Nuclear Moment of Inertia as an indicator of the phase transition in a finite system

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Introduction

- **TSD bands in Lu isotopes** (P.R.C73,034305(`06);C77,064318('08); C82,051303('10))

How far we can understand physics from the simple model.

$$H = H_{\rm rot} + H_{\rm sp},$$

$$H_{\rm rot} = \sum_{k=x,y,z} A_k (I_k - j_k)^2,$$

$$H_{\rm sp} = \frac{V}{j(j+1)}\left[\cos\gamma\left(3j_z^2 - \vec{j}^2\right) - \sqrt{3}\sin\gamma\left(j_x^2 - j_y^2\right)\right],$$

$$\mathcal{J}_k^{\rm rig} = \frac{\mathcal{J}_0}{1 + \left(\frac{5}{16\pi}\right)^{1/2}\beta_2}\left[1 - \left(\frac{5}{4\pi}\right)^{1/2}\beta_2 \cos\left(\gamma + \frac{2}{3}\pi k\right)\right]$$

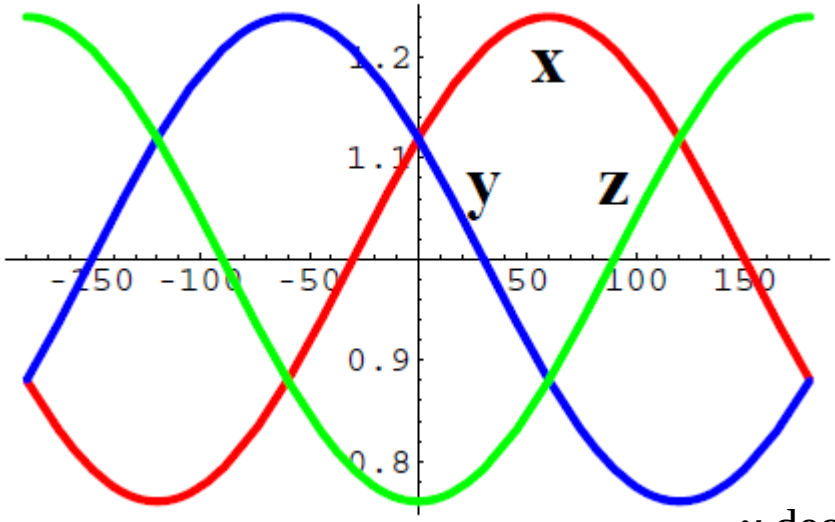
☆ $H_{\rm sp}$ and rigid MoI are derived from the same radius, and both have the same periodicity in γ with a span of $2\pi/3$.

☆Even when we change the sign of γ in hydrodynamical MoI, the periodicity is $\pi/3$, and does not agree with the periodicity of $H_{\rm sp}$ derived from Nilsson potential by Wigner-Eckart theorem.

☆Erot(I-1,1,0)-Erot(I,0,0)>0 for hydro MoI around $\gamma \sim 20^\circ$, which contradicts exp.data <0, while rigid MoI agrees with exp.data.

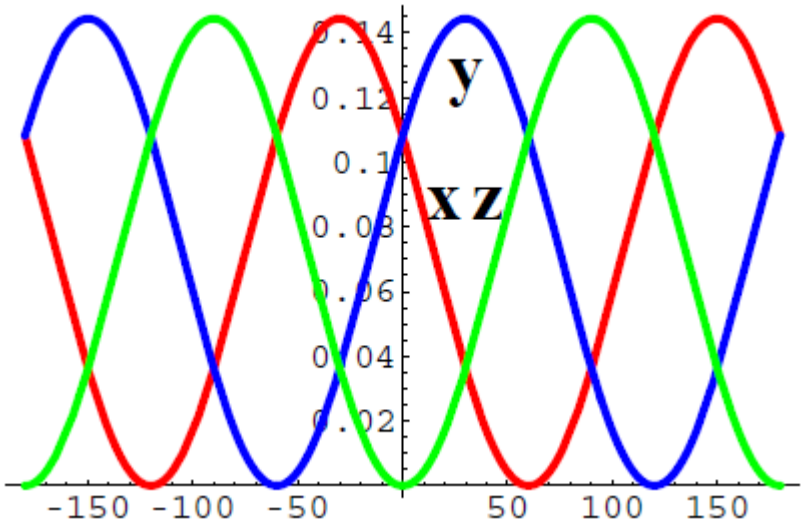
Moment of Inertia (MoI) versus γ (Lund Conv.)

Rigid- body MoI

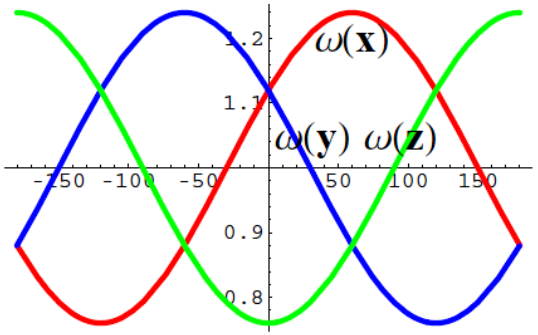


γ degree

Hydro. MoI



degree



$$R_k = R_0[1 + \sqrt{5/(4\pi)}\beta_2 \cos(\gamma + 2\pi k/3)]$$

Stability of the rotor (proved by Landav in classical mechanics)

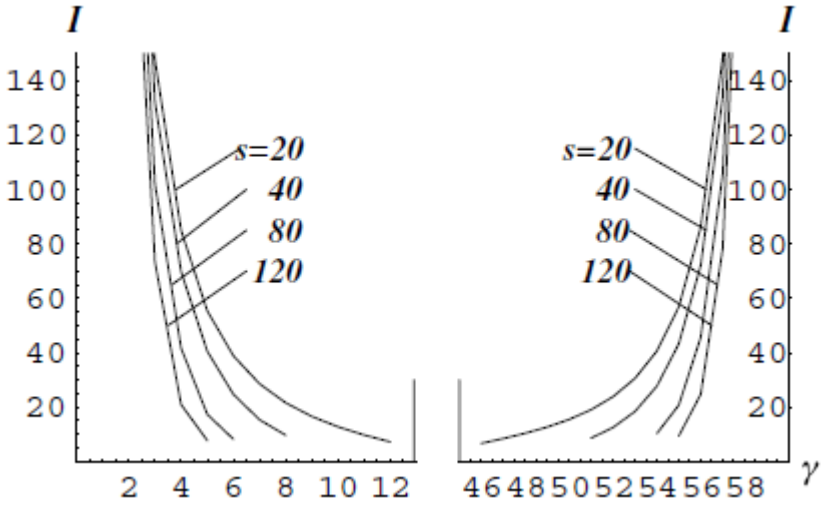
- **No wobbling about the intermediate axis, and no stable rotation around intermediate axis of the rotor.**

$$
\begin{aligned}
H_{\rm rot} &= \sum_{k=x,y,z} A_k R_k^2 \simeq A_x R(R+1) \\
&+\frac{R}{2}\begin{pmatrix} \hat{d}^\dagger & \hat{d} \end{pmatrix}\begin{pmatrix} A_y + A_z - 2A_x & A_y - A_z \\ A_y - A_z & A_y + A_z - 2A_x \end{pmatrix}\begin{pmatrix} \hat{d} \\ \hat{d}^\dagger \end{pmatrix}
\end{aligned}
$$

$$
\begin{vmatrix} A_y + A_z - 2A_x - \omega & A_y - A_z \\ -(A_y - A_z) & -(A_y + A_z - 2A_x) - \omega \end{vmatrix} = \omega^2 - 4(A_y - A_x)(A_z - A_x) = 0.
$$

- **When A_x is the maximum or the minimum, $\omega^2>0$, otherwise $\omega^2<0$ and ω is imaginary.**

Stability with potential V



P.R. C73,034305(2006)

$$s = \mathcal{J}_0 V$$

Hydro MoI with Copen conv.

