

# Three-body observables from the lattice

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July 15th, 2015

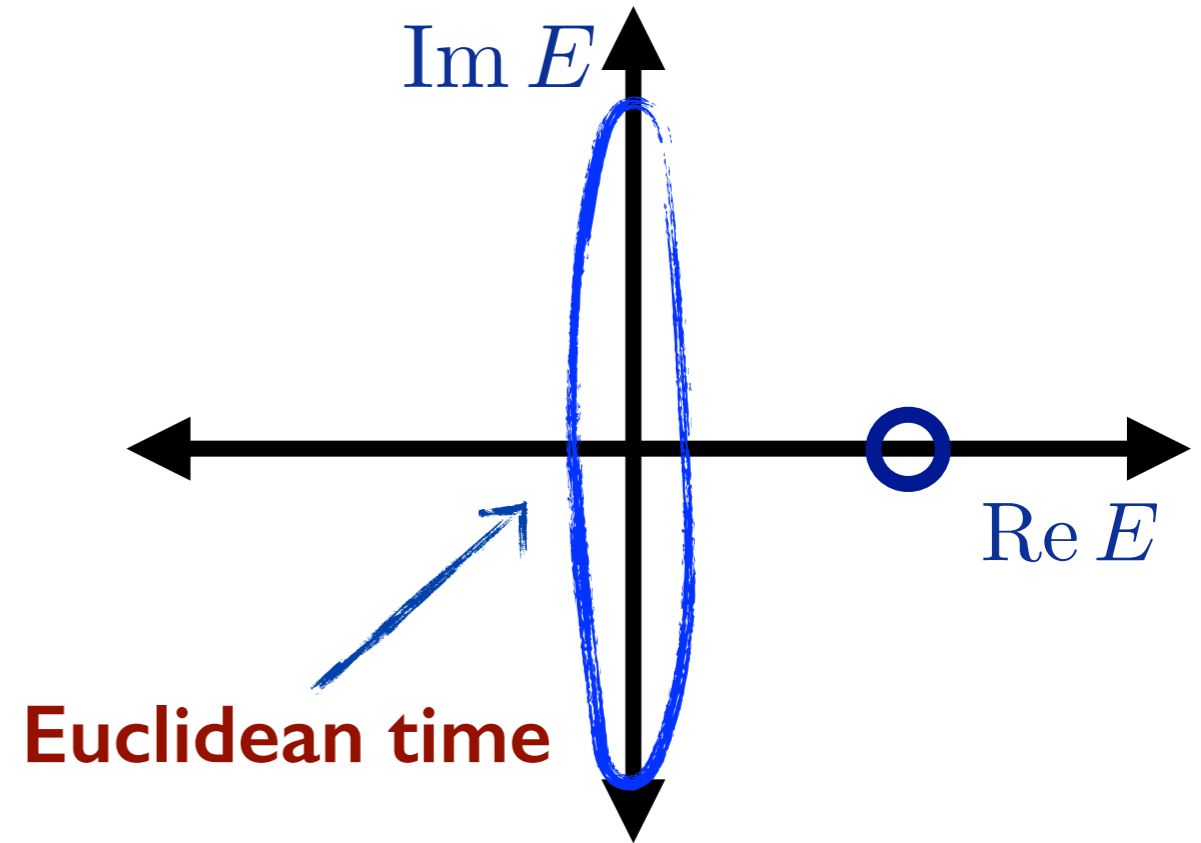
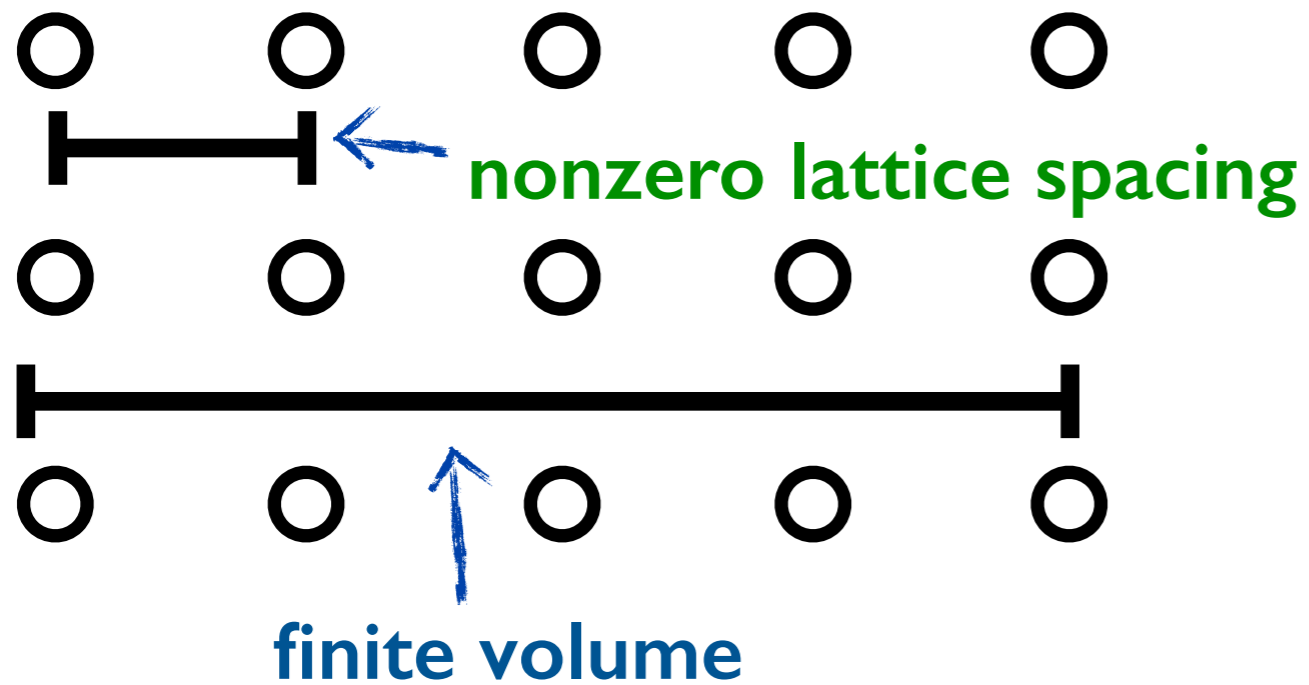


**HIM**

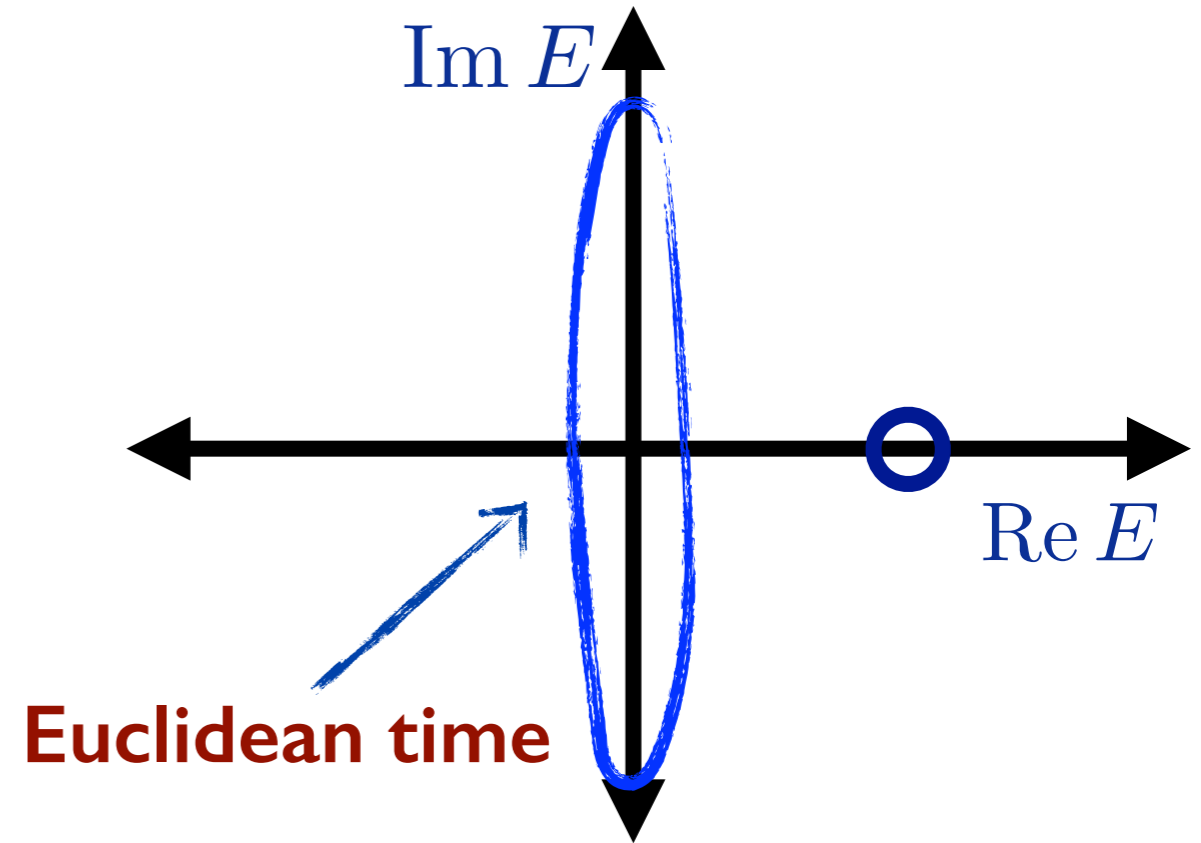
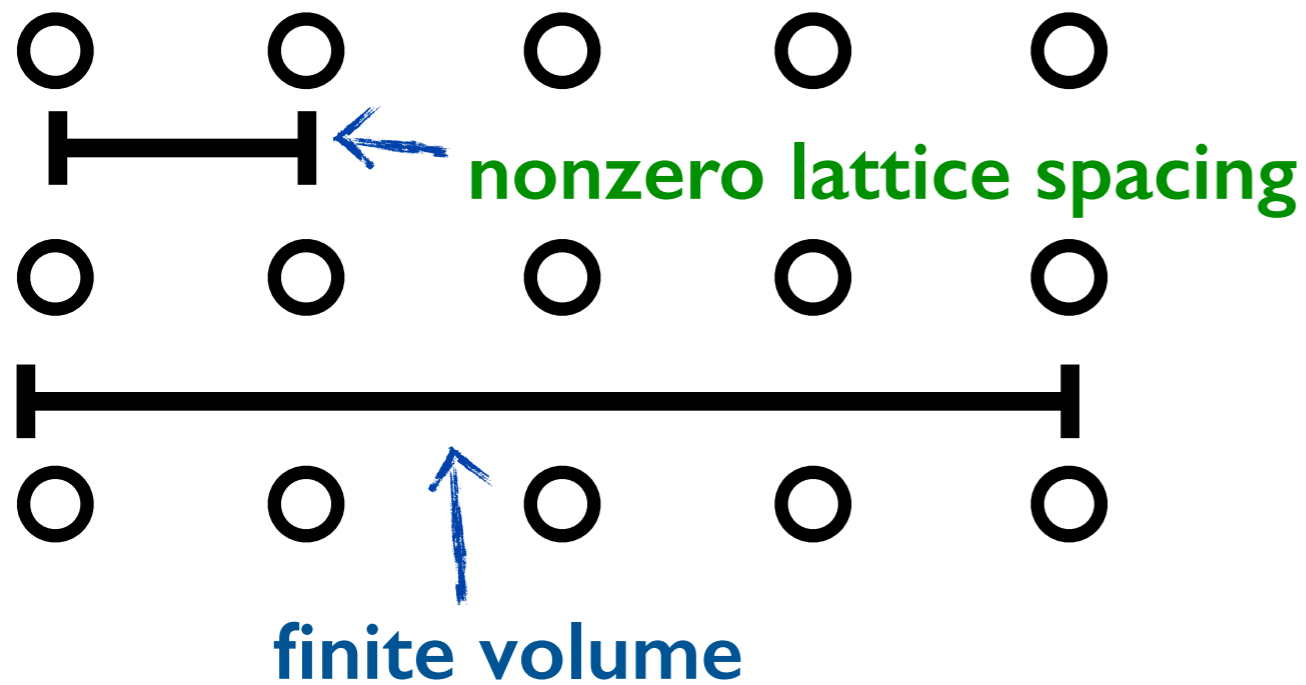
**Helmholtz-Institut Mainz**



