{r}q_\kappa(r) - q_\kappa'(r)\right] + e(\kappa) = V(r),$$

so that $q_\kappa(0) = 0$ and $\lim_{r\to 0} q_\kappa(r) = \frac{2M(e(\kappa)-V)}{(1-2\kappa)}r$ with regular potential $V(r)$.

- **Energy shifts** for PS doublets ($\kappa + \kappa' = 1$)
  - ⋆ For $\kappa < 0$, since the exact SUSY is achieved, it is required $E_1(\kappa) = 0$, i.e.,

$$e(\kappa) = E_{1\kappa}.$$

  - ⋆ For $\kappa > 0$, to fulfill $\lim_{r\to 0} q_\kappa(r) = \lim_{r\to 0} q_{\kappa'}(r)$, it is required Typel:2008

$$e(\kappa) = 2\, V|_{r=0} - e(\kappa').$$

Exact PSS limits

- The **exact PSS limits** indicate $E_{n\kappa_1} = E_{(n-1)\kappa_2}$, it is required

$$H_2(\kappa_1) + e(\kappa_1) = H_2(\kappa_2) + e(\kappa_2),$$

i.e.,

$$\frac{1}{2M}\left[q^2_{\kappa_1}(r) + \frac{2\kappa_1}{r}q_{\kappa_1}(r) + q'_{\kappa_1}(r)\right] + e(\kappa_1) = \frac{1}{2M}\left[q^2_{\kappa_2}(r) + \frac{2\kappa_2}{r}q_{\kappa_2}(r) + q'_{\kappa_2}(r)\right] + e(\kappa_2)$$

$$q'_{\kappa_1}(r) = q'_{\kappa_2}(r)$$

- Since $q_\kappa(0) = 0$, this leads to $q_{\kappa_1}(r) = q_{\kappa_2}(r)$, and finally

$$q_{\kappa_1}(r) = q_{\kappa_2}(r) = \frac{A}{2}\omega_{\{\kappa_1,\kappa_2\}} r \quad \text{with constants} \quad A \equiv 2M, \omega_{\{\kappa_1,\kappa_2\}} \equiv \frac{e(\kappa_1) - e(\kappa_2)}{\kappa_2 - \kappa_1}.$$

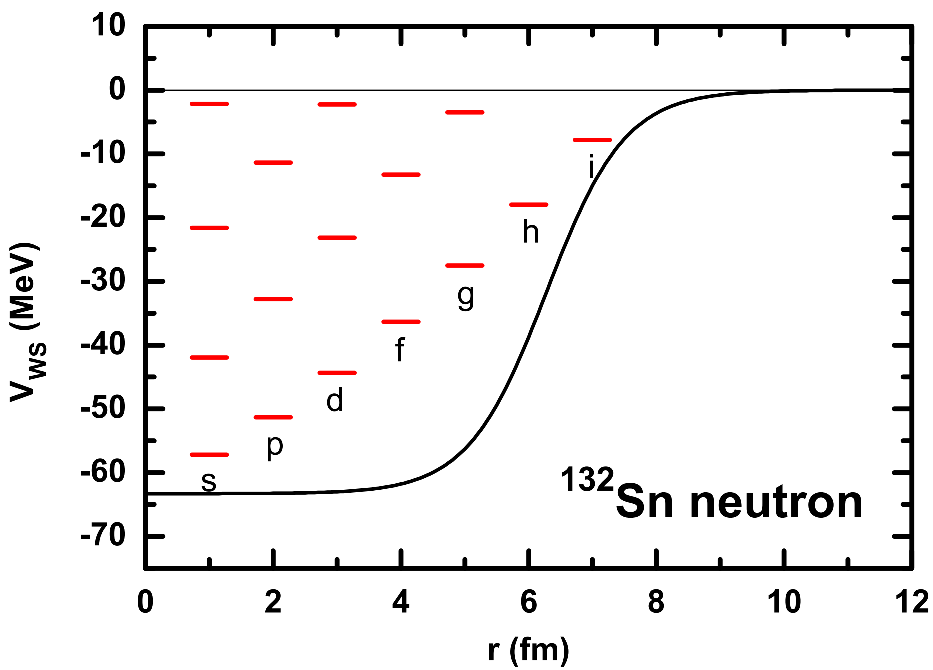
- This indicates the only possible PSS limits in the Schrödinger equations without spin-orbit term are those with harmonic oscillator (HO) potentials

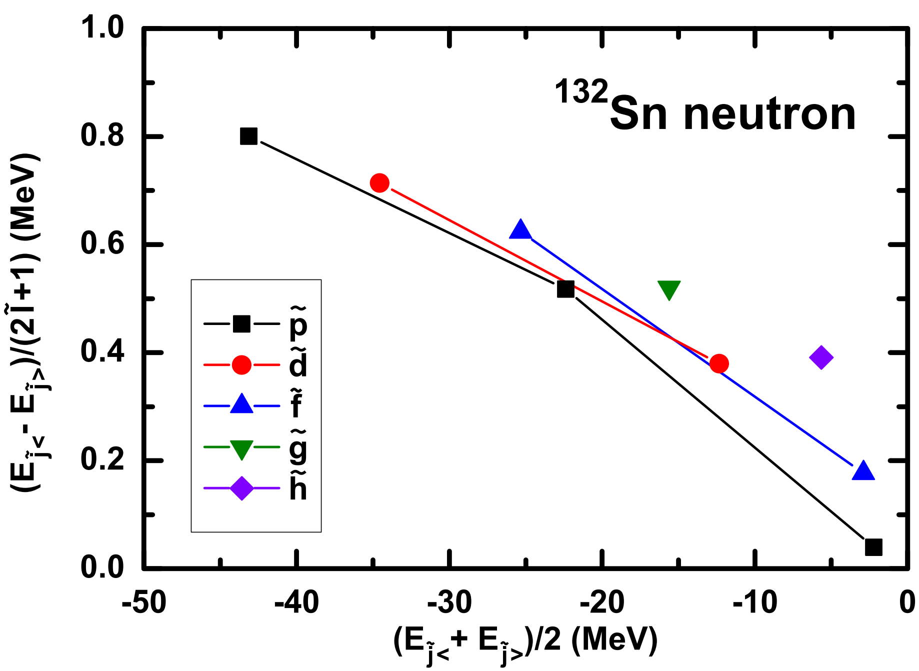
$$V_{\rm HO}(r) = \frac{A}{4}\omega^2_{\{\kappa_1,\kappa_2\}} r^2 + V(0).$$