Rotation about maximum MoI

Bohr symmetry

$$\left[\sqrt{\frac{2I+1}{16\pi^2}}\left[\mathcal{D}^I_{MK}(\theta_i)\phi^j_\Omega + (-1)^{I-j}\mathcal{D}^I_{M-K}(\theta_i)\phi^j_{-\Omega}\right];\right.$$

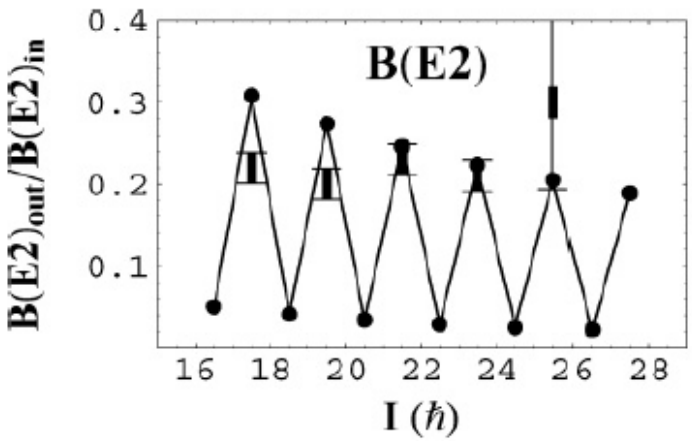
$$\left.|K-\Omega| = \text{even}, \quad \Omega > 0\right],$$

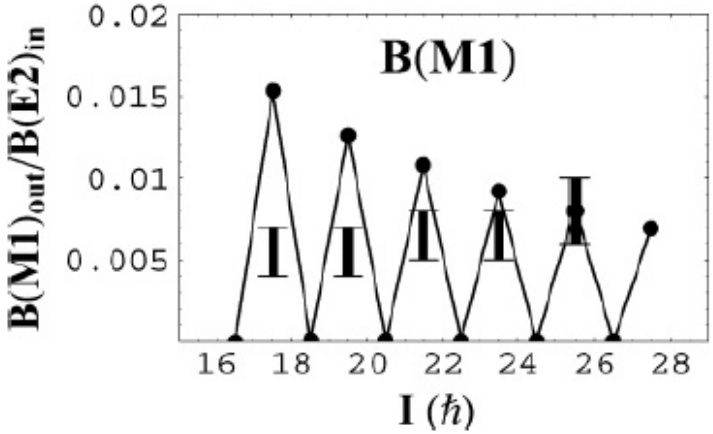
$$R = I - j + n_{\beta'}$$

$$n_{\beta'} = 0, 1, 2, \ldots, 2j-1, \text{or } 2j.$$

$$R_x = I_x - j_x = K - \Omega$$
$$= R - n_{\alpha'}$$

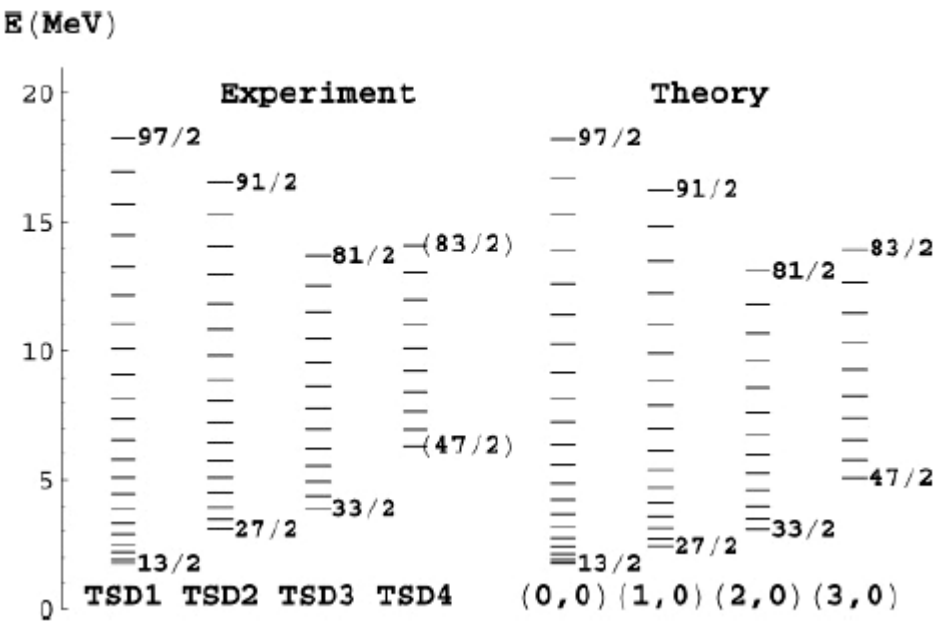
$$n_{\alpha'} = 0, 2, 4, \ldots, \text{or } 2R, \qquad \text{for} \quad R = \text{even},$$
$$n_{\alpha'} = 1, 3, 5, \ldots, \text{or } 2R-1, \qquad \text{for} \quad R = \text{odd}.$$

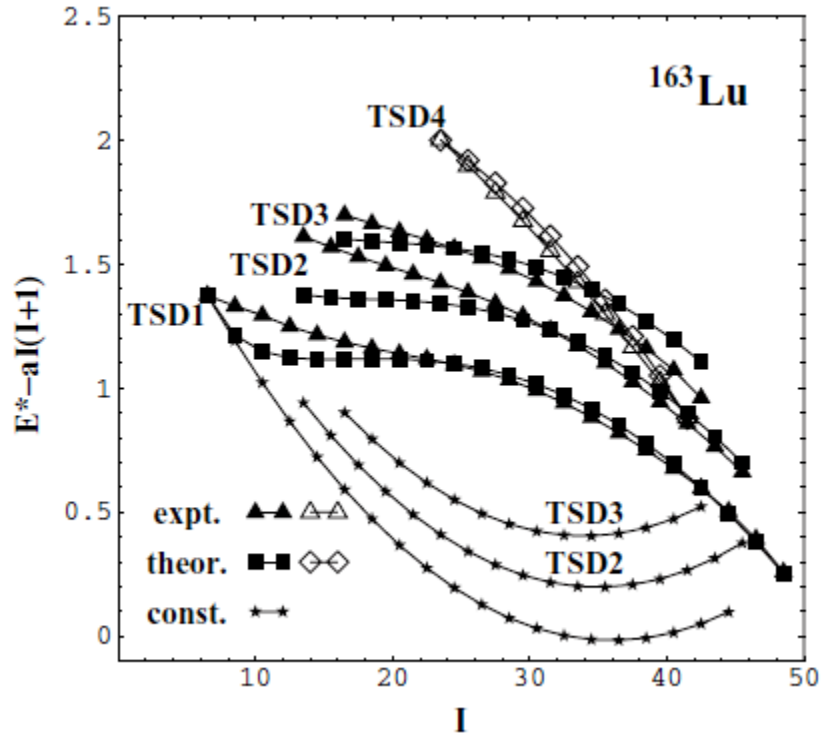




Angular-momentum dependence of MoI

^{163}Lu





$$\mathcal{J}_0 \rightarrow \mathcal{J}_0 \frac{I - c_1}{I + c_2}$$

Motivation for this work

- Discussion with Dr. Macchiavelli（Sept.　2011）

Why MoI has the angular momentum dependence? (top-on-top model: . K. Tanabe & K. S-T, Phys.Rev. C73 034305('06); C77 064318('08). K. S-T & K. T, Phys. Rev. C80 044307(R) ('10). K. S-T, K.T. & N.Yoshinaga, PTEP in press('14).)

He taught us the paper discussing gap-dependence in MoI by

D.Bengtsson and J.Helgessen: Lec. Note in Summer Sch. at OakRidge (1991)

A.Bohr and B.R. Mottelson: Nuclear Structure Vol.2 (Benjamin, MA, 1975)

- Coriolis anti-pairing effect

Mottelson-Valatin: Phys.Rev.Lett.5 (1960) 511.

Perturbation treatment of CAP: M. Sano & M. Wakai, Nucl.Phys.67 (1965)481.