# Compromises in numerical Lattice QCD

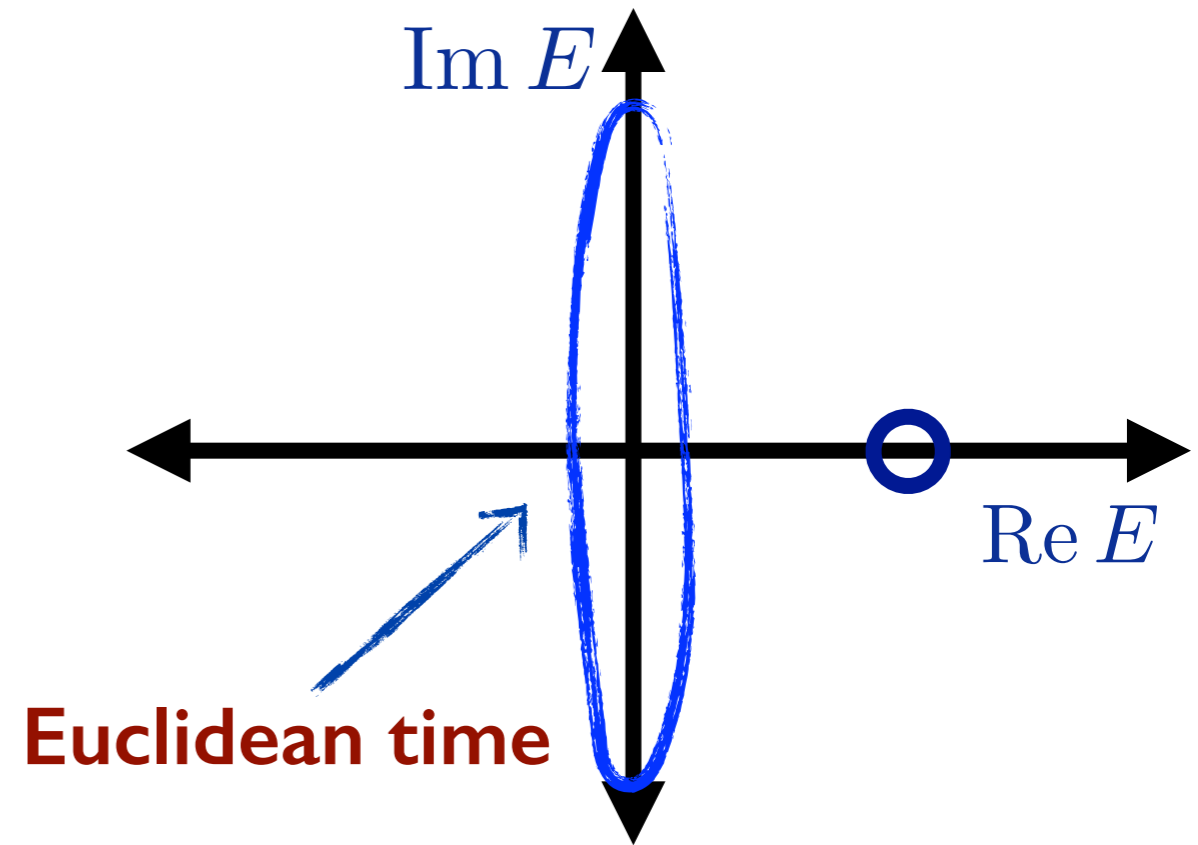
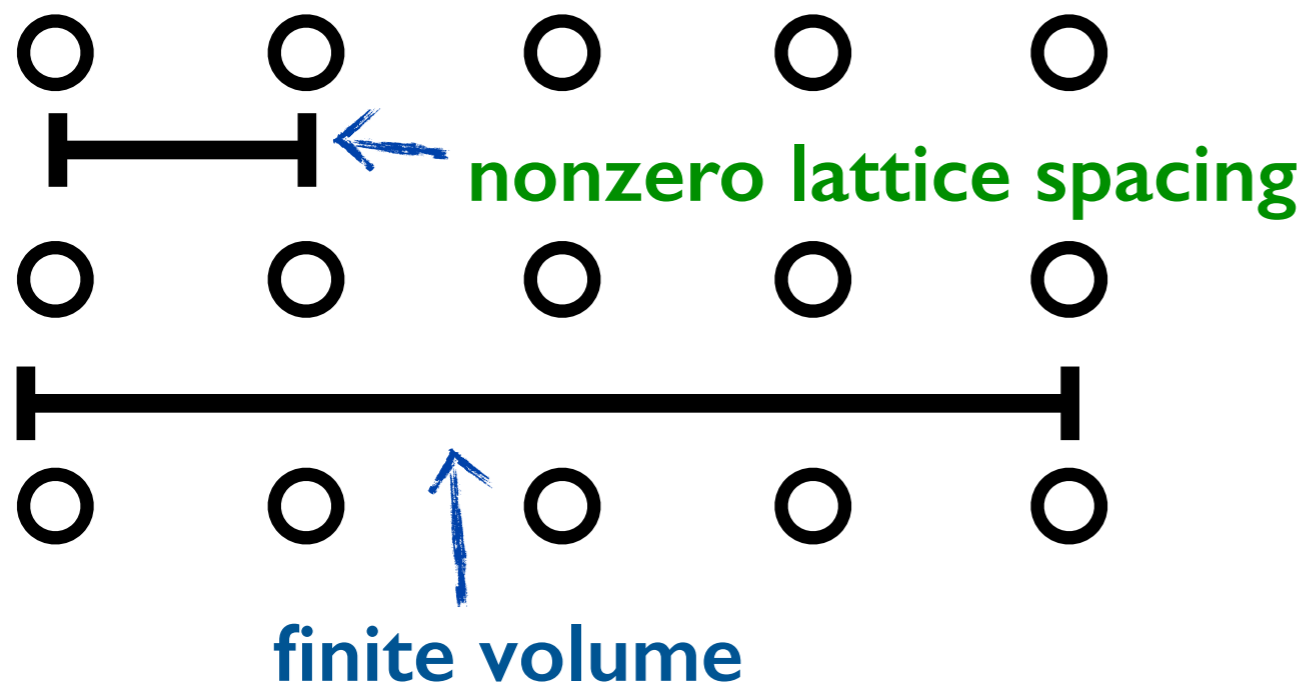


