Complex Langevin in low-dimensional QCD: the good and the not-so-good

Johannes Mahr, Sebastian Schmalzbauer, and Jacques Bloch

University of Regensburg

Lattice 2015
14 – 18 July 2015
Kobe, Japan

Introduction

- Sign problem in QCD at nonzero chemical potential: particularly serious in d=4, but already present in lower dimensions.
- Use QCD in 0+1 and 1+1 dimensions to study viability of the complex Langevin method
- Sign problem mild in 0+1d, but large in some regimes of 1+1d QCD
- Preliminary results for 1+1d QCD: strong coupling, 4×4 lattice

QCD in 0+1 dimensions

Dirac operator & determinant

- Consider 0+1d QCD: one spatial point and $N_t = 1/aT$ time slices.
- 0+1d QCD Dirac operator for quark of mass m at chemical potential μ:

$$D_{tt'} = m\,\delta_{tt'} + \frac{1}{2a}\left[e^{a\mu} U_t \delta_{t',t+1} - e^{-a\mu} U_{t-1}^{-1} \delta_{t',t-1}\right],$$

where $U_t \in \mathrm{SL}(3, \mathbf{C})$ and $\delta_{tt'}$ is anti-periodic Kronecker delta.

- Dirac determinant can be reduced to determinant of a 3×3 matrix:

$$\det(aD) = \frac{1}{2^{3N_t}} \det\left[e^{\mu/T} P + e^{-\mu/T} P^{-1} + 2\cosh(\mu_c/T)\ \mathbb{1}_3\right]$$

with Polyakov line $P = \prod_t U_t$ and *effective mass* $a\mu_c = \operatorname{arsinh}(am)$.

QCD in 0+1 dimensions

Partition function

- 0+1d QCD: no gauge action → partition function is one-link integral of Dirac determinant (set $N_f = 1$):

$$Z = \int \mathscr{D}P \det D(P),$$

  with SU(3) Haar measure $\mathscr{D}P$.
- For $\mu \neq 0$: $\mathrm{Re} \det D$ has fluctuating sign → sign problem in MC simulations.
- Analytic results available for 0+1d QCD (Bilic & Demeterfi, 1988), (Ravagli & Verbaarschot, 2007)
- Other works on 0+1d QCD with complex Langevin:
  - one-link formulation with mock-gauge action: Aarts & Stamatescu, 2008
  - U(N_c) in spectral representation: Aarts & Splittorff, 2010
- Numerical solution to 0+1d QCD using subsets (JB, Bruckmann, Wettig, 2013)

Eigenvalue representation

- 0+1d QCD partition function

$$Z = \int d\phi_1 d\phi_2 J(\phi_1, \phi_2) \det D(P),$$

with

$$P = \begin{pmatrix} e^{i\phi_1} & 0 & 0 \\ 0 & e^{i\phi_2} & 0 \\ 0 & 0 & e^{-i\phi_1 - i\phi_2}. \end{pmatrix}$$

where $\phi_1, \phi_2 \in \mathbf{C}$ and Haar measure

$$J(\phi_1, \phi_2) = \sin^2 \frac{\phi_1 - \phi_2}{2} \sin^2 \frac{2\phi_1 + \phi_2}{2} \sin^2 \frac{\phi_1 + 2\phi_2}{2}$$

Eigenvalue representation

Complex Langevin equations

- Complex action:

$$S(P) = -\log J - \log \det D$$

- Complex Langevin equation $\rightarrow$ complex evolution in SL(3,$\mathbf{C}$)

$$\frac{d\phi_i}{dt} = K_i(P) + \eta_i$$

with drift term

$$K_i(P) = -\frac{\partial S(P)}{\partial \phi_i} = \frac{1}{J}\frac{\partial J}{\partial \phi_i} + \mathrm{tr}\left[D^{-1}\frac{\partial D}{\partial \phi_i}\right].$$

and real Gaussian noise η with variance 2.