cf. $H = H_{\rm HO} + v_{ll}\mathbf{l}^2 + v_{ls}\mathbf{l}\cdot\mathbf{s}$; $\tilde{H} = \tilde{H}_{\rm HO} + v_{ll}\tilde{\mathbf{l}}^2 + (4v_{ll} - v_{ls})\tilde{\mathbf{l}}\cdot\tilde{\mathbf{s}}$ Bohr:1982

Outline

Single-particle energies and pseudospin-orbit splittings





⋆ Left: Woods-Saxon potential for ^{132}Sn and bound single-neutron energies.

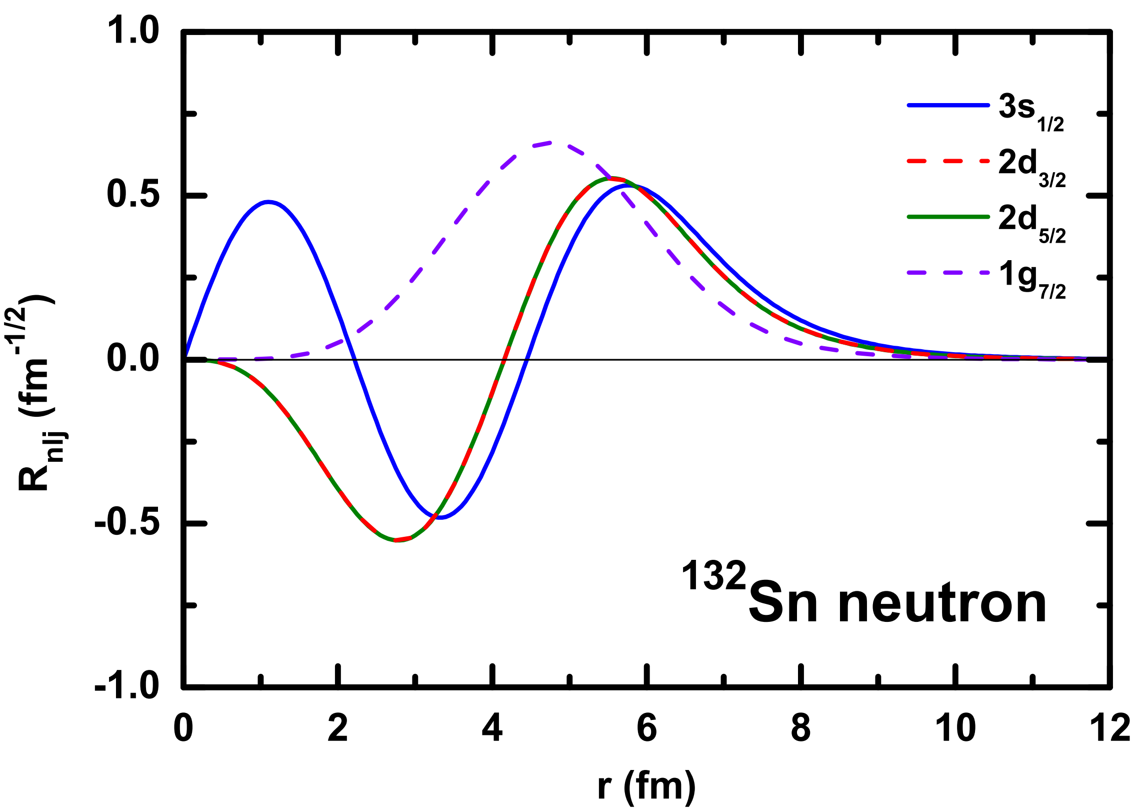
Right: pseudospin-orbit splittings $(E_{\tilde{j}_<} - E_{\tilde{j}_>})/(2\tilde{l}+1)$ *vs* $(E_{\tilde{j}_<} + E_{\tilde{j}_>})/2$.

HL, Shen, Zhao, Meng, *PRC* **87**, 014334 (2013)

- How to understand the amplitudes of PSS splittings?
- Why do pseudospin-orbit splittings ΔE_{PSO} decrease as single-particle energies E_{av} increase?

Single-particle wave functions

Normal representation



⋆ Single-particle wave functions of the $3s_{1/2}$, $2d_{3/2}$, $2d_{5/2}$, and $1g_{7/2}$ states.

- Wave functions of spin doublets are exact the same since there is no spin-orbit term.
- However, wave functions of the PS doublets are very different to each other, so it is difficult to analyze the origin of PSS and its breaking.