# Compromises in numerical Lattice QCD



**Not possible to directly calculate scattering amplitudes**

# Compromises in numerical Lattice QCD

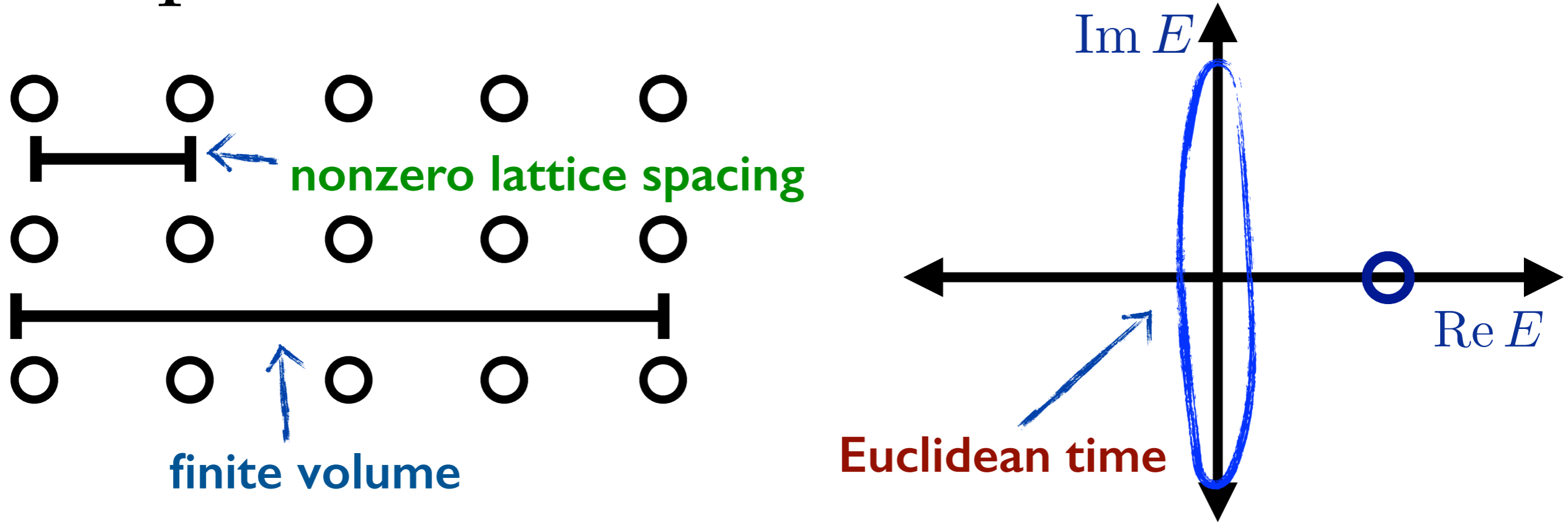


**Not possible to directly calculate scattering amplitudes**

Large **Euclidean time** limit is dominated by either **threshold or off-shell states**

L. Maiani and M. Testa, *Phys.Lett.* B245 (1990) 585–590

# Compromises in numerical Lattice QCD



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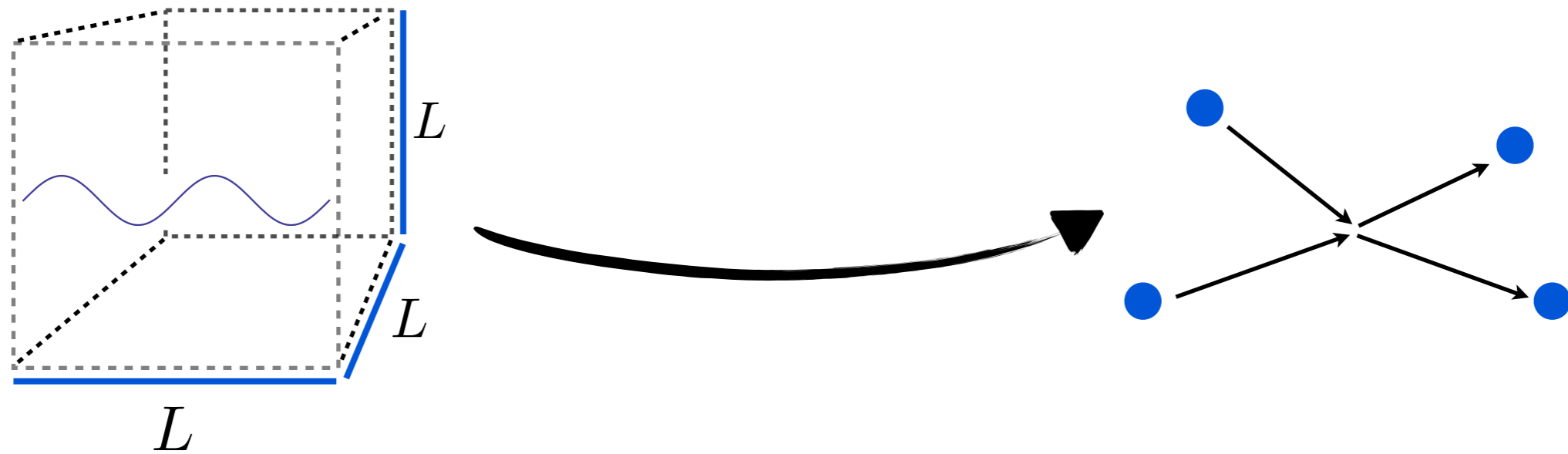
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L. Maiani and M. Testa, *Phys.Lett.* B245 (1990) 585–590

**Analytic continuation** of numerical **Euclidean** correlators is an **ill-posed** problem

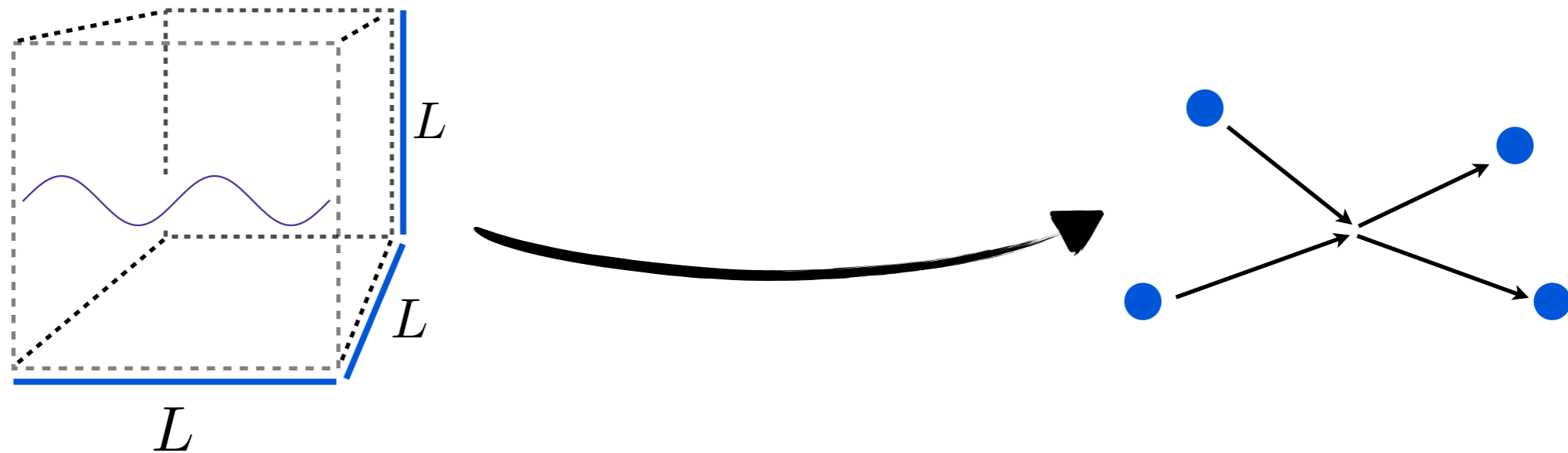
**Martin Lüscher** found a method to circumvent this issue and extract  $\pi\pi \rightarrow \pi\pi$  scattering from LQCD.

Lüscher, M. *Nucl. Phys* B354, 531-578 (1991)



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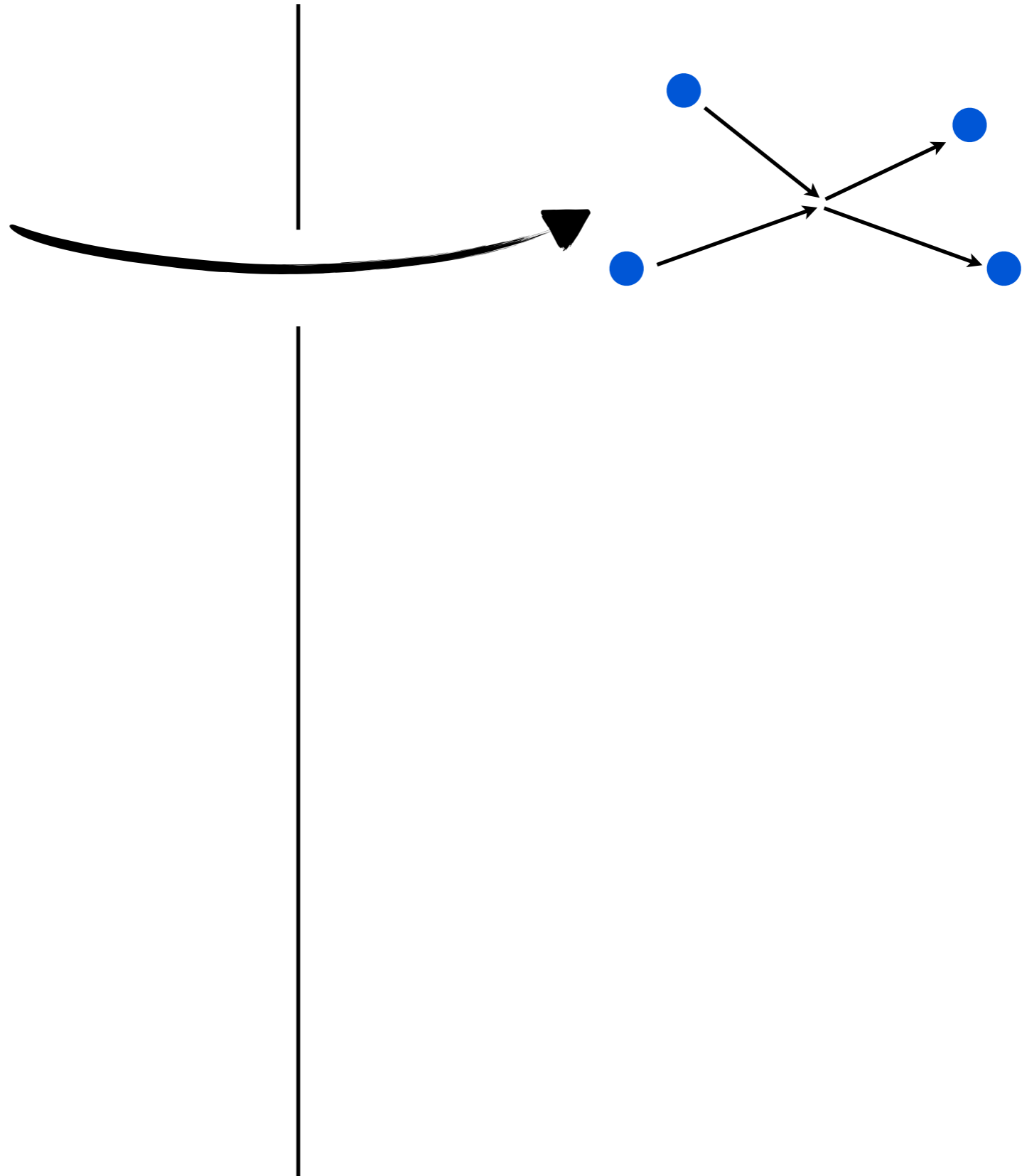
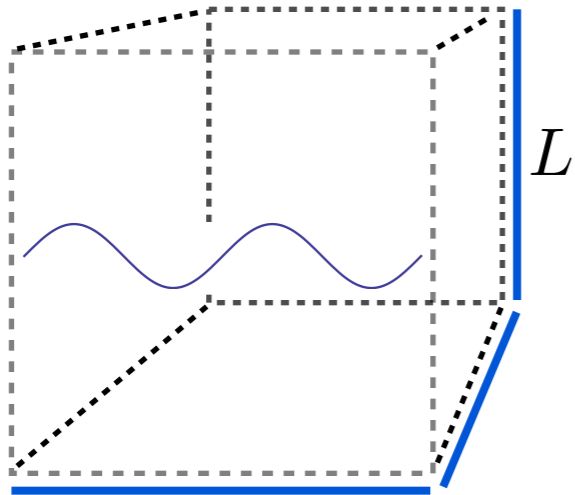
His key insight was to use **finite volume** as a tool.