K. Sugawara, Prog. Theor. Phys. 35(1966)44.

The gap dependence of MoI (Bohr-Mottelson)

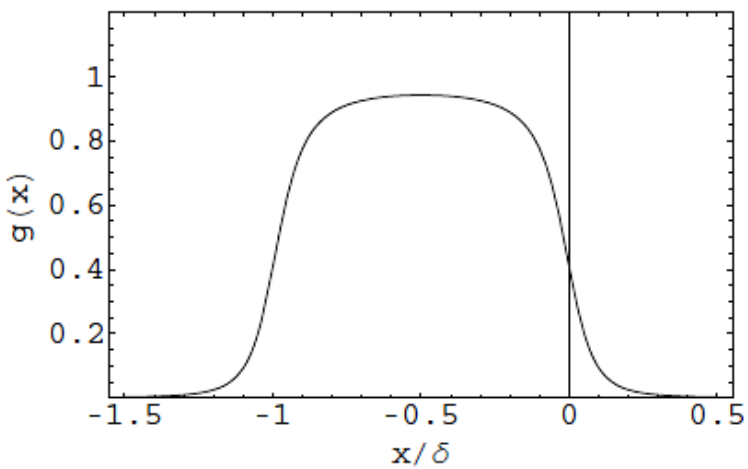
Cranking-formula : $(j_x)^2_{\alpha\beta}$ with a common exc. $\varepsilon_\beta = \varepsilon_\alpha + \delta$

$$\mathcal{J}_x = \sum_{\alpha,\beta\,(\varepsilon_\alpha<\varepsilon_\beta)} \frac{(j_x)^2_{\alpha\beta}}{E_\alpha + E_\beta}\left(1 - \frac{(\varepsilon_\alpha-\lambda)(\varepsilon_\beta-\lambda)+\Delta^2}{E_\alpha E_\beta}\right) \cong 2\sum_{\alpha,\beta\,(\varepsilon_\alpha<\varepsilon_\beta)} \frac{(j_x)^2_{\alpha\beta}}{\delta}\langle g(x)\rangle_{\rm av}$$

$$\mathcal{J}_x^{\rm rig} = 2\sum_{\alpha,\beta(\varepsilon_\alpha<\varepsilon_\beta)} \frac{(j_x)^2_{\alpha\beta}}{\varepsilon_\beta-\varepsilon_\alpha} \cong 2\sum_{\alpha(\varepsilon_\beta=\varepsilon_\alpha+\delta)} \frac{(j_x)^2_{\alpha\beta}}{\delta} \sim \boxed{\mathcal{J}_x^{\rm rig} < g(x) >_{\rm av}}$$

$$g(x) = \frac{\delta}{2(\sqrt{x^2+\Delta^2}+\sqrt{(x+\delta)^2+\Delta^2})}\left(1 - \frac{x(x+\delta)+\Delta^2}{\sqrt{x^2+\Delta^2}\sqrt{(x+\delta)^2+\Delta^2}}\right)$$

$x = \varepsilon_\alpha - \lambda$



$g(x)$ is symmetric function at $x = -\delta/2$

$\delta = 1$ and $\Delta = 0.1$

How to estimate $<g(x)>_{av}$

$$\xi = 2\Delta/\delta$$

Bohr-Mottelson:

$$< g(x) >_{\rm av} = \frac{\int_{-\infty}^{\infty} \rho g(x) dx}{\int_{-\delta/2}^{\delta/2} \rho dx} = 1 - \frac{\xi^2}{\sqrt{1+\xi^2}} \ln\left(\frac{1+\sqrt{1+\xi^2}}{\xi}\right)$$

Bengtsson-Helgessen: Triangle with height of g(x= -δ/2) and base of 2δ

$$< g(x) >_{\rm av} = \frac{1}{(1+\xi^2)^{3/2}}$$

This work: g(x)+g(x-δ) is symmetric about x=0

$$\int_{-3\delta/2}^{\delta/2} g(x)dx = \int_{-\delta/2}^{\delta/2} (g(x) + g(x-\delta))dx = 2\int_{-\delta/2}^{0} (g(x) + g(x-\delta))dx$$

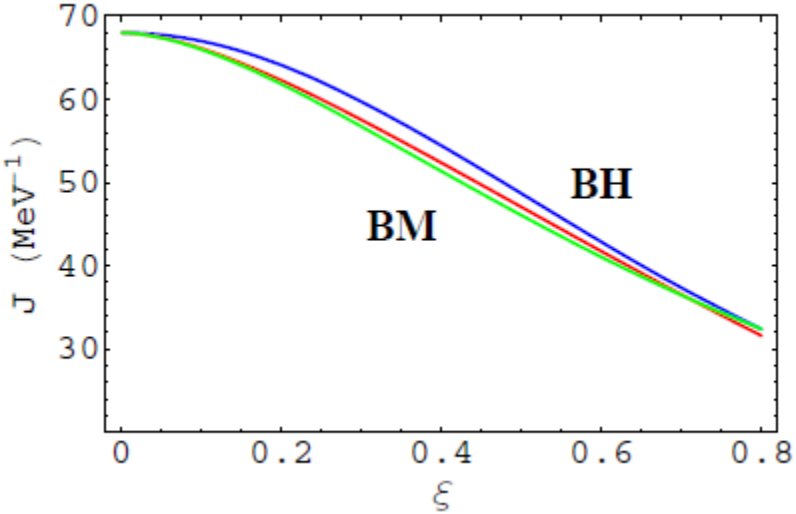
$$\begin{aligned} \langle g(x)\rangle_{\rm av} &= \frac{2}{\rho\delta}\int_{-\delta/2}^{0} \rho\big[g(x) + g(x-\delta)\big]dx \\ &= 1 - \xi^2 \ln\left(\frac{1+\sqrt{1+\xi^2}}{\xi}\right) + \frac{19}{90}\xi^2 \end{aligned}$$

we expand $g(x) + g(x-\delta)$ $\quad x = \varepsilon_\alpha - \lambda$
up to the second order

$$x/\delta, \ \sqrt{x^2+\Delta^2}/\delta \ \text{ and } \ \Delta/\delta$$

Three $<g(x)>_{av}$ go to 1 in the limit of ξ=0.

Comparison of three methods for moment of inertia



$\beta \sim 0.3$, $\delta \sim \hbar(\omega_y - \omega_z) \sim 2.3$ MeV

the average of δ' for

$h_{11/2}$, $g_{7/2}$, $2d_{5/2}$, $2d_{3/2}$ and $i_{13/2}$ is 2.1 MeV

1.4 ~ 1.5 MeV for $i_{13/2}$ level around $\beta \sim 0.3$

δ=2.0 MeV

$\mathcal{J}_x^{\rm rig}$=68 MeV^{-1}

Three MoI go to the Rigid value at the limit of ξ=0

Coriolis Anti-Pairing effect and CHFB equation in even nuclei

$$H = H_0 - H_\Omega,\ H_0 = \sum_\alpha (\varepsilon_\alpha - \lambda) c^\dagger_\alpha c_\alpha - \frac{G}{4}\sum_{\alpha,\beta} c^\dagger_{\tilde{\alpha}} c^\dagger_\alpha c_\beta c_{\tilde{\beta}},$$

$$H_\Omega = \Omega_x \hat{I}_x,\ \text{with}\ \hat{I}_x = \sum_{\alpha,\beta} (j_x)_{\alpha\beta} c^\dagger_\alpha c_\beta,$$

$$< \hat{I}_x >= I = \mathcal{J}_x \Omega_x.$$

$$\alpha_i = \sum (A_{i\alpha} c^\dagger_\alpha + B_{i\alpha} c_\alpha)$$

CHFB equation

$$\Lambda_i A_{i\alpha} = \sum_\beta ((\varepsilon_\alpha - \lambda)\delta_{\alpha\beta} - \Omega_x (j_x)_{\alpha\beta}) A_{i\beta} + \delta_{\alpha\tilde{\beta}} \Delta B_{i\beta}$$

$$-\Lambda_i B_{i\alpha} = \sum ((\varepsilon_\alpha - \lambda)\delta_{\alpha\beta} - \Omega_x (j_x)_{\alpha\beta}) B_{i\beta} + \delta_{\alpha\tilde{\beta}} \Delta A_{i\beta}$$

CHFB in perturbation; M.Sano &M.Wakai,N.P.67,481(1965),K. Sugawara, PTP35,44(1966)

2nd order perturbation for H_Ω based on BCS solution

$$\Delta = \frac{G}{4}\sum_\alpha \frac{\Delta}{E_\alpha}\Big[1 - \Omega_x^2 \sum_\beta \frac{(j_x)^2_{\alpha\beta}}{(E_\alpha + E_\beta)}\Big(\frac{E_\alpha E_\beta - (\varepsilon_\alpha - \lambda)(\varepsilon_\beta - \lambda) - \Delta^2}{E_\alpha E_\beta (E_\alpha + E_\beta)} + \frac{(\varepsilon_\alpha - \lambda)(\varepsilon_\alpha - \varepsilon_\beta)}{E_\alpha^2 E_\beta}\Big)\Big].$$

Moment of inertia has the same form as cranking formula within the second order perturbation, but Δ and λ are different from BCS.