Eigenvalue representation

Discretizations

- Stochastic Euler algorithm (order 1):

$$\phi_i(t+1) = \phi_i(t) + \epsilon K_i(P) + \sqrt{\epsilon}\,\eta_i.$$

- Stochastic Runge-Kutta method of order 3/2 (Chang, 1987), (Aarts & James, 2012):

$$\phi_i(t+1) = \phi_i(t) + \frac{\epsilon}{3}\Big[K(P') + 2K(P'')\Big] + \sqrt{\epsilon}\,\eta_i$$

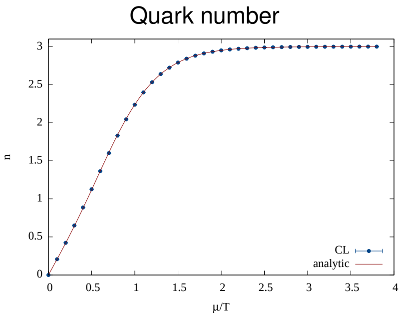
with two intermediate steps P' and P'' computed from:

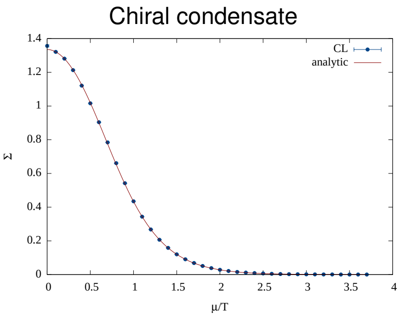
$$\begin{cases} \phi_i' = \phi_i + \dfrac{1}{2}\epsilon K_i(P) \\ \phi_i'' = \phi_i' + \dfrac{3}{2}\sqrt{\epsilon}\left(\dfrac{1}{2}\eta_i + \dfrac{\sqrt{3}}{6}\eta_i'\right) \end{cases}$$

and η_i and η_i' are independent real Gaussian noises with variance 2.

Eigenvalue representation

Results (m=0.5)





So far so good

- Polyakov line:

$$P = \exp\left[i \sum_a z_a \lambda_a \right] \qquad (\lambda_a\text{: Gell-Mann matrices})$$

- Increased number of degrees of freedom
- Discrete time evolution of P in SL(3,$\mathbf{C}$)

$$P(t+1) = R(t)\,P(t)$$

with $R \in \mathrm{SL}(3,\mathbf{C})$

- Drift term:

$$K_a(P) = -D_a S(P) = -\partial_\alpha S(e^{i\alpha\lambda_a} P)|_{\alpha=0}$$

- Euler discretization:

$$R = \exp\left[i \sum_a \lambda_a (\epsilon K_a + \sqrt{\epsilon}\, \eta_a) \right],$$

- Runge-Kutta discretization (Chang, 1987; Batrouni et al., 1985; Bali et al., 2013):

$$R = \exp\left[i \sum_a \lambda_a \left(\epsilon \left[k\, K_a(P') + (1-k)\, K_a(P'') \right] + \sqrt{\epsilon}\, \eta_a \right) \right],$$

with intermediate steps

$$P' = R' P \quad , \quad P'' = R'' P$$

Gauge cooling in 0+1 d QCD

- SL(3,$\mathbf{C}$) Gauge transformation of Polyakov line:

$$P' = GPG^{-1}$$

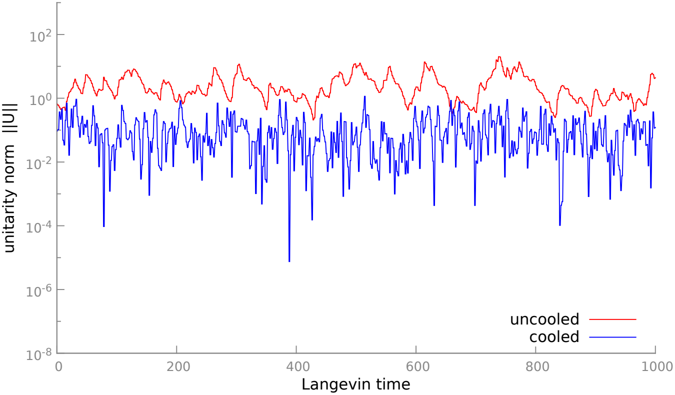
- Gauge cooling: choose $G \in \text{SL}(3, \mathbf{C})$ to minimize the unitarity norm