He gave a mapping between the **finite-volume energy spectrum** and **elastic pion scattering amplitude**.

The same problem was addressed earlier in perturbative **non-relativistic quantum mechanics**

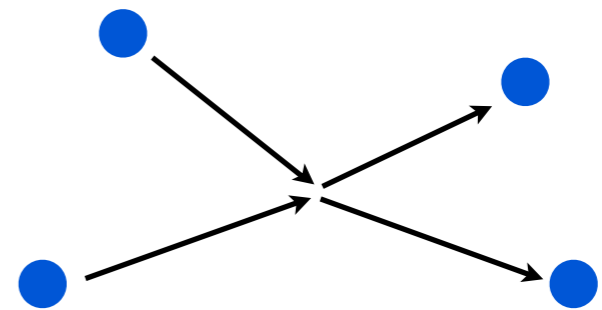
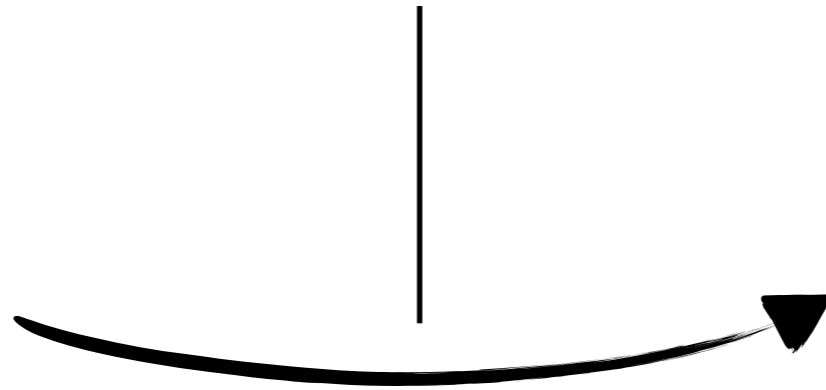
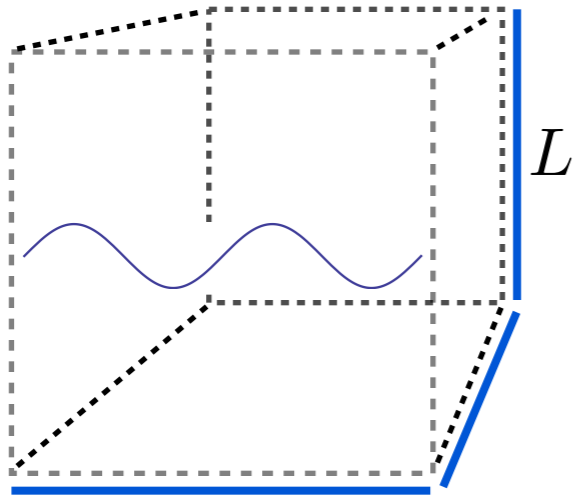
K. Huang and C. Yang, *Phys. Rev.* 105 (1957) 767-775

# Understanding Lüscher's Result



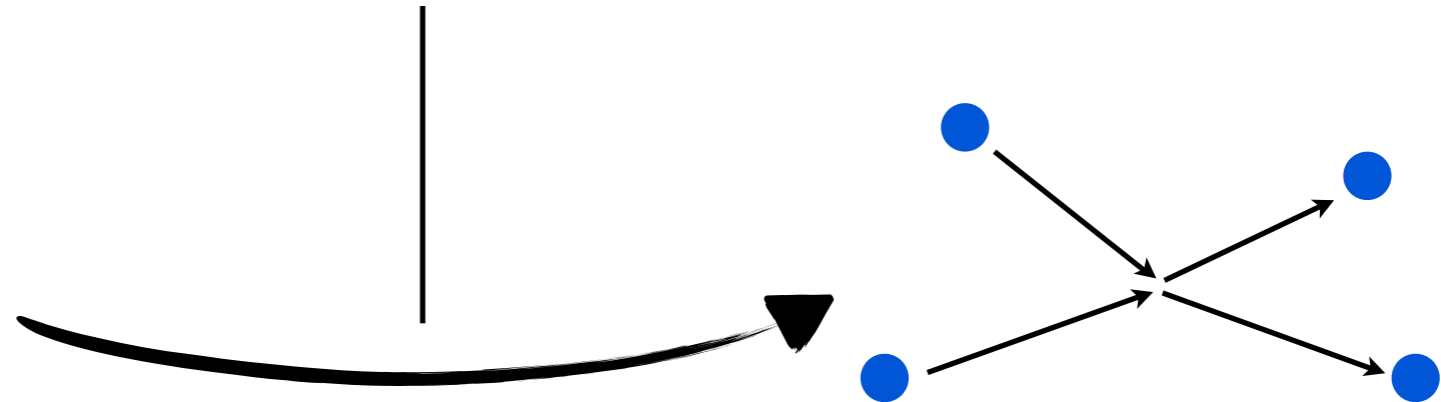
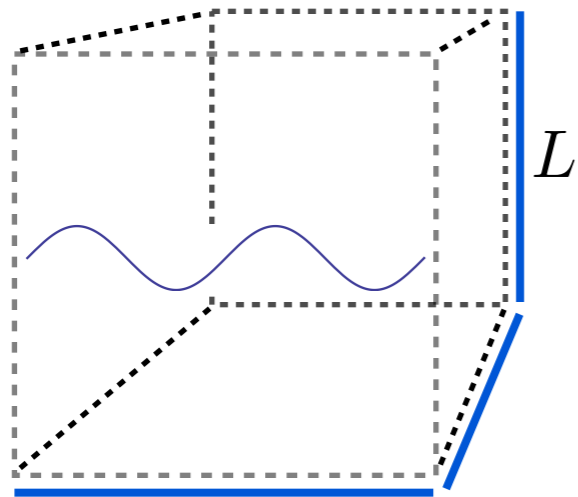


# Understanding Lüscher's Result



**Infinite volume**

# Understanding Lüscher's Result



**Infinite volume**  
Decompose scattering amplitude  
in partial waves

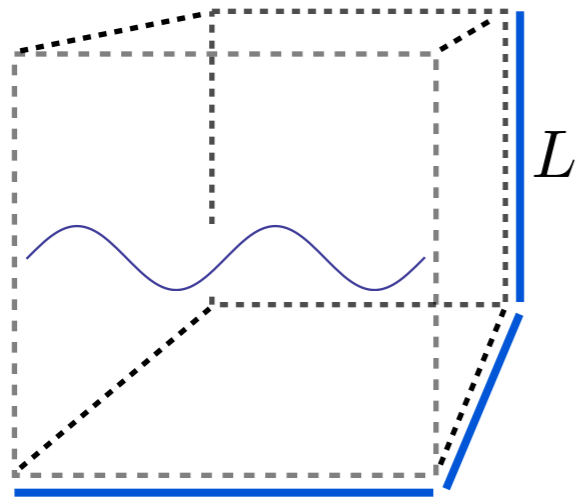
One real observable...

$$\delta_\ell(E^*)$$

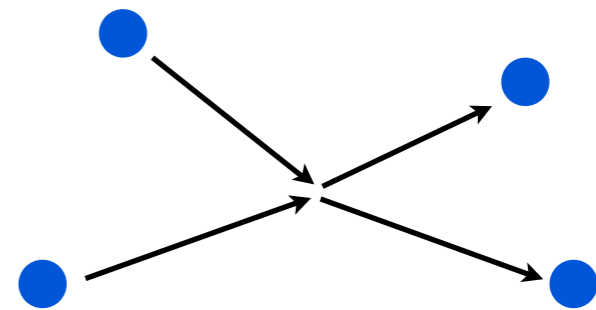
in each  
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at each  
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**Infinite volume**