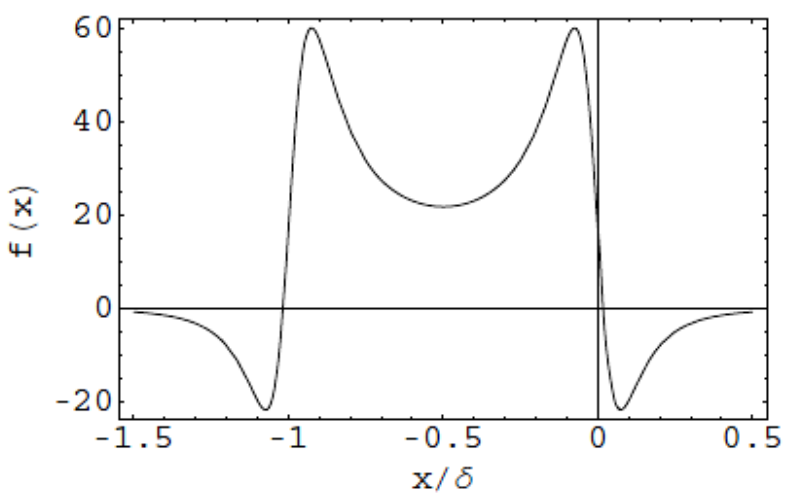
The same technique for gap as applied to MoI

The second term in Gap equation

$$\Omega_x^2 \sum_{\alpha\,(\varepsilon_\beta=\varepsilon_\alpha+\delta)} (j_x)^2_{\alpha\beta} \frac{1}{\delta} \int_{-3\delta/2}^{\delta/2} f(x)dx$$

$$f(x) = \frac{1}{(E_x + E_{x+\delta})E_x E_{x+\delta}} \left[1 - \frac{x(x+\delta)+\Delta^2}{E_x E_{x+\delta}} + \frac{\delta^2(\Delta^2 - x(x+\delta))}{(E_x E_{x+\delta})^2}\right]$$

$$\int_{-3\delta/2}^{\delta/2} f(x)dx = \int_{-\delta/2}^{\delta/2} (f(x) + f(x-\delta))dx = \int_{-\delta/2}^{\delta/2} F(x)dx$$



f(x) is symmetric with respect to x = - δ/2, while *f*(x)+*f*(x-δ) is symmetric with respect to x=0. We expand *F*(x) around x=0 up to the second order of

$$\boxed{x/\delta,\ \sqrt{x^2+\Delta^2}/\delta \text{ and } \Delta/\delta}$$

$$\delta = 1 \text{ and } \Delta = 0.1$$

$$\int_{\frac{-3\delta}{2}}^{\frac{\delta}{2}} f(x)dx = 2\int_{-\delta/2}^{0} F(x)dx \cong 2\int_{-\delta/2}^{0} G(x)dx + \frac{\delta}{5}\left[F(-\frac{\delta}{2}) - G(-\frac{\delta}{2})\right]$$
$$= \frac{1}{\delta^2}\left[16(1-\xi^2)\ln\frac{1+\sqrt{1+\xi^2}}{\xi} + \frac{319}{27}\xi^2 - \frac{371}{45}\right] \equiv \frac{\bar{F}}{\delta^2}.$$

The first term in the gap equation (Bohr-Mottelson)

$$\sum_\alpha \frac{1}{E_\alpha} \cong \int_{-S/2}^{S/2} \frac{2\rho\, dx}{\sqrt{x^2+\Delta^2}} = 4\rho \sinh^{-1}\frac{S}{2\Delta}, \quad \text{with} \quad x = \varepsilon_\alpha - \lambda.$$

S is the cut-off energy

The initial value Δ_0 at I=I_0 (without rotation).

$$\xi_0 \equiv 2\Delta_0/\delta = (2S/\delta)\exp[-1/(G\rho)]$$

Constraint for I is replaced by I-I_0

$$\boxed{I - I_0 = \left[8\delta^2\rho\mathcal{J}_x^{\text{rig}}\frac{\ln(\xi_0/\xi) < g(x) >_{\text{av}}^2}{\bar{F}}\right]^{1/2}}$$

$$< \hat{I}_x > = \mathcal{J}_x\Omega_x = I - I_0$$

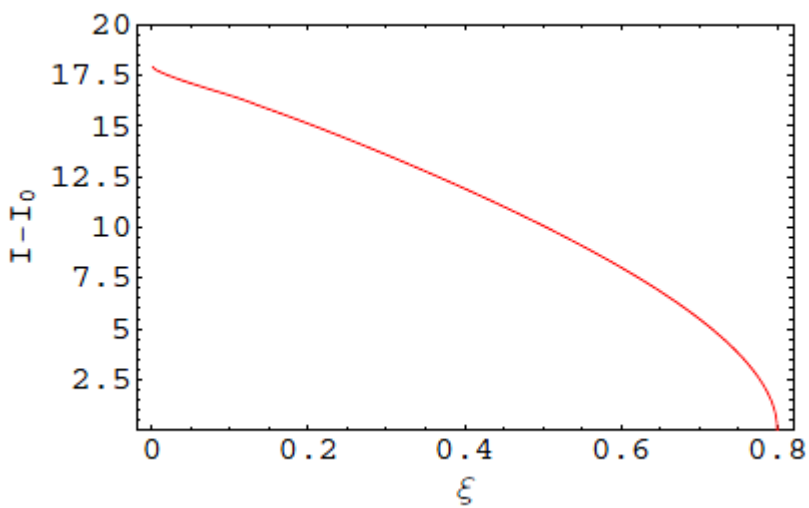
In the limit of ξ=0, denominator diverges as ln(2/ξ) and numerator diverges as ln(ξ_0/ξ) , and I_c - I_0 is finite (phase transition is sharp.)

The relation between I-I_0 and Δ

$$\rho = 2.5\mathrm{MeV}^{-1},\ \mathcal{J}_x^{\mathrm{rig}} = 68\mathrm{MeV}^{-1},\ \delta = 2.0\mathrm{MeV} \quad \xi_0 = 0.8$$

$$\xi = 2\Delta/\delta$$

$$I - I_0 = \left[8\delta^2\rho\mathcal{J}_x^{\mathrm{rig}}\frac{\ln(\xi_0/\xi) < g(x) >_{\mathrm{av}}^2}{\bar{F}}\right]^{1/2}$$