Implicit PSS limit: Schrödinger equations with HO potentials

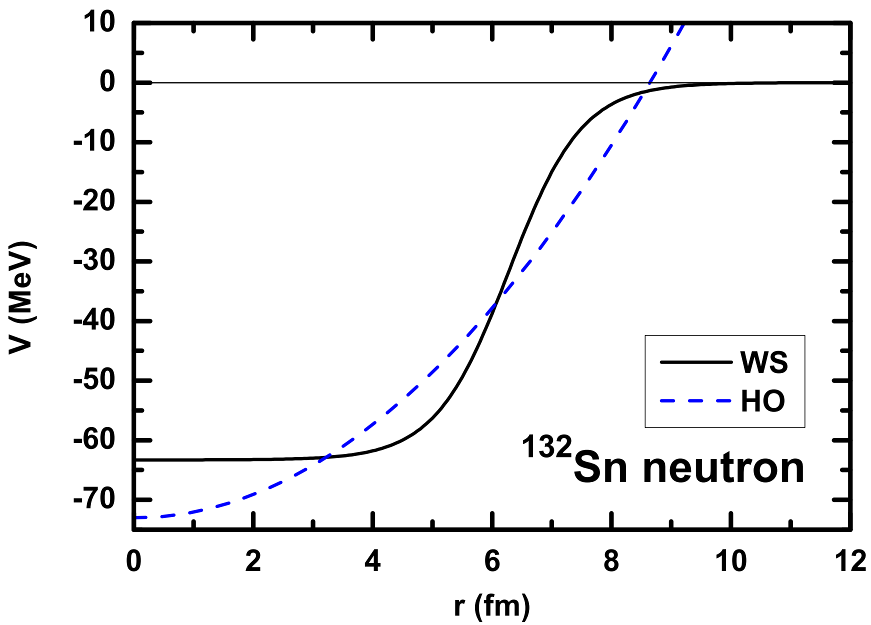
- Perturbative interpretation of PSS by using Rayleigh-Schrödinger perturbation theory

  HL, Zhao, Zhang, Meng, Giai, *PRC* **83**, 041301(R) (2011)

- Hamiltonian can be divided as

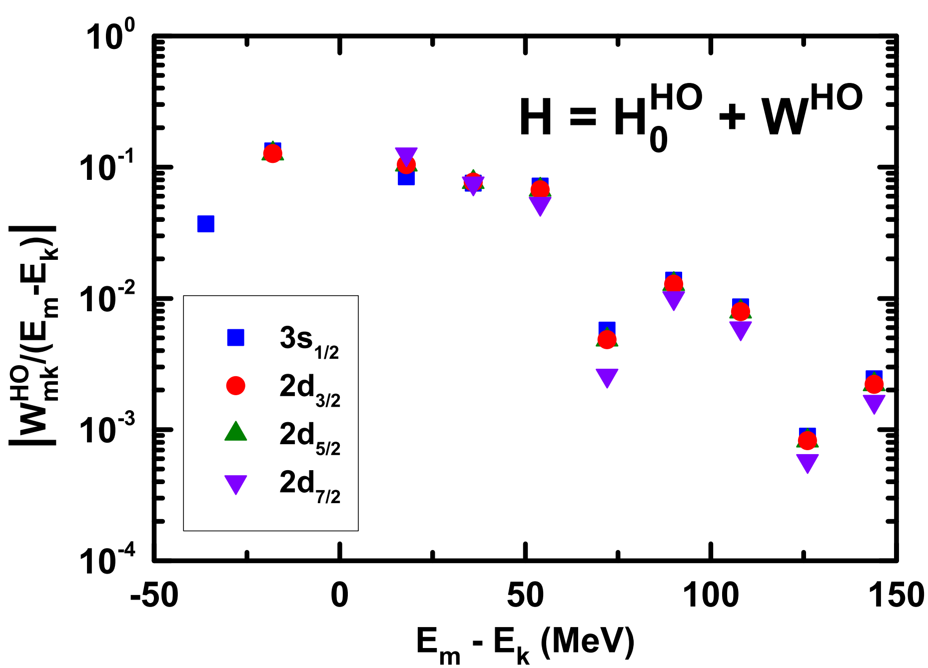
$$H = H_0^{\mathrm{HO}} + W^{\mathrm{HO}}$$

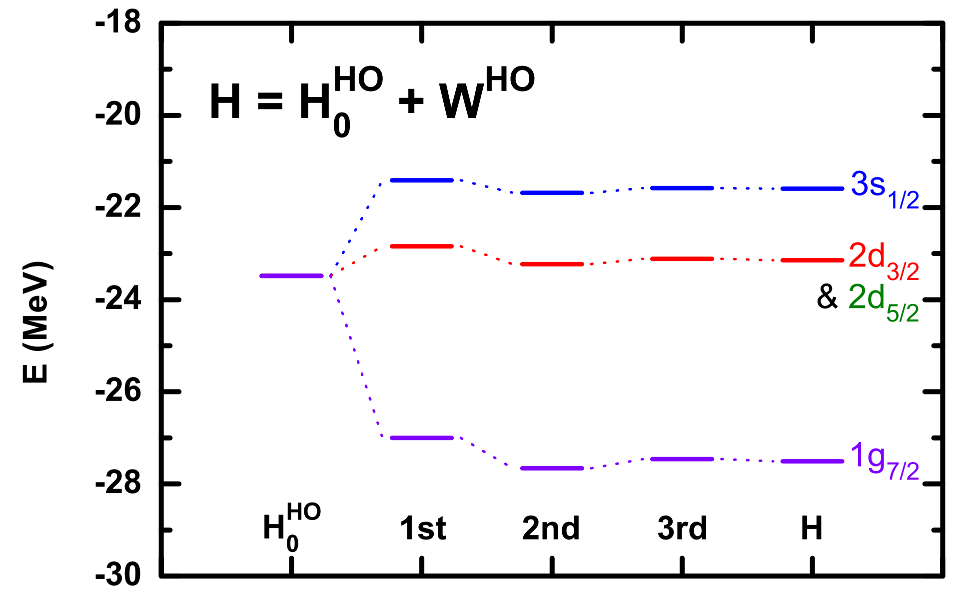
  ⋆ H_0^{HO} leading to PSS
  ⋆ W^{HO} symmetry breaking potential



$$H_0^{\mathrm{HO}} = -\frac{1}{2M}\left[\frac{d^2}{dr^2} + \frac{\kappa(\kappa+1)}{r^2}\right] + \frac{A}{4}\omega^2 r^2 + V(0)$$

Validity of perturbation theory and perturbation corrections





HL, Shen, Zhao, Meng, *PRC* **87**, 014334 (2013)

- The biggest perturbations $\sim$ 0.13.
- Pseudospin-orbit splittings are reproduced by the 3rd-order perturbation calculations.