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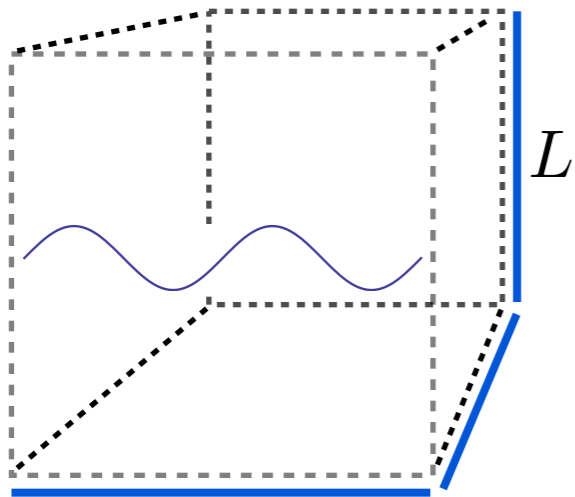
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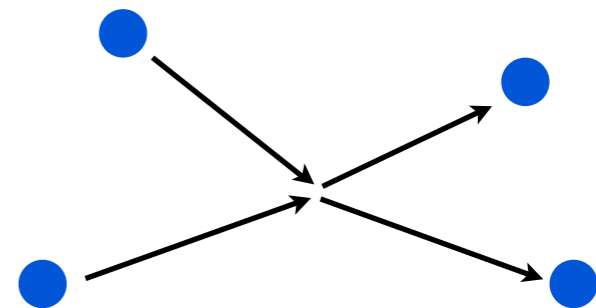
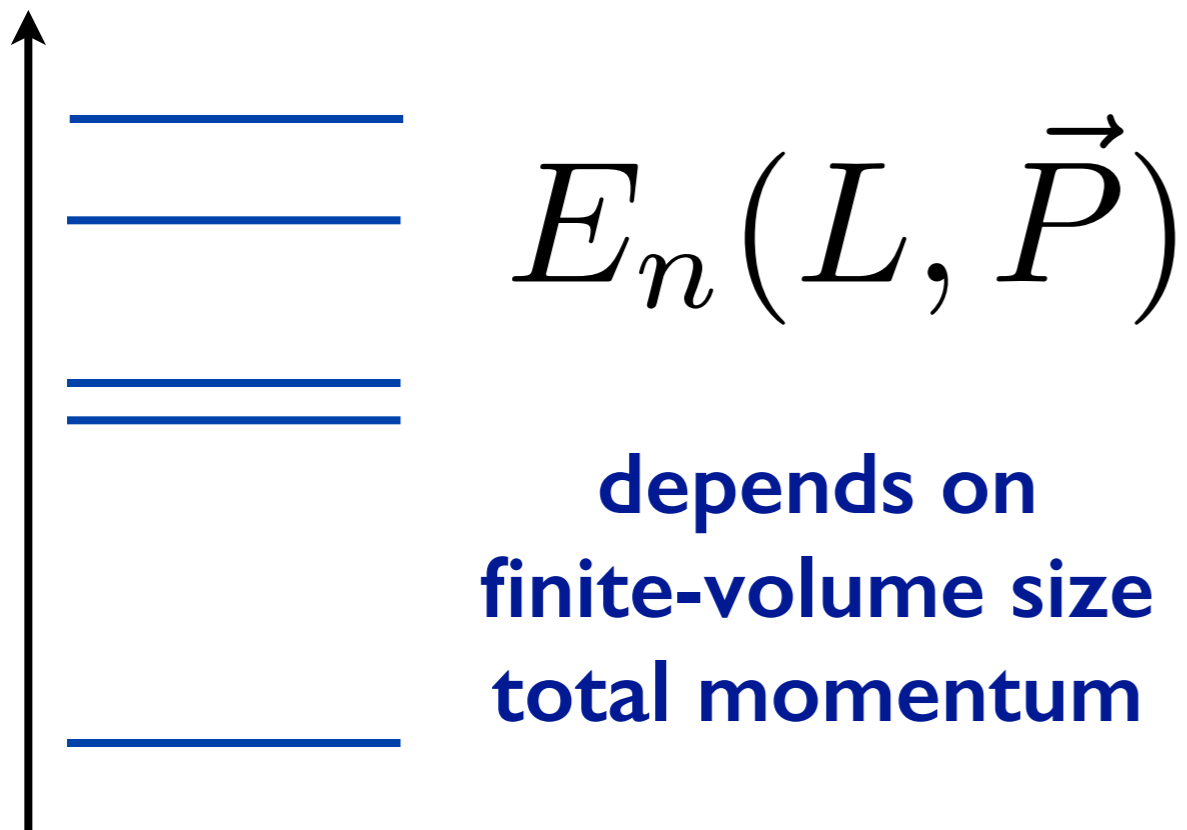
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# Understanding Lüscher's Result



**Finite volume**

**Discrete tower of energy levels**

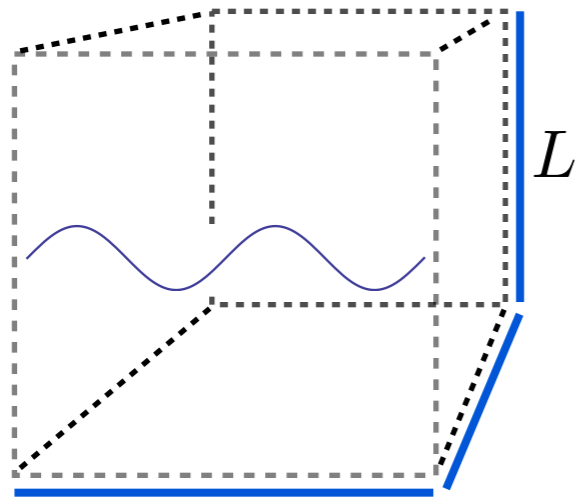


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**Decompose scattering amplitude in partial waves**

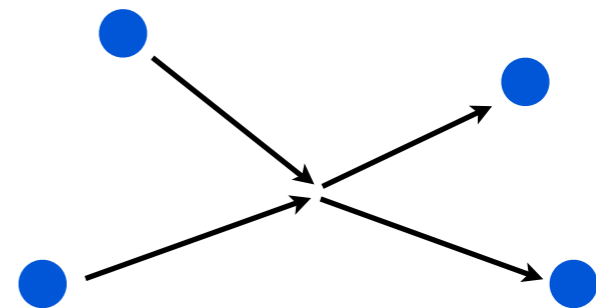
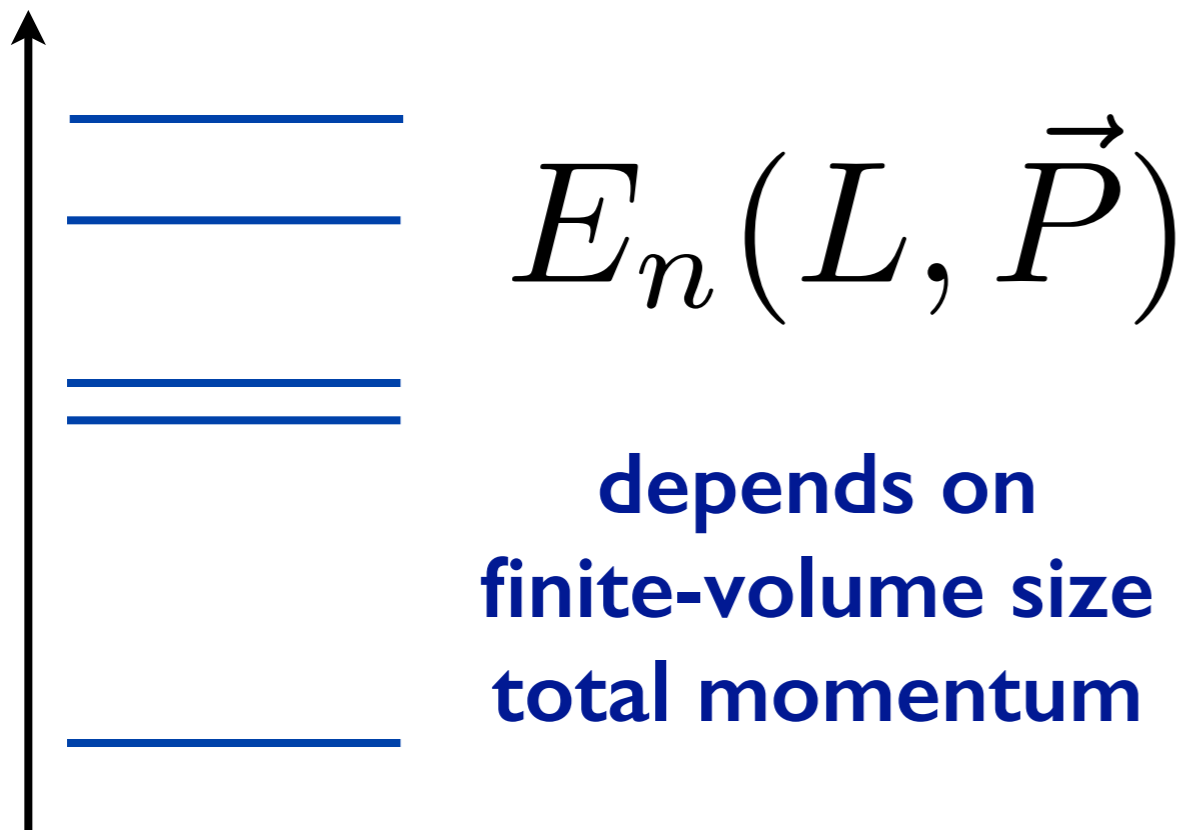
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# Understanding Lüscher's Result



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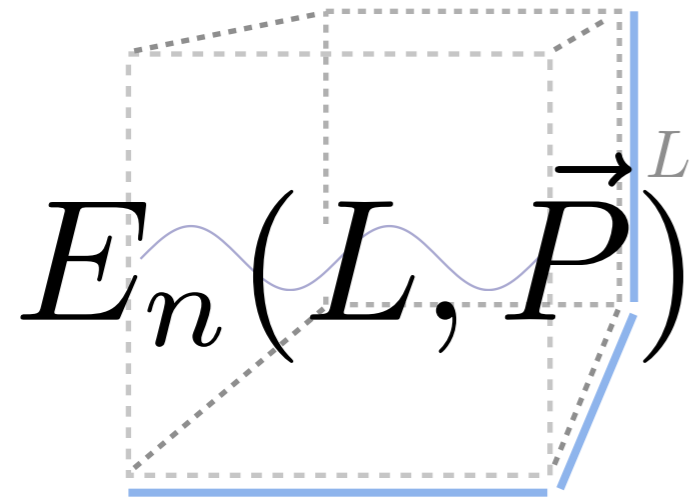