Moment of inertia with our formula

Around I-I_0 ~18, Δ=0

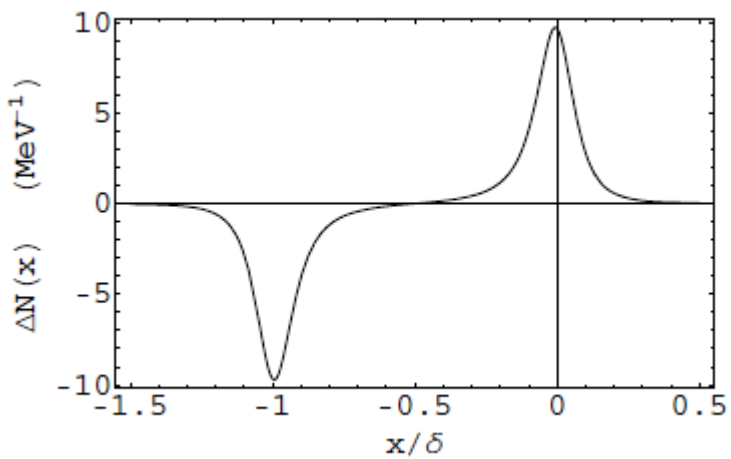
$$\Delta \geq 1/\rho \equiv d$$

Sharp phase transition

Number constraint
up to the second order perturbation

$$N = \sum_\alpha v_\alpha^2 + \frac{\Omega_x^2}{2} \sum_{\alpha,\beta} \frac{(j_x)^2_{\alpha\beta}}{E_\alpha + E_\beta} \left[\frac{\varepsilon_\alpha - \lambda}{E_\alpha (E_\alpha + E_\beta)} \left(1 - \frac{(\varepsilon_\alpha - \lambda)(\varepsilon_\beta - \lambda) + \Delta^2}{E_\alpha E_\beta} \right) - \frac{\Delta^2 (\varepsilon_\alpha - \varepsilon_\beta)}{E_\alpha^3 E_\beta} \right].$$

$$= \sum_\alpha v_\alpha^2 + \frac{\Omega_x^2}{2} \sum_{\alpha<\beta(\varepsilon_\beta = \varepsilon_\alpha + \delta)} \frac{(j_x)^2_{\alpha\beta}}{\delta} \Delta N(x).$$



$$\delta = 1 \text{ and } \Delta = 0.1$$

$$x = \varepsilon_\alpha - \lambda.$$

Odd function of x, and the rotational effect on λ is small.

The case when gap is much less than single-particle level distance $\Delta \ll d = 1/\rho$ (direct summation)

As sum is replaced by integral,

the contribution from $1/\sqrt{x^2+\Delta^2}$ show divergence at $x \to 0$ for $\Delta \to 0$.

Assume ***n***-levels in the interval **$0 \leq x \leq \delta/2$** (picket fence approx.), expand up to $(2\Delta/d)^2$

$$\sum_{x>0}^{\delta/2} \frac{1}{\sqrt{x^2+\Delta^2}} \sim \rho(\Gamma_n - \xi^2 Z_n), \qquad \Gamma_n \sim \gamma + \ln 4n, \qquad Z_n \cong (\rho\delta)^2 \frac{7}{8}\zeta(3)$$

$\gamma(= 0.577\cdots)$ is Euler constant and $\zeta(3)(= 1.202\cdots)$

Riemann Zeta function

$$\ln((1+\sqrt{1+\xi^2})/\xi) \to \Gamma_n - \xi^2 Z_n$$

$$< g(x) >_{\text{av}} \to 1 - \xi^2\left(\Gamma_n - \xi^2 Z_n - \frac{19}{90}\right)$$

Both $<g(x)>_{av}$ and F_n go to finite value in the limit of $\xi=0$

$$\bar{F} \to \bar{F}_n = 16(1-\xi^2)(\Gamma_n - \xi^2 Z_n) + \frac{319}{27}\xi^2 - \frac{371}{45}$$

Difference between integral and direct summation

$$I - I_0 = \left[8\delta^2 \rho \mathcal{J}_x^{\rm rig} \frac{\ln(\xi_0/\xi)\langle g(x)\rangle_{\rm av}^2}{\bar{F}}\right]^{1/2} \qquad \bar{F} = \left[16(1-\xi^2)\ln\left(\frac{1+\sqrt{1+\xi^2}}{\xi}\right) + \frac{319}{27}\xi^2 - \frac{371}{45}\right].$$

Integral

$\xi \to 0$, numerator diverges as ln(2/ξ), denominator diverges as ln(ξ_0/ξ), and I_c-I_0 is finite. Sharp phase transition.

$$I - I_0 = \left[8\delta^2 \rho \mathcal{J}_x^{\rm rig} \frac{\ln(\xi_0/\xi)\langle g(x)\rangle_{\rm av}^2}{\bar{F}_n}\right]^{1/2}. \qquad \bar{F}_n = 16(1-\xi^2)(\Gamma_n - \xi^2 Z_n) + \frac{319}{27}\xi^2 - \frac{371}{45}$$

Direct summation

$\xi \to 0$, numerator diverges as ln(2/ξ), denominator is finite, I_c-I_0 becomes infinite. Slow phase transition originates from the finiteness of nucleus.

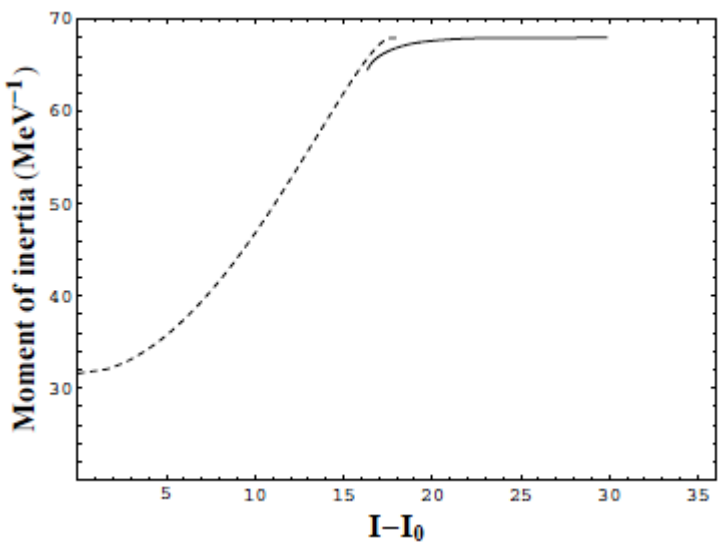
(Projection of number or angular momentum prevents the rapid decrease of Δ.: J.L.Egido, P.Ring, S.Iwasaki and H.J.Mang,P.L.154B,1(1985); Y.Sun and J.L.Egido,P.R.C50,1839(1994))

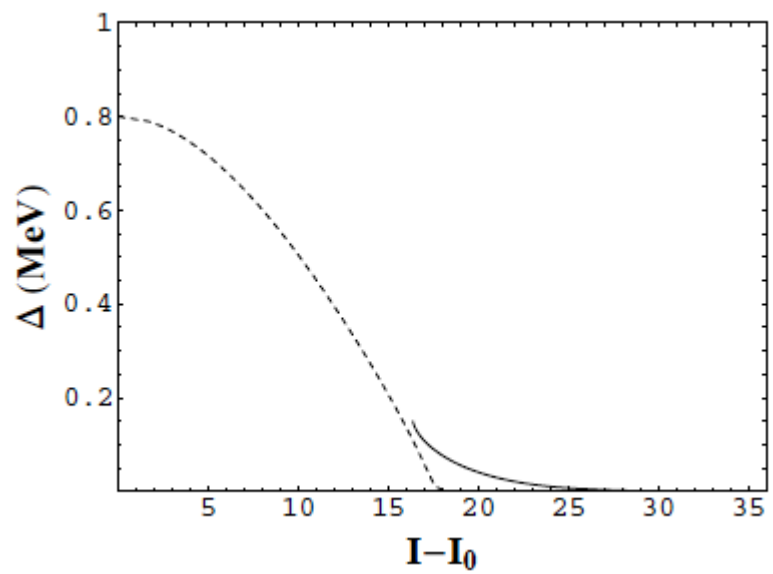
Both Δ ≥d and <<d in even nucleus

$$\rho = 2.5\text{MeV}^{-1},\ \mathcal{J}_x^{\text{rig}} = 68\text{MeV}^{-1} \text{ and } \delta = 2.0\text{MeV} \text{ and } \xi_0{=}0.8$$

$$I - I_0 = \left[8\delta^2\rho\mathcal{J}_x^{\text{rig}}\frac{(\ln\xi_0 - \ln\xi) < g(x) >_{\text{av}}^2}{\bar{F}}\right]^{1/2} \qquad I - I_0 = \left[8\delta^2\rho\mathcal{J}_x^{\text{rig}}\frac{(\ln\xi_0 - \ln\xi) < g(x) >_{\text{av}}^2}{\bar{F}_n}\right]^{1/2}$$

$$< g(x) >_{\text{av}} = 1 - \xi^2 \ln\frac{1+\sqrt{1+\xi^2}}{\xi} + \frac{19}{90}\xi^2, \qquad < g(x) >_{\text{av}} = 1 - \xi^2(\Gamma_n - \xi^2 Z_n - \frac{19}{90})$$





The MoI in the direct summation shows upward convexity before it reaches to rigid value. The gap shows concave before it reaches to 0.