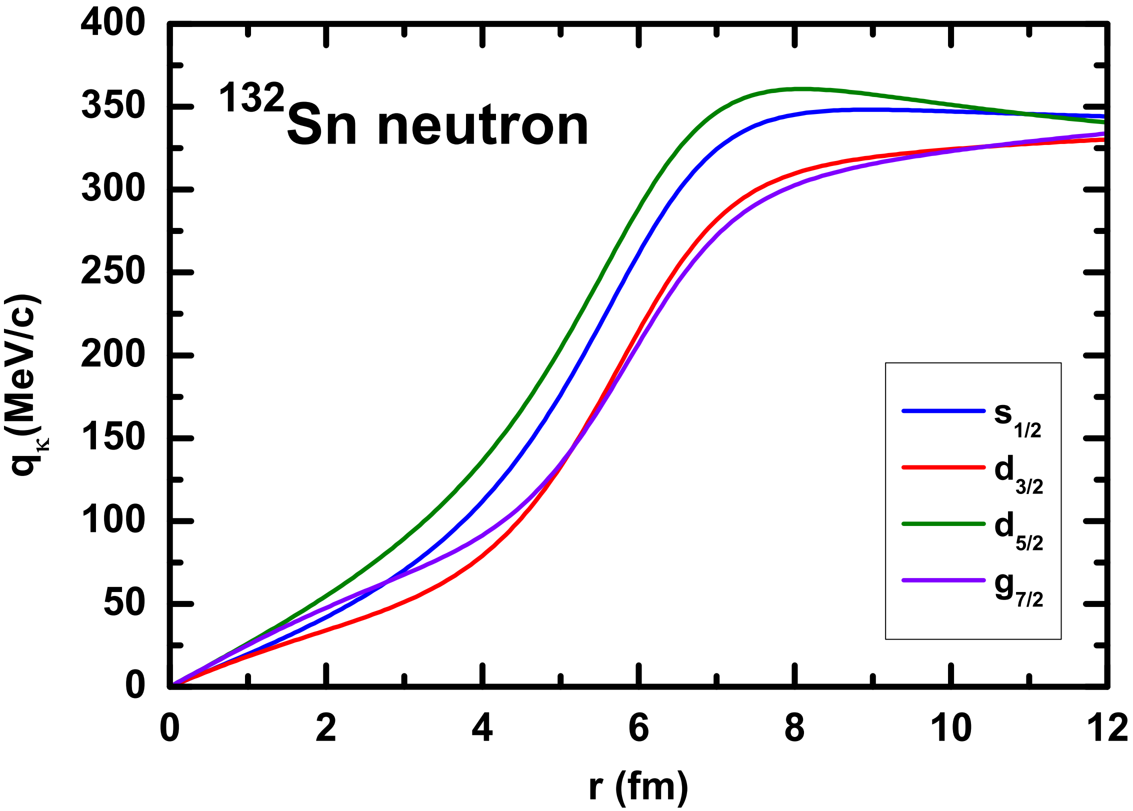
Conclusion

The nature of PSS is perturbative, and its breaking can be understood in such implicit way.

Reduced supermomenta

SUSY representation

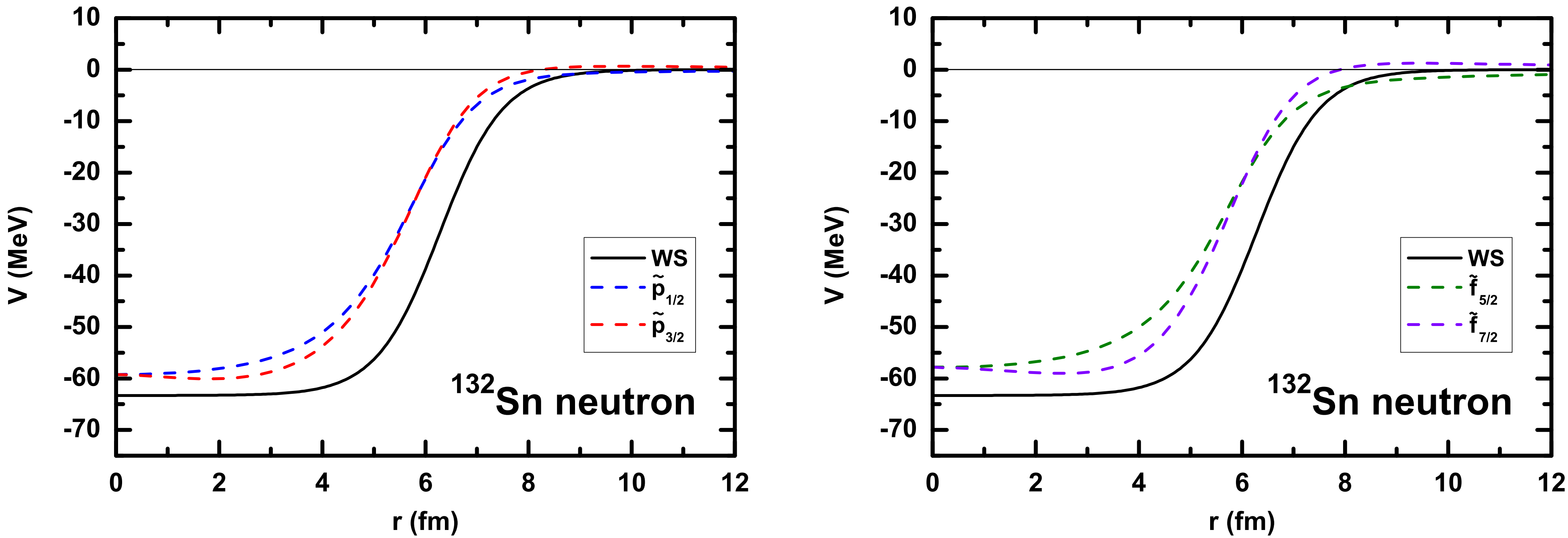
$$\frac{1}{2M}\left[q_\kappa^2(r)+\frac{2\kappa}{r}q_\kappa(r)-q'_\kappa(r)\right]+e(\kappa)=V(r)$$



⋆ Reduced supermomenta $q_\kappa(r)$ for the $s_{1/2}$, $d_{3/2}$, $d_{5/2}$, and $g_{7/2}$ blocks.