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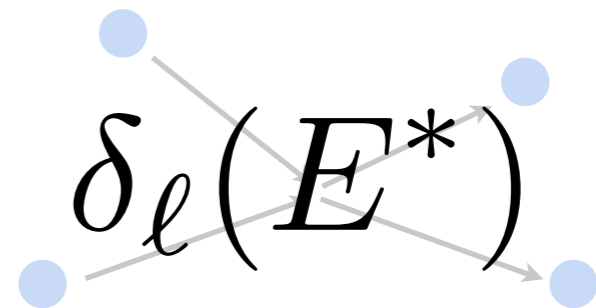
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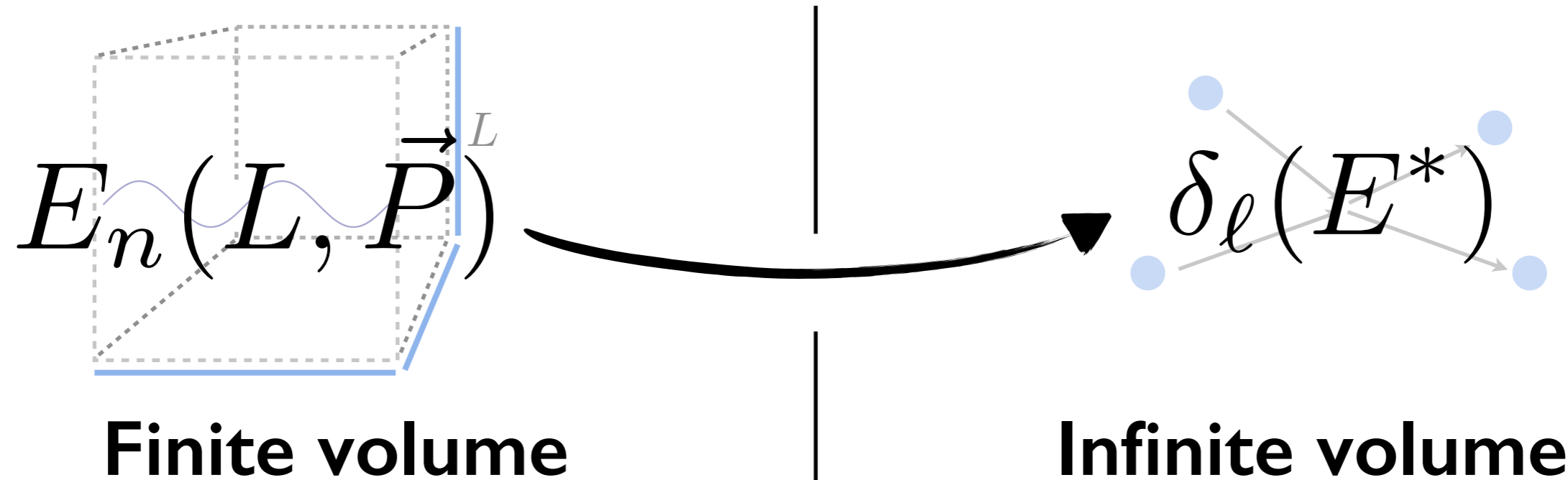


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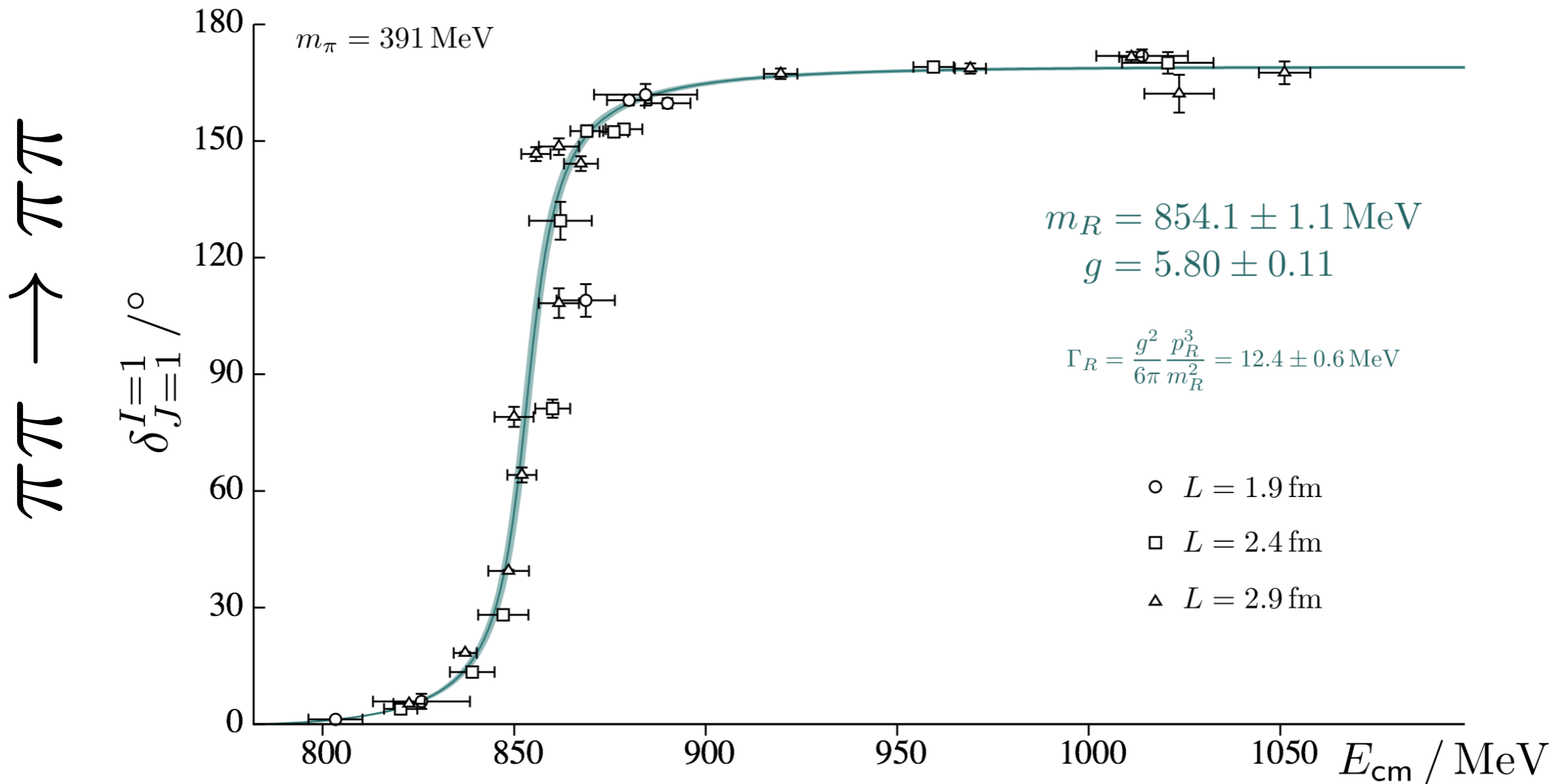
# Understanding Lüscher's Result



$$\det \left[ \cot \delta(E_n^*) + \cot \phi(E_n, \vec{P}, L) \right] = 0$$

$$\left( \cot \phi = -\frac{2}{\pi\gamma} \sum_{\ell m} \frac{1}{q^{*(\ell+1)}} \mathcal{Z}_{\ell m}^P[1, q^{*2}] \int d\Omega Y_{\ell_1 m_1}^* Y_{\ell m}^* Y_{\ell_2 m_2} \right)$$