$$N(P) = \text{tr}[PP^\dagger + (PP^\dagger)^{-1} - 2].$$

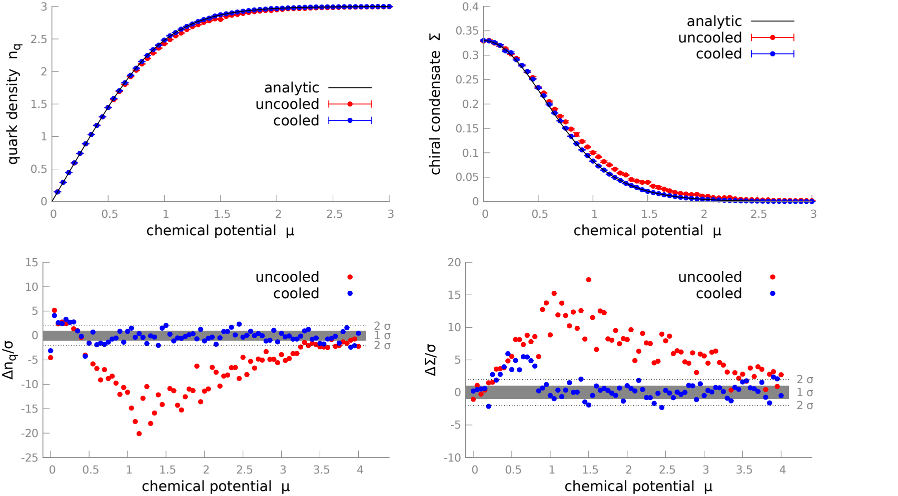
- In 0+1d QCD: maximal cooling = diagonalize Polyakov line.
- Observables invariant under gauge tf
- noise distribution not invariant under gauge tf
  $\longrightarrow$ gauge cooling and Langevin step do not commute
  $\longrightarrow$ different trajectories

Gauge cooling

Unitarity norm ($\mu = 1.0, m = 0.1$)

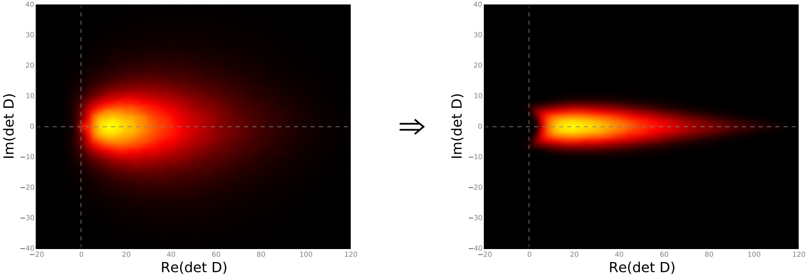


0+1d QCD with Gell-Mann: Results

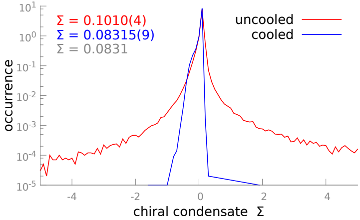


→ uncooled: significant deviation

0+1d QCD: Effect of gauge cooling on determinant

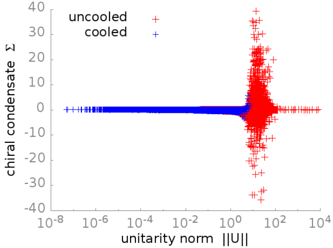


0+1d QCD: Effect of gauge cooling on observable

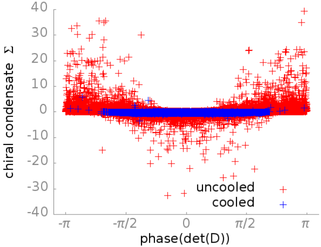


⇒ cooling "sharpens" distribution

Correlation between observables, unitarity norm and phase



no skirts for small unitarity norm



relation skirts and branch-cuts

1+1d QCD

- Staggered Dirac operator for 1+1d QCD:

$$D_{rs} = m\,\delta_{rs} + \frac{1}{2}\left[e^{\mu}U_t(r)\delta_{s,r+\hat{t}} - e^{-\mu}U_t^{\dagger}(r-\hat{t})\delta_{s,r-\hat{t}}\right] + \frac{1}{2}\eta_r\left[U_x(r)\delta_{s,r+\hat{x}} - U_x^{\dagger}(r-\hat{x})\delta_{s,r-\hat{x}}\right],$$

  with staggered fermion phase $\eta_r = (-)^t$ at site $r = (x,t)$.
- Simulations in strong coupling limit, i.e. $e^{-S_G} = 1$
- Preliminary results: 4×4 lattice
- Validation by comparison with subset method
- In preparation: validation for $N_s \times N_t = 4 \times \{2,6,8,10\}, 6 \times \{2,4,6,8\}, 8 \times \{2,4,6,8\}$

1+1d QCD: Gauge cooling

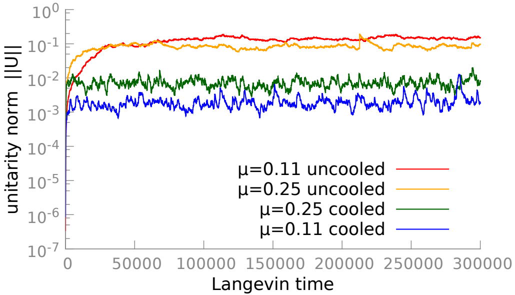
- Gauge trafos

$$U'_\nu(r) = G(r)\, U_\nu(r)\, G(r+\hat{\nu})^{-1}, \quad G \in \mathrm{SL}(3,\mathbf{C})$$

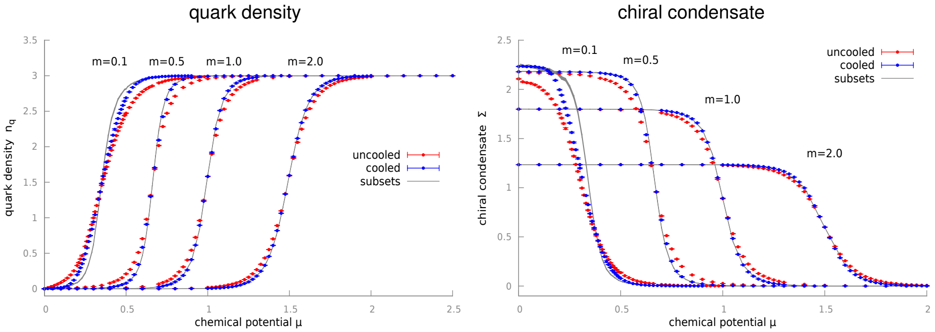
- gauge cooling → minimization of unitarity norm

$$||\mathscr{U}|| = \sum_{r,\nu} \mathrm{tr}\left[U_\nu(r)^\dagger U_\nu(r) + \left(U_\nu^\dagger(r) U_\nu(r) \right)^{-1} - 2 \right]$$

via steepest descent



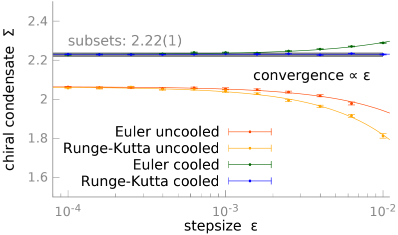
1+1d QCD: Results



Convergence with step size (m=0.1)

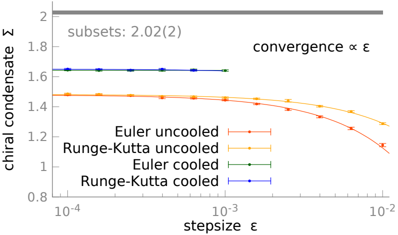
- Convergence to continuum limit $\epsilon \to 0$