- $q_\kappa(r)$ are block-dependent.
- Asymptotic behaviors: $\lim_{r\to 0} q_\kappa(r)=\frac{2M(e(\kappa)-V)}{(1-2\kappa)}r$ and $\lim_{r\to\infty} q_\kappa(r)=\sqrt{-2Me(\kappa)}$.

Central potentials in SUSY partner Hamiltonians

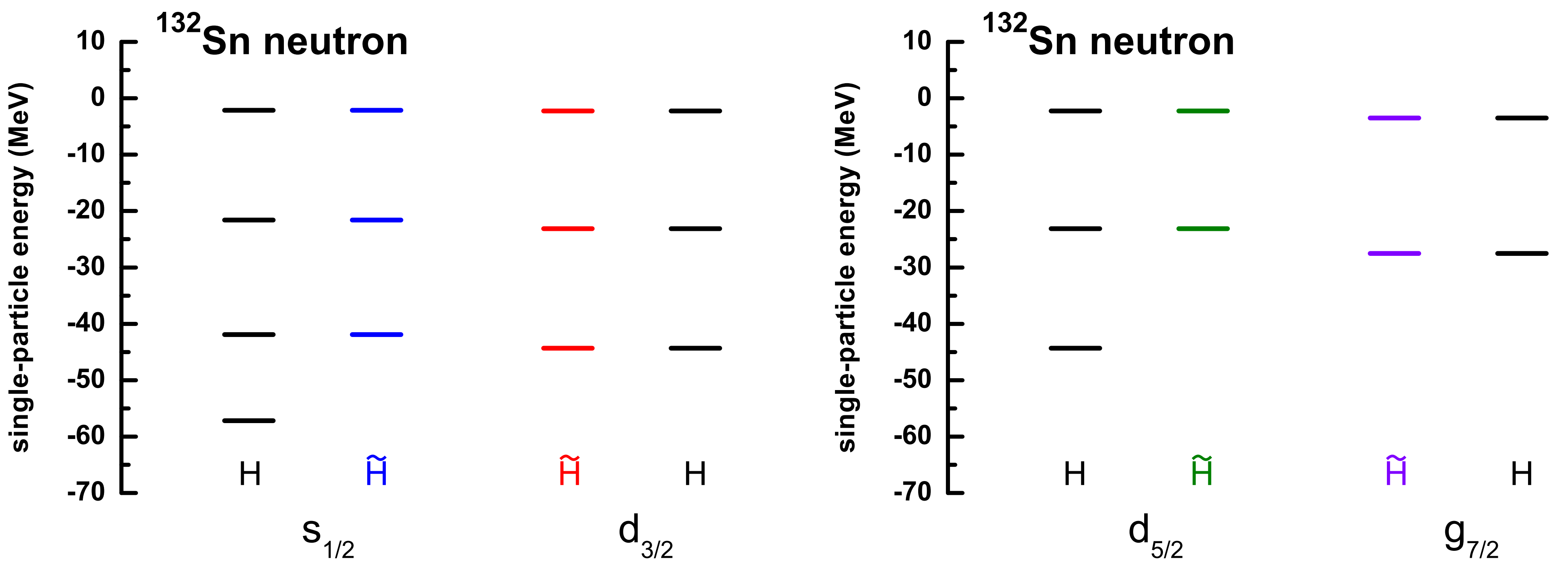


⋆ Central potentials $\tilde{V}_\kappa(r)$ in $\tilde{H} = H_2 + e(\kappa)$ for the $\tilde{p}_{1/2}$, $\tilde{p}_{3/2}$, $\tilde{f}_{5/2}$, and $\tilde{f}_{7/2}$ blocks.

HL, Shen, Zhao, Meng, *PRC* **87**, 014334 (2013)

- $\tilde{V}_\kappa(r) = V(r) + q'_\kappa(r)/M$ are regular and block-dependent.
- Asymptotic behaviors: $\lim_{r\to 0} \tilde{V}_\kappa(r) = V + \frac{2(e(\kappa)-V)}{(1-2\kappa)}$ and $\lim_{r\to\infty} \tilde{V}_\kappa(r) = 0$.