# Lüscher's method has led to a large body of work extracting phase shifts from Lattice QCD.



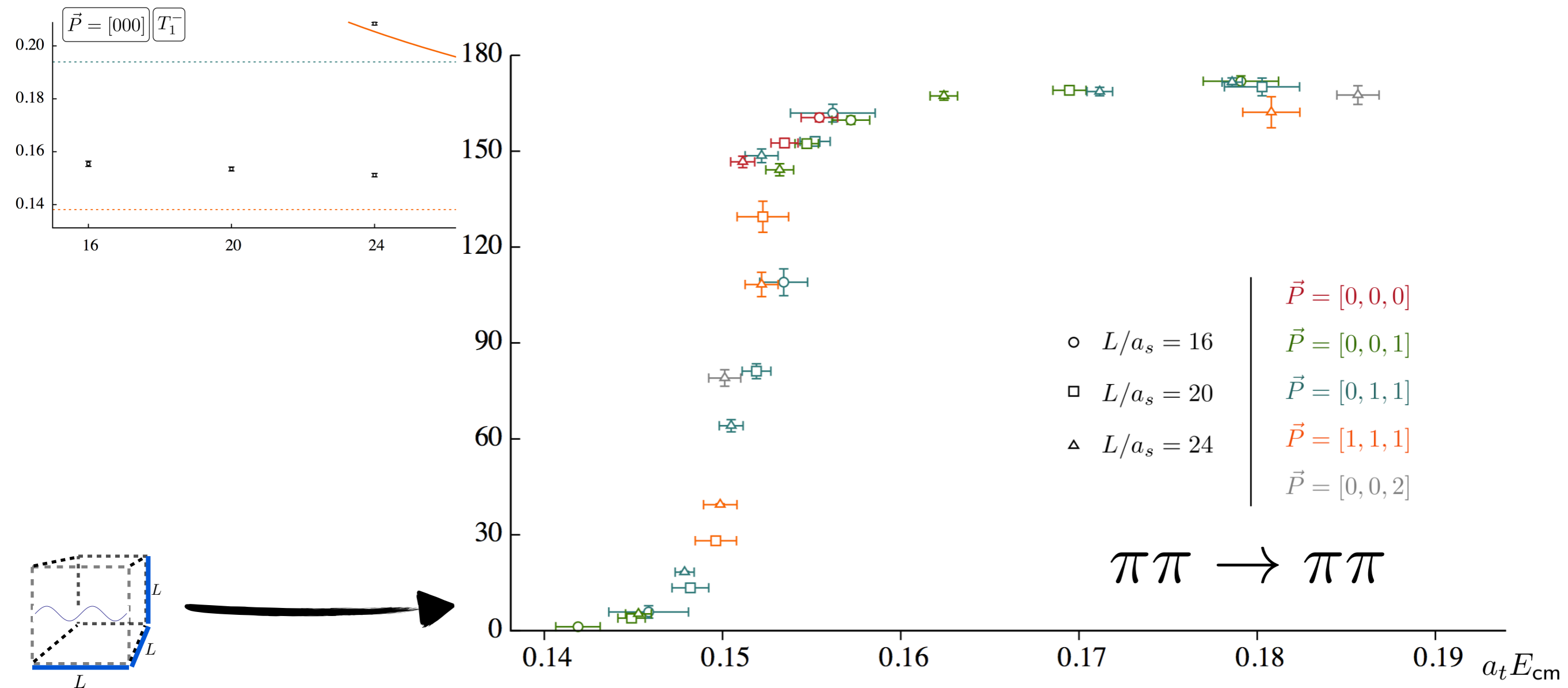
$$m_\pi = 391 \text{ MeV}$$

from Dudek, Edwards, Thomas in *Phys.Rev.* D87 (2013) 034505



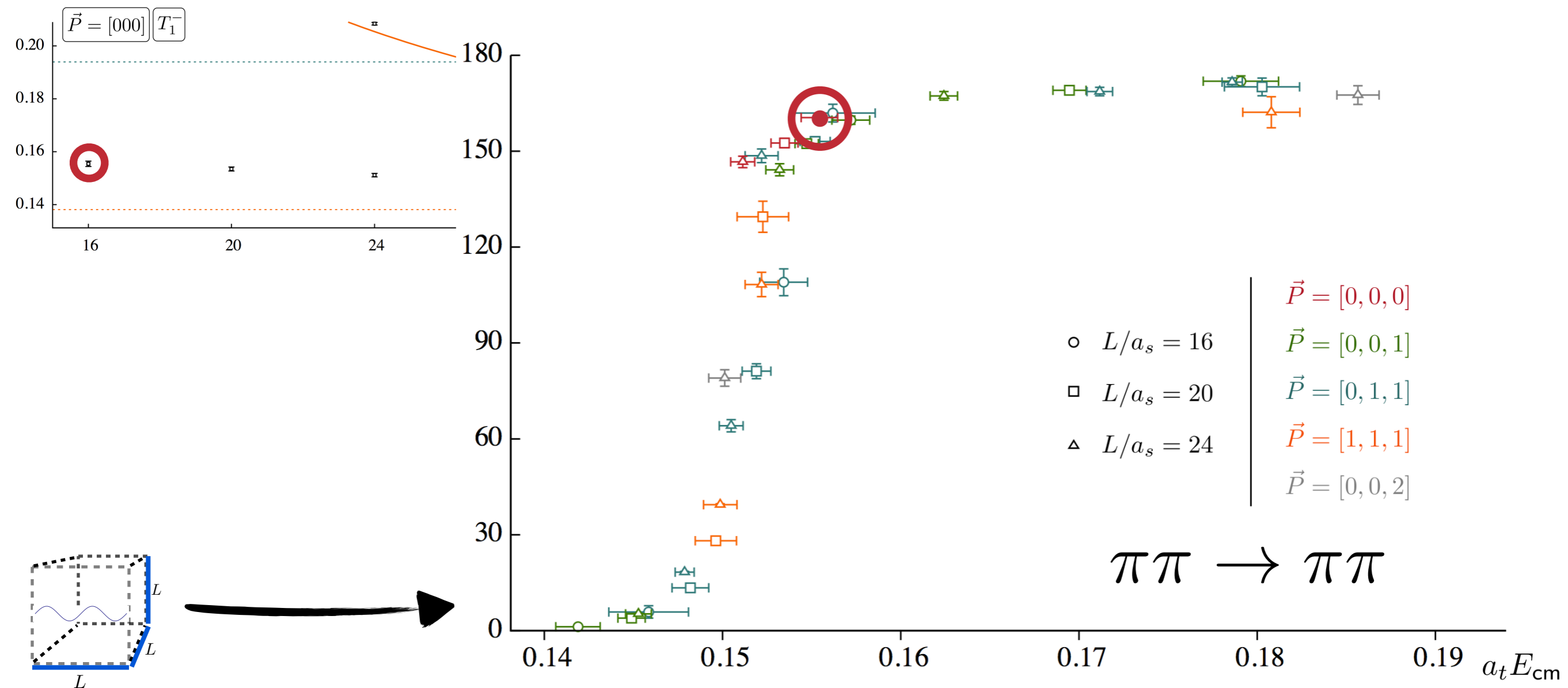
# Using Lüscher's Result (p-wave)

$$\cot \delta_{\ell=1}(E_n^*) = \frac{1}{\pi^{3/2} \gamma q_n^*} \mathcal{Z}_{00}^P [1, q_n^{*2}]$$