$\mu = 0.07$



correct convergence

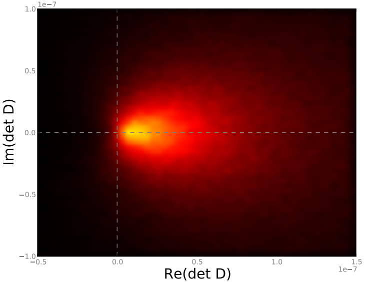
$\mu = 0.25$



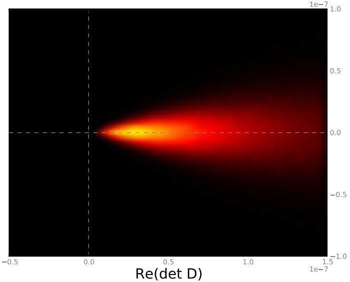
wrong convergence

1+1d QCD: Effect of gauge cooling on det D (m=0.1)

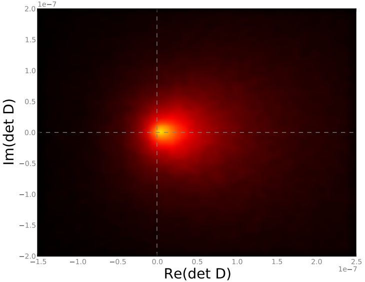
$\mu = 0.07$



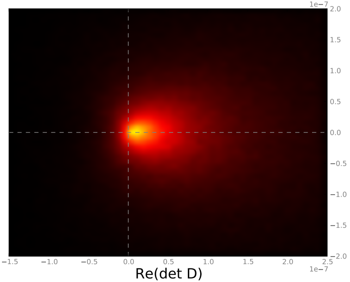
$\Rightarrow$



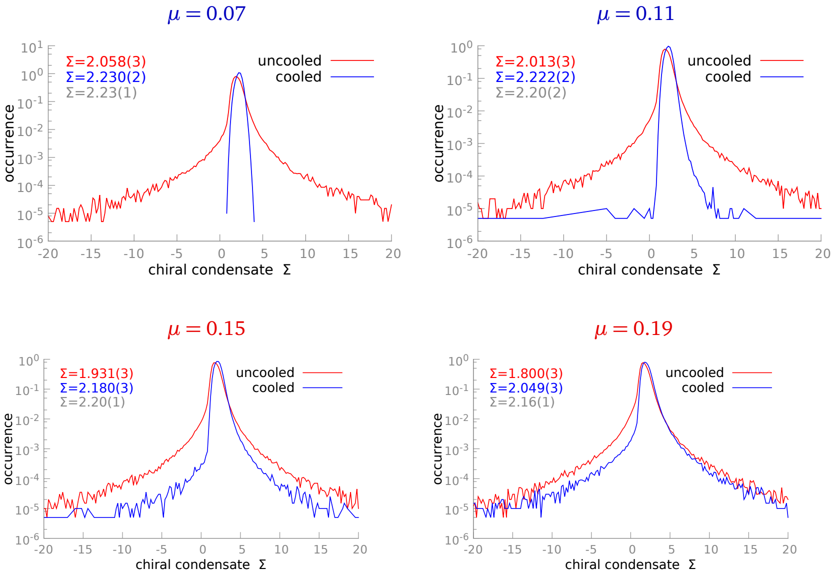
$\mu = 0.25$



$\Rightarrow$



Distribution of observables – skirts



Summary

Complex Langevin in 0+1d QCD at nonzero μ

- Correct in diagonal representation
- Small discrepancies in Gell-Mann representation
- Improvement using gauge cooling

Complex Langevin in 1+1d QCD at nonzero μ

- Incorrect results without gauge cooling
- Gauge cooling: correct results for some m, μ ranges
- For light quarks: cooling does NOT help "enough" in some μ-region
- No clear correlation with branch cut crossings.

Conclusion and Outlook

Conclusions

- No systematic runaways
- Gauge cooling necessary but not sufficient to get correct results
- Signal for wrong convergence:
  - skirts in distributions of observables
  - distribution of determinant is not squeezed but remains broad

Outlook

- Compact vs non-compact group: try multirate integration to handle real and imaginary directions differently
- Replace gauge cooling by alternative gauge fixing
- Validate method for larger lattices
- Include gauge action: improved convergence?