Single-particle energies of SUSY Hamiltonians

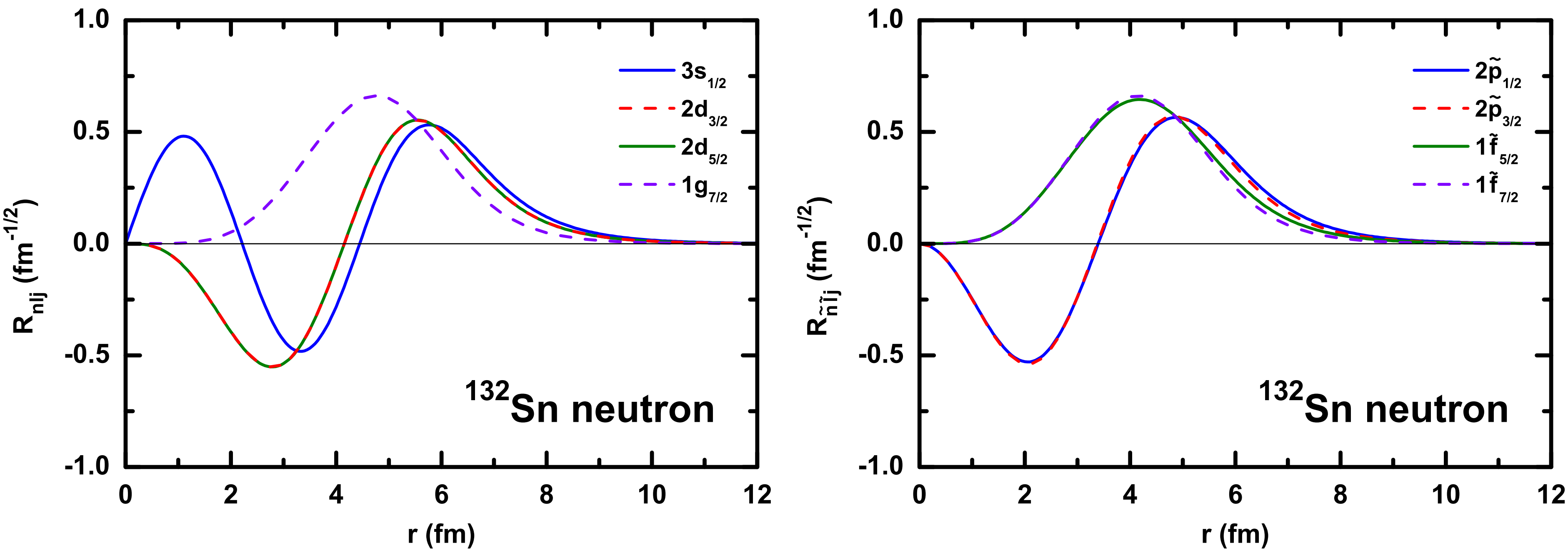


⋆ Single-particle energies of both H and $\tilde{H}$ for the $s_{1/2}$, $d_{3/2}$, $d_{5/2}$, and $g_{7/2}$ blocks.

- H and $\tilde{H}$ have **identical spectra**, expect **an additional eigenstate** with $E_1 = 0$, corresponding to the states without pseudospin partners.
- The pseudospin-orbit splittings $\Delta E_{\rm PSO}$ can be explicitly understood as the splitting appearing in $\tilde{H}$ with the SUSY representation.

Single-particle wave functions in $\tilde{H}$

- Wave function transformation: $\psi_2(n) = \frac{B^-}{\sqrt{E_S(n)}}\psi_1(n)$



⋆ Single-particle wave functions in H and $\tilde{H}$ of the $2\tilde{p}_{1/2}$, $2\tilde{p}_{3/2}$, $1\tilde{f}_{5/2}$, and $1\tilde{f}_{7/2}$ states.

- Single-particle wave functions of PS doublets are almost identical to each other.
- It is a natural result as they are quasi-degenerate.

Explicit PSS limit: symmetry conserving and breaking terms

- SUSY Hamiltonian can be divided as

$$\tilde{H} = \tilde{H}_0^{\mathrm{PSS}} + \tilde{W}^{\mathrm{PSS}}$$

where

- ⋆ symmetry conserving term

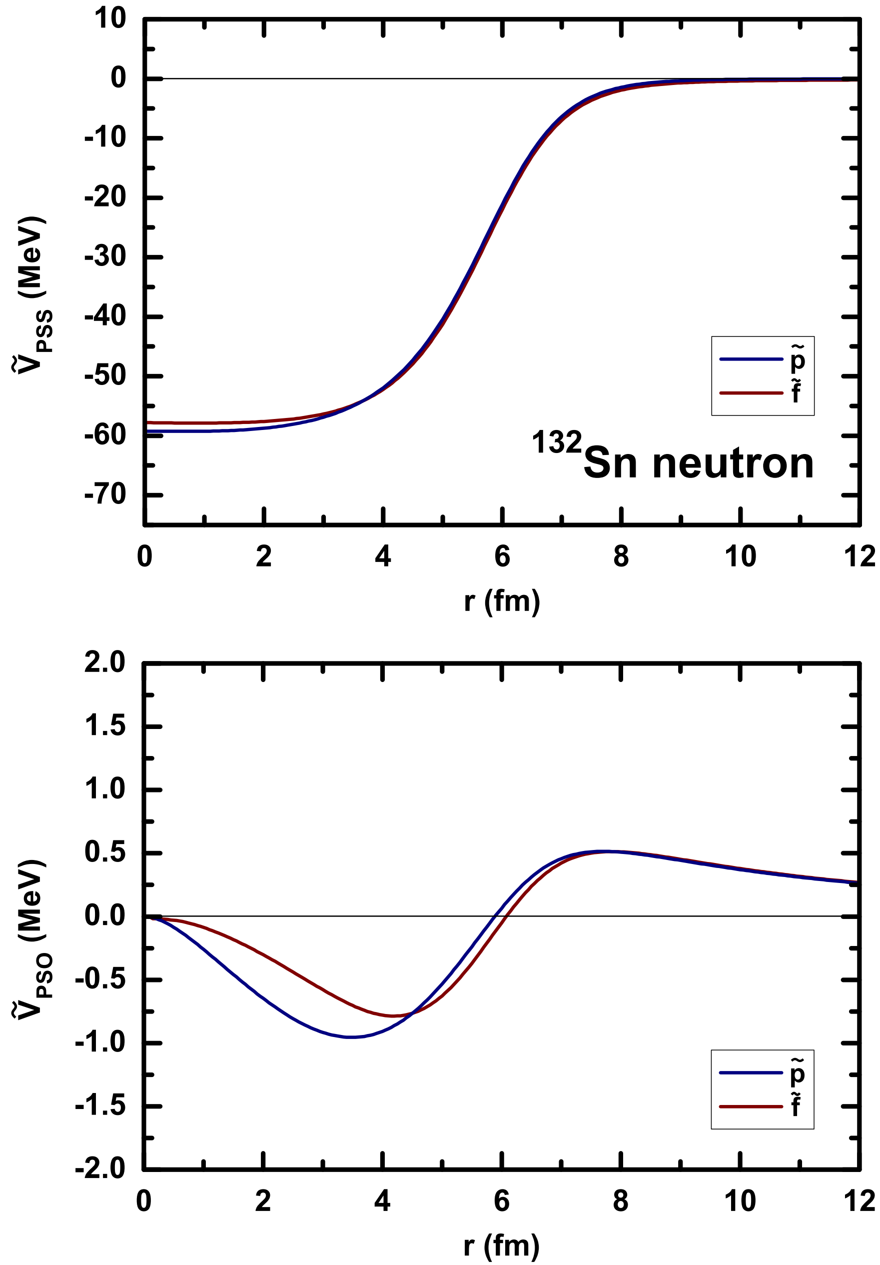
$$\tilde{H}_0^{\mathrm{PSS}} = \frac{1}{2M}\left[-\frac{d^2}{dr^2} + \frac{\kappa(\kappa-1)}{r^2}\right] + \tilde{V}_{\mathrm{PSS}}$$