Around I >30, Δ~0.

Odd nucleus (the last nucleon in ℓ state)

Quasi-vacuum $|>> \equiv \alpha_\ell^\dagger|>$

CHFB equation is the same as even case, if we adopt $-\Lambda_\ell$ instead of Λ_ℓ or $A_{\ell\alpha}$ and $B_{\ell\alpha}$ are interchanged.

CAP effect and blocking effect is taken into account in 2nd-order perturbation theory. (K. Sugawara, PTP 35,44(1966))

In order to apply the same technique as used in even nucleus, we rewrite equations for gap, moments of inertia, and number constraints.

In order to apply the same technique as used in even nucleus, we rewrite equations for gap, moments of inertia, and number constraint.

$$\frac{4}{G} = -\frac{2}{E_\ell} + \sum_\alpha \frac{1}{E_\alpha}\Bigg[1 - \Omega_x^2 \sum_\beta \frac{(j_x)^2_{\alpha\beta}}{(E_\alpha + E_\beta)}$$
$$\times\left(\frac{E_\alpha E_\beta - (\varepsilon_\alpha - \lambda)(\varepsilon_\beta - \lambda) - \Delta^2}{E_\alpha E_\beta (E_\alpha + E_\beta)} + \frac{(\varepsilon_\alpha - \lambda)(\varepsilon_\alpha - \varepsilon_\beta)}{E_\alpha^2 E_\beta}\right)\Bigg]$$
$$-2\Omega_x^2 \sum_{\alpha \neq \ell, \tilde{\ell}} \frac{(j_x)^2_{\alpha\ell}}{E_\ell (E_\ell^2 - E_\alpha^2)}\left[3 - \frac{(\varepsilon_\alpha - \lambda)(\varepsilon_\ell - \lambda) + \Delta^2}{E_\ell^2}\right].$$

$$\mathcal{J}_x = \sum_{\varepsilon_\alpha < \varepsilon_\beta} \frac{(j_x)^2_{\alpha\beta}}{E_\alpha + E_\beta}\left(1 - \frac{(\varepsilon_\alpha - \lambda)(\varepsilon_\beta - \lambda) + \Delta^2}{E_\alpha E_\beta}\right)$$
$$+2 \sum_{\alpha \neq \ell, \tilde{\ell}} \frac{(j_x)^2_{\alpha\ell}}{E_\alpha^2 - E_\ell^2}\left(E_\ell + \frac{(\varepsilon_\alpha - \lambda)(\varepsilon_\ell - \lambda) + \Delta^2}{E_\ell}\right).$$

$$N - 1 = \sum_{\alpha \neq \ell, \tilde{\ell}} v_\alpha^2 + \frac{\Omega_x^2}{2} \sum_{\alpha, \beta \neq \ell, \tilde{\ell}} \frac{(j_x)^2_{\alpha\beta}}{E_\alpha + E_\beta}\Bigg[\frac{\varepsilon_\alpha - \lambda}{E_\alpha (E_\alpha + E_\beta)}$$
$$\times\left(1 - \frac{(\varepsilon_\alpha - \lambda)(\varepsilon_\beta - \lambda) + \Delta^2}{E_\alpha E_\beta}\right) - \frac{\Delta^2 (\varepsilon_\alpha - \varepsilon_\beta)}{E_\alpha^3 E_\beta}\Bigg]$$
$$+\frac{\Omega_x^2}{2} \sum_{\alpha \neq \ell, \tilde{\ell}} \frac{(j_x)^2_{\alpha\ell}}{E_\alpha E_\ell} \frac{\Delta^2 E_\ell (\varepsilon_\alpha - \varepsilon_\ell)}{(E_\alpha^2 - E_\ell^2)^2} \frac{(5E_\alpha^2 - E_\ell^2)}{E_\alpha^2}.$$

Moment of inertia

$$\mathcal{J}_x = \mathcal{J}_x^{\rm rig}\left[1 - \xi^2 \ln\frac{1+\sqrt{1+\xi^2}}{\xi} - \frac{19}{90}\xi^2\right] + X_1\sqrt{\xi^2+\eta^2} + \frac{X_2}{\sqrt{\xi^2+\eta^2}},$$

$$\eta = 2(\varepsilon_\ell - \lambda)/\delta$$

$$\xi = 2\Delta/\delta$$

$$X_1 = \frac{1}{2\delta}\left(\frac{j_>^2}{1+\eta} + \frac{j_<^2}{1-\eta}\right),\ X_2 = \frac{\eta}{2\delta}\left(\frac{j_>^2}{1+\eta} - \frac{j_<^2}{1-\eta}\right).$$

$j_>^2/4 = (j_x)^2_{\alpha\ell}$ for $\varepsilon_\alpha > \varepsilon_\ell$

$j_<^2/4$ $\varepsilon_\alpha < \varepsilon_\ell$

At the limit of ξ=0,

$$\mathcal{J}_x^{\rm rig}(odd) - \mathcal{J}_x^{\rm rig}(even) = \frac{1}{2\delta}(j_>^2 - j_<^2).$$

$$j_>^2 - j_<^2 = 2$$

$$\mathcal{J}_x^{\rm rig}(odd) - \mathcal{J}_x^{\rm rig}(even) \sim 0.5{\rm MeV}^{-1}$$

How to estimate $\boxed{j_>^2}$ $\boxed{j_<^2}$

Bohr-Mottelson text book

$$< Nn_z'\Lambda \pm 1|\ell_x \pm i\ell_y|Nn_z\Lambda > \;=\; \frac{\omega_z+\omega_\perp}{2\sqrt{\omega_z\omega_\perp}}\Big(\sqrt{(n_z+1)(N-n_z\mp\Lambda)}\,\delta_{n_z' n_z+1} + \sqrt{n_z(N-n_z\pm\Lambda+2)}\,\delta_{n_z' n_z-1}\Big)$$

If	[521]3/2 in odd particle,	$j_>^2$ = 12 to [512]5/2
		$j_<^2$ = 12 to [510]1/2
	[532]3/2	$j_>^2$ = 18 to [523]5/2
		$j_<^2$ = 16 to [541]1/2
	[512]5/2	$j_>^2$ = 8 to [503]7/2
		$j_<^2$ = 4 to [523]3/2

Blocking effect (Ω_x=0)

$$N-1=\sum_{\alpha\neq\ell,\tilde{\ell}} v_\alpha^2+\frac{\Omega_x^2}{2}\sum_{\alpha,\beta\neq\ell,\tilde{\ell}}\frac{(j_x)^2_{\alpha\beta}}{E_\alpha+E_\beta}\left[\frac{\varepsilon_\alpha-\lambda}{E_\alpha(E_\alpha+E_\beta)}\right.$$
$$\left.\times\left(1-\frac{(\varepsilon_\alpha-\lambda)(\varepsilon_\beta-\lambda)+\Delta^2}{E_\alpha E_\beta}\right)-\frac{\Delta^2(\varepsilon_\alpha-\varepsilon_\beta)}{E_\alpha^3 E_\beta}\right]+\frac{\Omega_x^2}{2}\sum_{\alpha\neq\ell,\tilde{\ell}}\frac{(j_x)^2_{\alpha\ell}}{E_\alpha E_\ell}\frac{\Delta^2 E_\ell(\varepsilon_\alpha-\varepsilon_\ell)}{(E_\alpha^2-E_\ell^2)^2}\frac{(5E_\alpha^2-E_\ell^2)}{E_\alpha^2}.$$

$(\Omega_x=0)$

$$\boxed{\Delta=\frac{G}{4}\sum_{\alpha\neq\ell,\tilde{\ell}}\frac{\Delta}{E_\alpha},\qquad N-1=\sum_{\alpha\neq\ell,\tilde{\ell}} v_\alpha^2.}$$

$$\boxed{\Delta^e=\frac{G}{4}\sum_{\alpha}\frac{\Delta^e}{E_\alpha^e},\qquad N-1=\sum_{\alpha}(v_\alpha^e)^2,}$$

$$\Delta=\Delta^e+\delta\Delta,\qquad \lambda=\lambda^e+\delta\lambda.$$

up to the first order in $\delta\Delta$ and $\delta\lambda$

$$\delta\Delta=-\frac{A\frac{(\Delta^e)^2}{E_\ell^e}-2B(v_\ell^e)^2}{\Delta^e\left((A\Delta^e)^2+B^2\right)},\qquad \delta\lambda=\frac{B\frac{1}{E_\ell^e}+2A(v_\ell^e)^2}{(A\Delta^e)^2+B^2}$$

$$A=\sum_{\alpha>0\neq\ell}\frac{1}{(E_\alpha^e)^3},$$

$$B=\sum_{\alpha>0\neq\ell}\frac{\varepsilon_\alpha-\lambda^e}{(E_\alpha^e)^3}.$$

$$\boxed{\varepsilon_\ell\sim\lambda^e}$$

$$\boxed{\delta\Delta=-\frac{1}{A(\Delta^e)^2}\quad\text{and}\quad\delta\lambda=-\delta\Delta.}$$

odd gap is smaller than even gap because of blocking effect.

Nilsson-Prior:Mat.Fys.Dan.32(1961)

Number constraint ($\Omega_x \neq 0$)

$$N = \frac{\varepsilon_\ell - \lambda}{E_\ell} + \sum_\alpha v_\alpha^2 + \frac{\Omega_x^2}{2} \sum_{\alpha,\beta} \frac{(j_x)^2_{\alpha\beta}}{E_\alpha + E_\beta} \left[\frac{\varepsilon_\alpha - \lambda}{E_\alpha(E_\alpha + E_\beta)} \times \left(1 - \frac{(\varepsilon_\alpha - \lambda)(\varepsilon_\beta - \lambda) + \Delta^2}{E_\alpha E_\beta}\right) - \frac{\Delta^2(\varepsilon_\alpha - \varepsilon_\beta)}{E_\alpha^3 E_\beta} \right] + \frac{\Omega_x^2 \Delta^2}{E_\ell^3} \sum_\alpha \frac{(j_x)^2_{\alpha\ell}(\varepsilon_\alpha - \varepsilon_\ell)(5E_\ell^2 - E_\alpha^2)}{(E_\alpha^2 - E_\ell^2)^2}.$$

$$\sum_\alpha v_\alpha^2 + \frac{\varepsilon_\ell - \lambda}{E_\ell}$$

Blocking effect

$$\frac{\Omega_x^2}{2} \sum_{\alpha,\beta} \frac{(j_x)^2_{\alpha\beta}}{E_\alpha + E_\beta} \left[\frac{\varepsilon_\alpha - \lambda}{E_\alpha(E_\alpha + E_\beta)} \left(1 - \frac{(\varepsilon_\alpha - \lambda)(\varepsilon_\beta - \lambda) + \Delta^2}{E_\alpha E_\beta}\right) - \frac{\Delta^2(\varepsilon_\alpha - \varepsilon_\beta)}{E_\alpha^3 E_\beta} \right]$$

$\Delta N(x)$:no effect on λ at $\Omega_x \neq 0$

$$\frac{\Omega_x^2 \xi^2}{2\delta^2} \left[\frac{1}{\sqrt{\xi^2 + \eta^2}} \left(\frac{j_>^2}{(1+\eta)^2} - \frac{j_<^2}{(1-\eta)^2} \right) - \frac{2\delta X_2}{\eta(\sqrt{\xi^2 + \eta^2})^3} \right].$$

If we assume $j_>^2 = j_<^2$

$$\left(\frac{\xi \Omega_x}{\delta}\right)^2 \frac{j_>^2 \eta(1 - 3\eta^2)}{(\xi^2 + \eta^2)^{3/2}(1 - \eta^2)^2},$$

Zero for $\xi \sim 0$ (high $I - I_0$) , $\eta \sim 0$ or $\eta^2 \sim 1/3$ ($\eta \sim 0.6$)

Gap equation and Moment of Inertia as a function of angular momentum

when $\Delta \geq$ d (integral)

$$(I-I_0)^2 = \delta^2 \mathcal{J}_x^2 \rho \left(\ln \frac{\xi_0}{\xi} - \frac{1}{\rho\delta\sqrt{\xi^2+\eta^2}} + \frac{1}{\rho\delta\sqrt{\xi_0^2+\eta^2}} \right) \left(\frac{\mathcal{J}_x^{\mathrm{rig}} \bar{F}}{8} - \frac{X_1}{\sqrt{\xi^2+\eta^2}} + \frac{X_2}{\sqrt{\xi_0^2+\eta^2}} \right)^{-1}$$

$$\mathcal{J}_x = \mathcal{J}_x^{\mathrm{rig}} \left[1 - \xi^2 \ln \frac{1+\sqrt{1+\xi^2}}{\xi} - \frac{19}{90}\xi^2 \right] + X_1\sqrt{\xi^2+\eta^2} + \frac{X_2}{\sqrt{\xi^2+\eta^2}},$$

when $\Delta \ll$ d, (direct summation) ξ starts from 0.15

$$(I-I_0)^2 = \delta^2 \mathcal{J}_x^2 \rho \left(\ln \frac{\xi_0}{\xi} - \frac{1}{\rho\delta\sqrt{\xi^2+\eta^2}} + \frac{1}{\rho\delta\sqrt{\xi_0^2+\eta^2}} \right) \left(\frac{\mathcal{J}_x^{\mathrm{rig}} \bar{F}_n}{8} - \frac{X_1}{\sqrt{\xi^2+\eta^2}} + \frac{X_2}{\sqrt{\xi_0^2+\eta^2}} \right)^{-1},$$

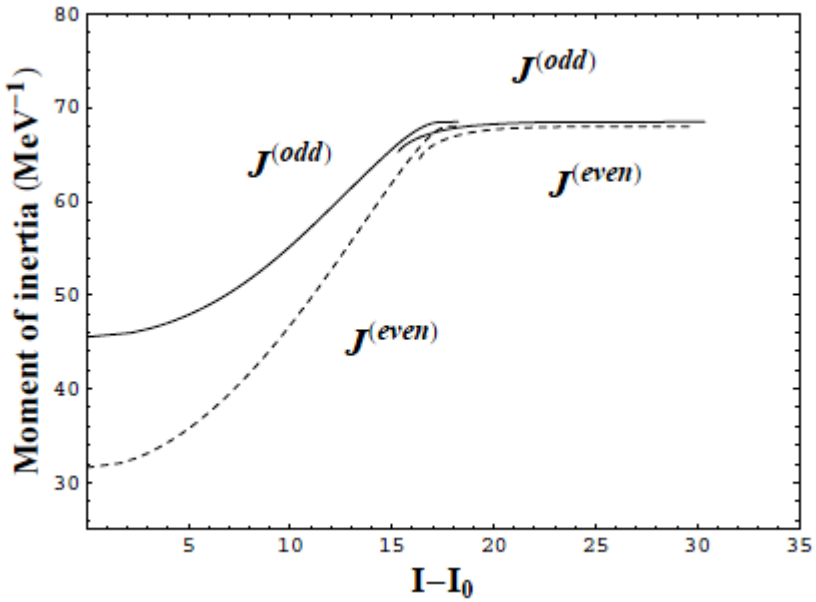
$$\mathcal{J}_x = \mathcal{J}_x^{\mathrm{rig}} \left[1 - \xi^2 \left(\Gamma_n - \xi^2 Z_n - \frac{19}{90} \right) \right] + X_1\sqrt{\xi^2+\eta^2} + \frac{X_2}{\sqrt{\xi^2+\eta^2}}$$