- ⋆ symmetry breaking term

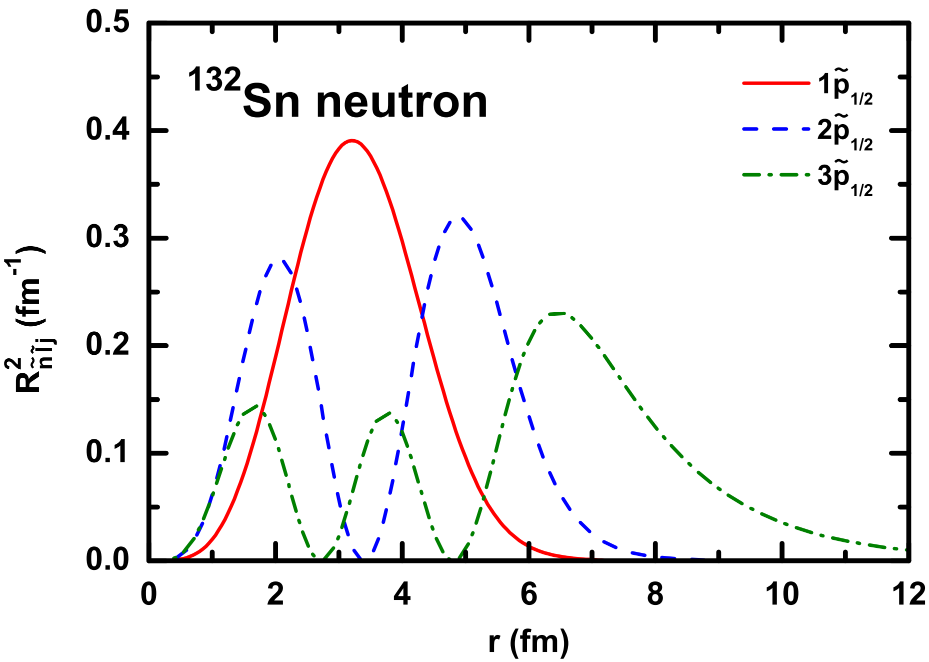
$$\tilde{W}^{\mathrm{PSS}} = \kappa \tilde{V}_{\mathrm{PSO}}$$

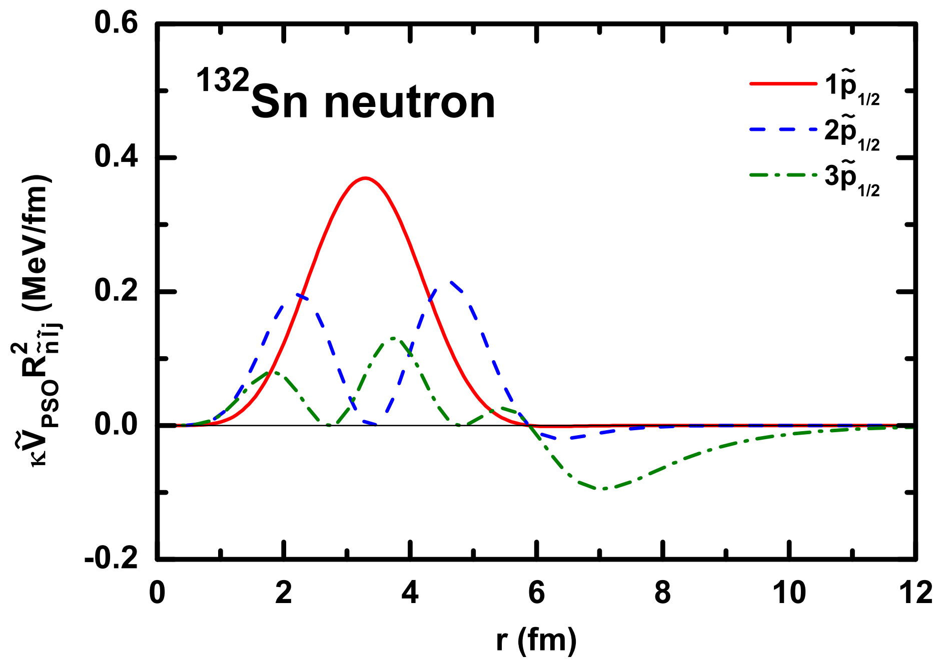
- $\tilde{V}_{\mathrm{PSO}}(r)$ with amplitudes of ~ 1 MeV are negative inside and positive outside.
- This is why ΔE_{PSO} decrease as main quantum numbers n increase.

HL, Shen, Zhao, Meng, *PRC* **87**, 014334 (2013)



General pattern of PSO splittings



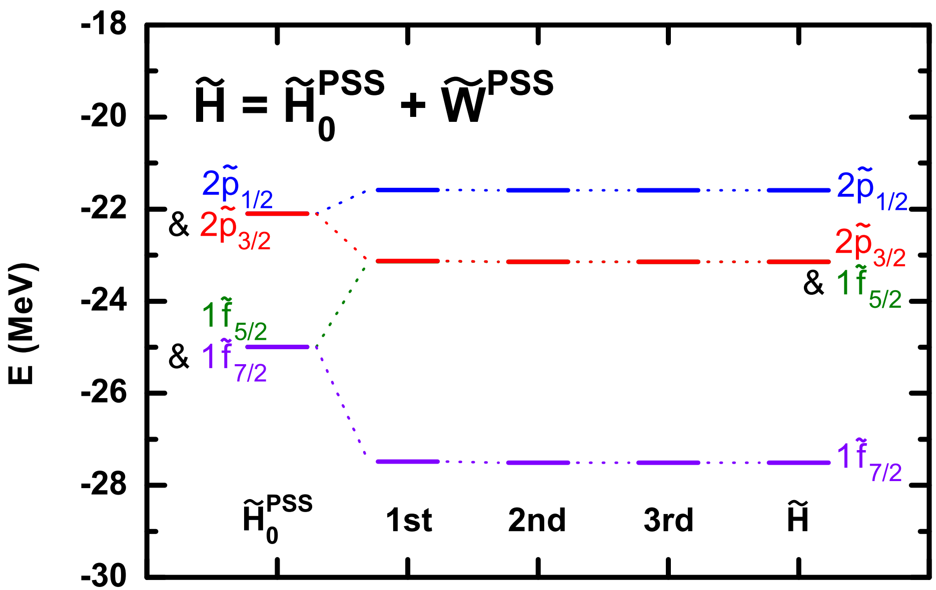


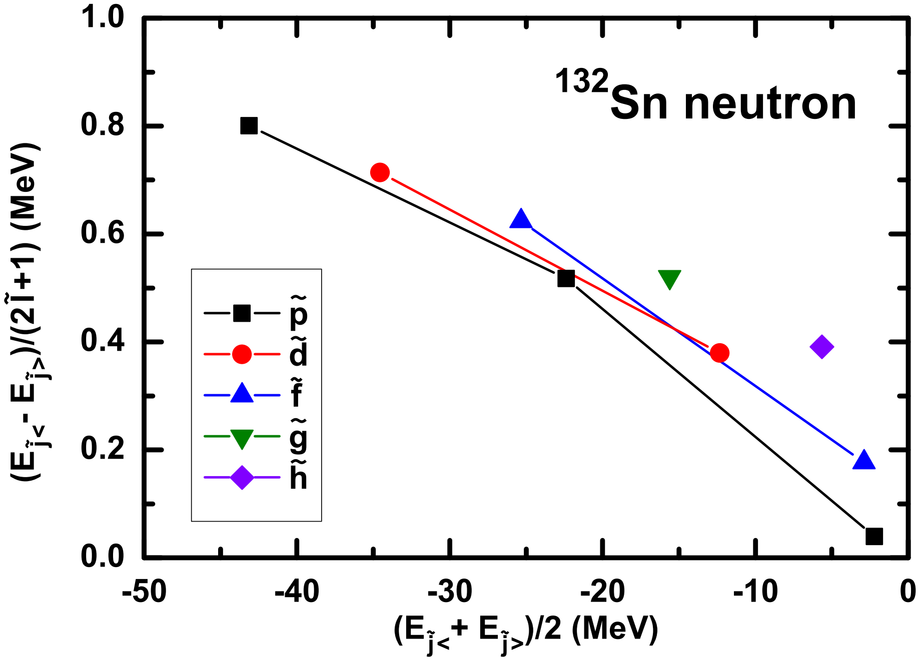
⋆ Left: $\tilde{R}^2(r)$ for the $1\tilde{p}_{1/2}$, $2\tilde{p}_{1/2}$, and $3\tilde{p}_{1/2}$ states.

Right: $\kappa\tilde{V}_{\rm PSO}(r)\tilde{R}^2(r)$ for the $1\tilde{p}_{1/2}$, $2\tilde{p}_{1/2}$, and $3\tilde{p}_{1/2}$ states.

- Main quantum numbers n increase $\Rightarrow$ wave functions $\tilde{R}(r)$ move outward $\Rightarrow E_{\rm PSO} = \int \kappa\tilde{V}_{\rm PSO}(r)\tilde{R}^2(r)dr$ decrease

Validity of perturbation theory and perturbation corrections





- Pseudospin-orbit splittings are reproduced by the 1st-order perturbation calculations.

Conclusions

The nature of PSS is perturbative. In the SUSY representation $\tilde{H}$:

- ⋆ both single-particle energies and wave functions of PS doublets are quasi-degenerate.
- ⋆ $\tilde{W}^{\mathrm{PSS}}$ can be explicitly identified.
- ⋆ shape of $\tilde{W}^{\mathrm{PSS}} \Rightarrow \Delta E_{\mathrm{PSO}}$ decrease as main quantum numbers n increase

Outline

Summary and Perspectives

Summary

★ We deem it promising to understand PSS and its breaking mechanism in a fully quantitative way by combining the similarity renormalization group (SRG) technique, SUSY quantum mechanics, and perturbation theory.

✓ **SRG**: to transform the Dirac Hamiltonian into a Schrödinger-like form yet keeping all operators Hermitian.

✓ **SUSY**: to identify the PSS conserving and breaking terms naturally; to clarify the reason why the intruder states have no pseudospin partners.

✓ **Perturbation theory**: to understand the behavior of pseudospin-orbit splitting in a quantitative way.

Perspectives

?' Schrödinger equations with spin-orbit term Shen, HL, Zhao, Zhang, Meng, in preparation

?' Dirac equations and/or Schrödinger-like equations

?' Why $\Delta E_{\rm PSO} \lesssim \Delta E_{\rm SO}$ in realistic nuclei?

?' ......

Acknowledgments

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- Shihang Shen, Peking University, China
- Nguyen Van Giai, IPN-Orsay, France
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Thank you!