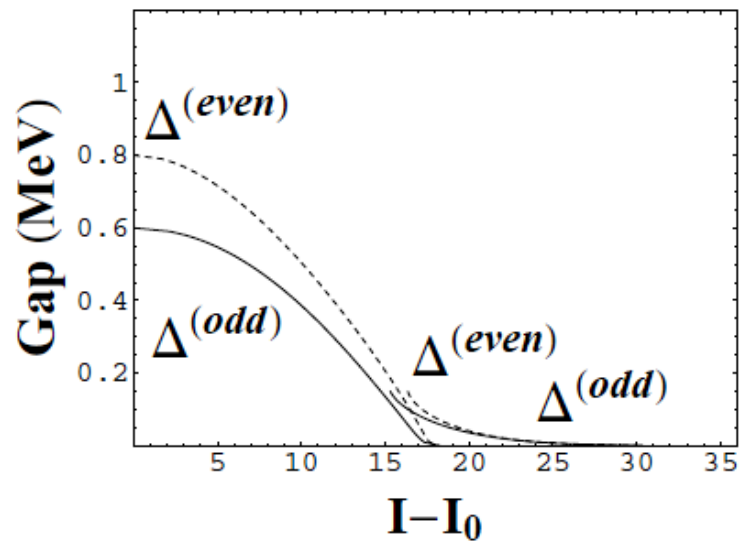
Comparison between even and odd cases

common parameters(ρ=2.5MeV^{-1}, δ=2.0MeV, J^{rig} =68MeV^{-1})

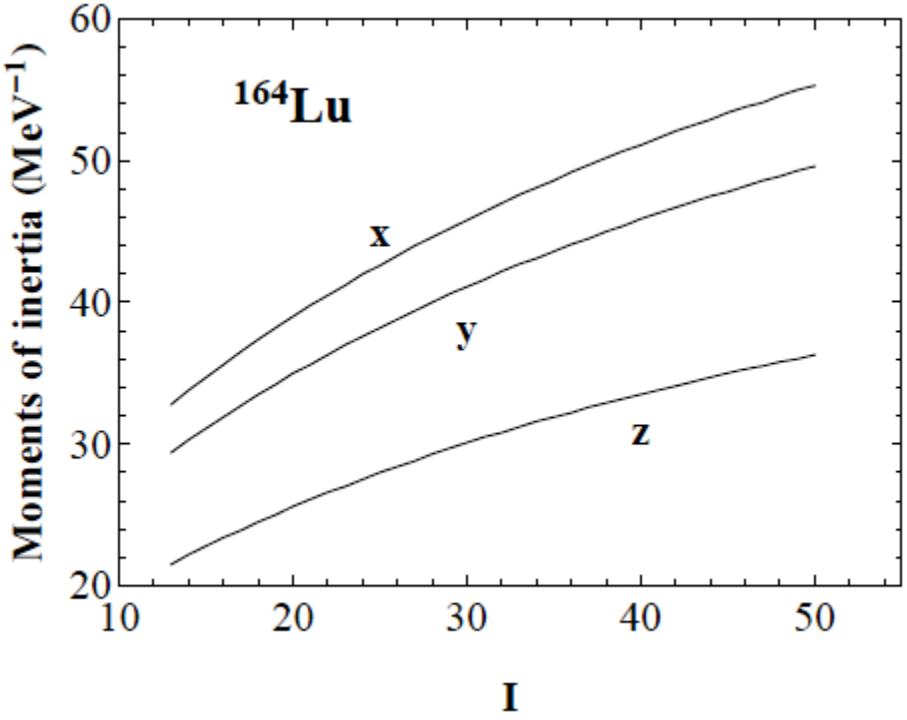
(even: ξ_0=0.8, odd: ξ_0=0.6, η=0.6, $j_>^2$=12, $j_<^2$ =10

In the case of Δ<<d, ξ starts from 0.15in both cases.)

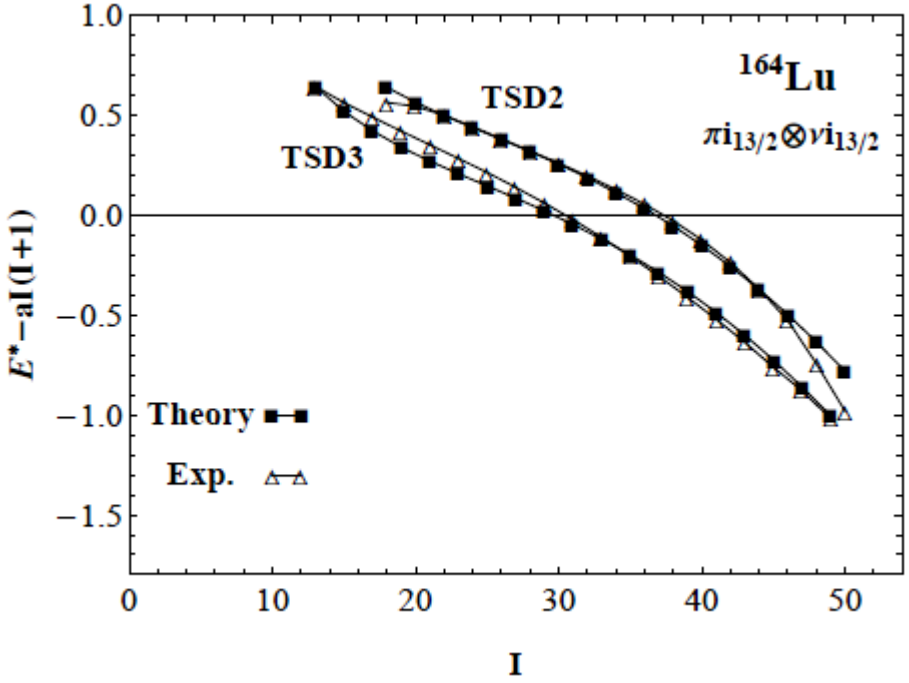




Odd moment of inertia starts from larger value, increases gradually, and reaches to rigid value later than even. Before it reaches to rigid, the curve shows upward convexity. Gap shows concave before it reaches to 0.



upward convexity



K.S.-T.,K.T. &N.Yoshinaga, PTEP in print (2014)

Excited bands in even nucleus and odd-odd nucleus

The last nucleons in ℓ_1 and ℓ_2 levels:

$$\Delta = \frac{G}{4} \sum_{\alpha \neq \ell_1, \ell_2, \tilde{\ell}_1, \tilde{\ell}_2} \frac{\Delta}{E_\alpha} \Big[1 - \Omega_x^2 \sum_{\beta \neq \ell_1, \ell_2, \tilde{\ell}_1, \tilde{\ell}_2} \frac{(j_x)^2_{\alpha\beta}}{(E_\alpha + E_\beta)} \Big(\frac{E_\alpha E_\beta - (\varepsilon_\alpha - \lambda)(\varepsilon_\beta - \lambda) - \Delta^2}{E_\alpha E_\beta (E_\alpha + E_\beta)}$$

$$+ \frac{(\varepsilon_\alpha - \lambda)(\varepsilon_\alpha - \varepsilon_\beta)}{E_\alpha^2 E_\beta} \Big) + \Omega_x^2 \Big(\frac{(j_x)^2_{\alpha\ell_1}}{E_\alpha^2 - E_{\ell_1}^2} (3 - \frac{(\varepsilon_\alpha - \lambda)(\varepsilon_{\ell_1} - \lambda) + \Delta^2}{E_\alpha^2})$$

$$+ \frac{(j_x)^2_{\alpha\ell_2}}{E_\alpha^2 - E_{\ell_2}^2} (3 - \frac{(\varepsilon_\alpha - \lambda)(\varepsilon_{\ell_2} - \lambda) + \Delta^2}{E_\alpha^2}) \Big) \Big].$$

They prevent the reduction of gap much more than the case with only one blocked level.

Conclusion

- We discuss the angular momentum dependence of moments of inertia based on the perturbation treatment of the Coriolis anti-pairing effect both for even and odd nuclei, by applying the approximation method developed by Bohr-Mottelson for the gap-dependence of moment of inertia.
- We developed the integral method ($\Delta \geq$ d (level distance)) and direct summation method ($\Delta \ll$ d) for the gap and the moment of inertia as functions of I. Because of the finiteness of nucleus, the phase transition from super to normal becomes slow and infinite. Moment of inertia shows upward convexity in direct summation (high-spin states).
- Because of the blocking effect, the gap in odd nucleus starts from smaller value than even nucleus, decreases gradually, the moment of inertia starts from larger value than even nucleus, increases gradually and show slightly convex upward curve before it reaches the rigid-body value which is similar to those in top-on-top model.