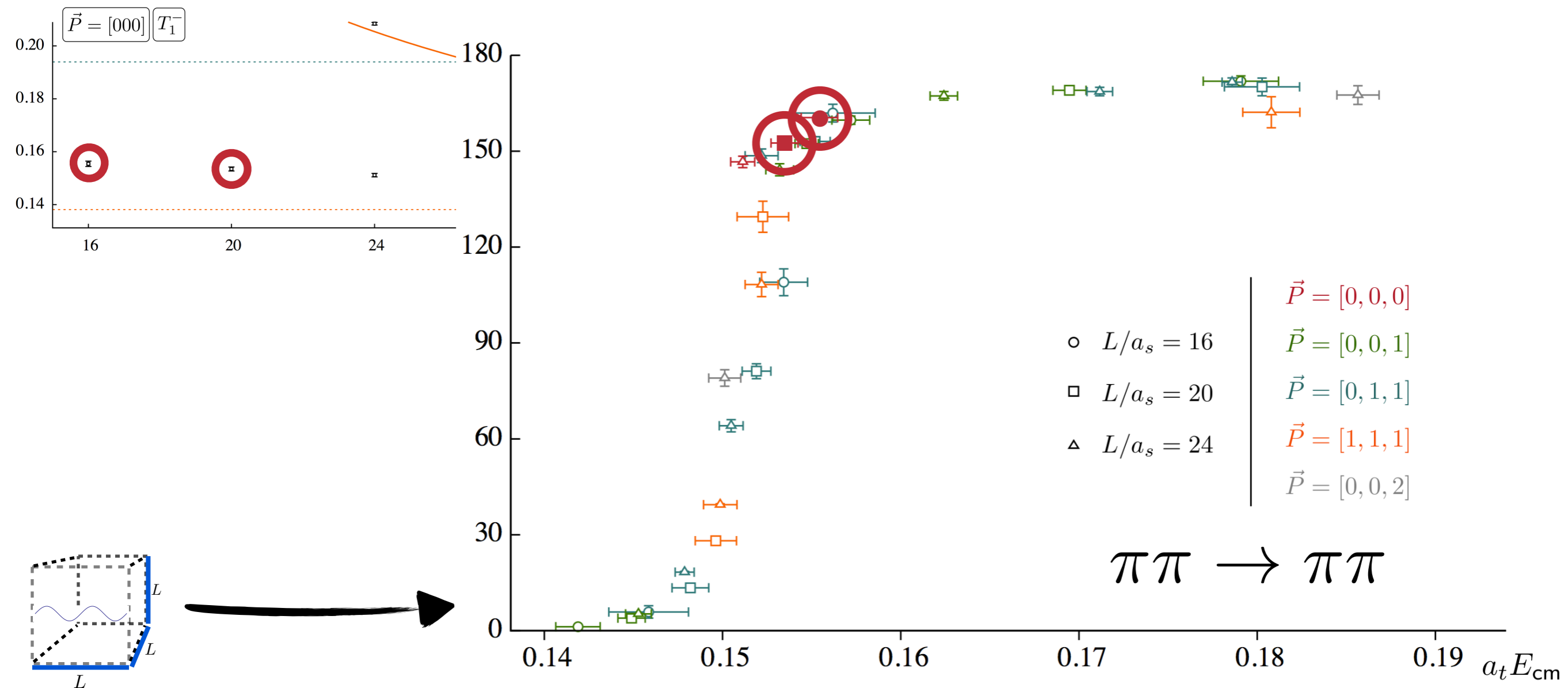
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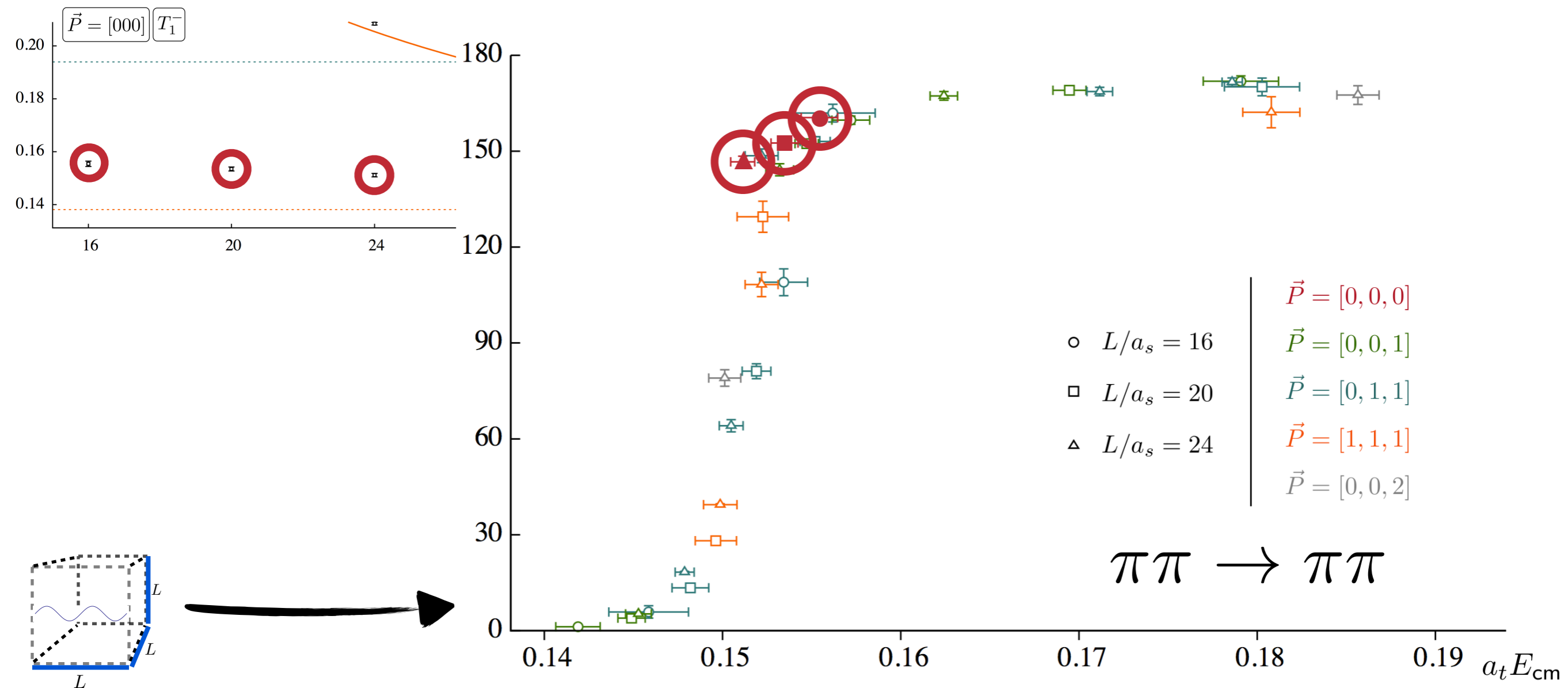
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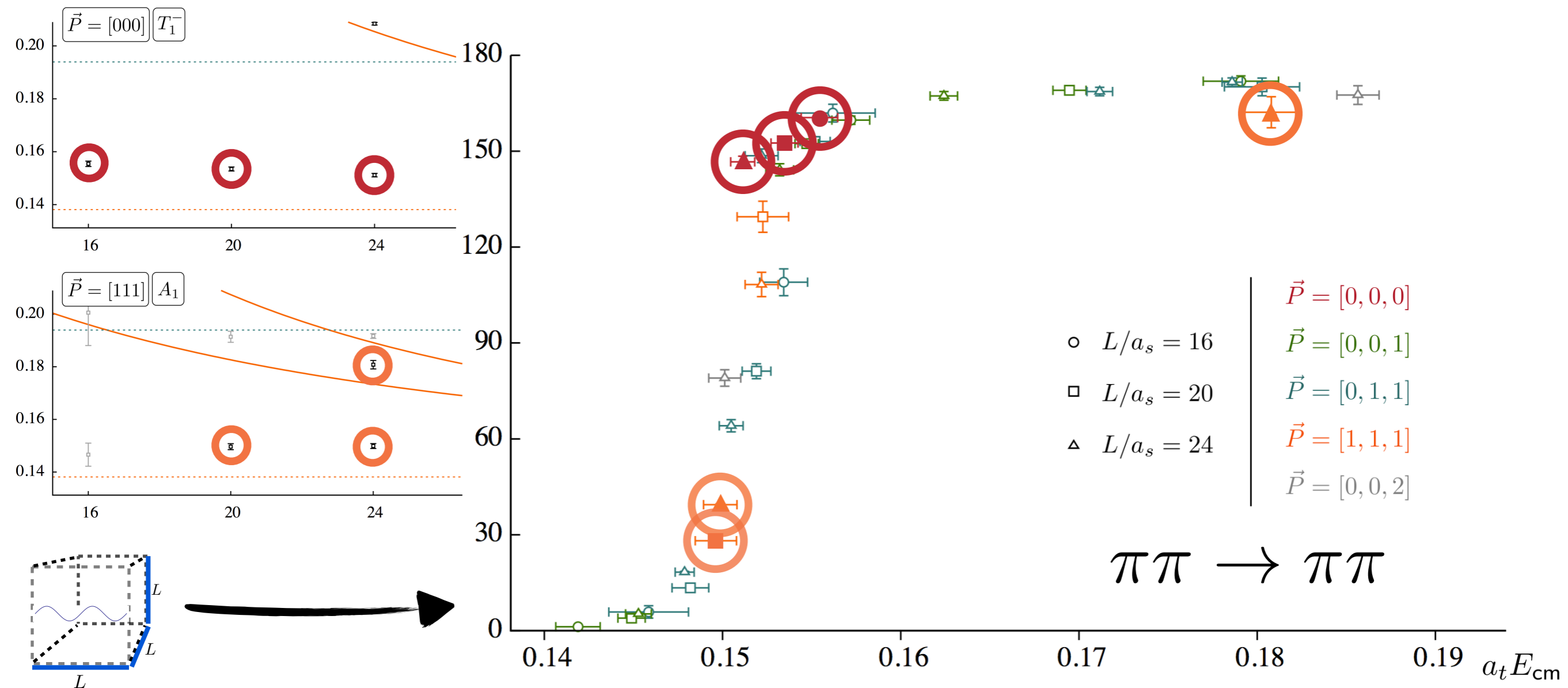
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Lüscher's result has since been generalized to accommodate  
**moving frames, non-identical particles,  
multiple two-particle channels, particles with spin**

Rummukainen and Gottlieb, *Nucl. Phys.* B450, 397 (1995)

Beane, Bedaque, Parreno, and Savage, *Nucl. Phys.* A747, 55 (2005)

Kim, Sachrajda, and Sharpe, *Nucl. Phys.* B727, 218 (2005)

Christ, Kim, Yamazaki, *Phys. Rev.* D72, 114506 (2005)

Bernard, Lage, Meißner, and Rusetsky, *JHEP*, 1101, 019 (2011)

MTH and Sharpe, *Phys.Rev.* D86 (2012) 016007

Briceño and Davoudi, *Phys.Rev.* D88 (2013) 094507

Li and Liu, *Phys. Rev.* D87, 014502 (2013)

Briceño, *Phys. Rev.* D 89, 074507 (2014)

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**The numerical implementation of the formalism  
has reached an impressive level**

Wilson, et.al. (2015) 1507.02599

Talks yesterday by Ben Hörz, John Bulava, Tadeusz Janowski, Gordon Donald, Dehua Guo

However, there is no general method for extracting scattering amplitudes involving more than two hadrons.

This limits LQCD investigation of...

**resonances which decay into more than two hadrons**

$$\omega(782) \rightarrow \pi\pi\pi \qquad N(1440) \rightarrow N\pi\pi$$

**two-particle scattering above three-particle thresholds**

$$\pi K \rightarrow \pi K, \pi\pi K$$



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**weak decays and transitions ala Lellouch-Lüscher**

$$K \rightarrow \pi\pi\pi$$

Lellouch and Lüscher, *Com.Math.Phys.* 219, 31 (2001)

Meyer, (2012) 1202.6675

Agadjanov, Bernard, Meißner, Rusetsky, *Nucl. Phys.* B886 (1014) 1199

Briceño, MTH, Walker-Loud, *Phys. Rev. D* 91, 034501 (2015)

MTH and Briceño, (2015) 1502.04314

# Outline

$1/L$  expansions

Nonperturbative studies in  
non-relativistic quantum theories

Three-particle bound state

Relativistic QFT in finite volume

# $1/L$ expansions

In 1957, Huang and Yang determined energy shift for  $n$  identical bosons in a box

K. Huang and C. Yang, *Phys. Rev.* 105 (1957) 767-775

$$E_0(n, L) = \frac{4\pi a}{M L^3} \left\{ \binom{n}{2} - \left(\frac{a}{\pi L}\right) \binom{n}{2} \mathcal{I} + \left(\frac{a}{\pi L}\right)^2 \left\{ \binom{n}{2} \mathcal{I}^2 - \left[ \binom{n}{2}^2 - 12 \binom{n}{3} - 6 \binom{n}{4} \right] \mathcal{J} \right\} \right\} + \mathcal{O}(L^{-6})$$

where  $a$  is the two-particle scattering length and

$$\mathcal{I} = \lim_{\Lambda \rightarrow \infty} \sum_{\substack{|\mathbf{i}| \leq \Lambda \\ \mathbf{i} \neq \mathbf{0}}} \frac{1}{|\mathbf{i}|^2} - 4\pi\Lambda = -8.91363291781$$

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**In 2007 Beane, Detmold and Savage pushed the order to  $1/L^6$  and the latter two calculated to  $1/L^7$  the next year**

Beane, S., Detmold, W. & Savage, M. *Phys. Rev.* D76 (2007) 074507

Detmold, W. & Savage, M. *Phys. Rev.* D77 (2008) 057502

**At  $1/L^6$  a three-particle contact term appears**

# $1/L$ expansions

Last year Detmold and Flynn performed a similar calculation for  
**matrix elements**

Detmold and Flynn, *Phys. Rev. D* 91, 074509 (2015)

$$\begin{aligned} \langle n|J|n\rangle = & n\alpha_1 + \frac{n\alpha_1 a^2}{\pi^2 L^2} \binom{n}{2} \mathcal{J} + \frac{\alpha_2}{L^3} \binom{n}{2} \\ & + \frac{2n\alpha_1 a^3}{\pi^3 L^3} \binom{n}{2} \left\{ \mathcal{K} \binom{n}{2} - \left[ \mathcal{I} \mathcal{J} + 4\mathcal{K} \binom{n-2}{1} + \mathcal{K} \binom{n-2}{2} \right] \right\} - \frac{2\alpha_2 a}{\pi L^4} \binom{n}{2} \mathcal{I} \\ & + \frac{n\alpha_1 a^4}{\pi^4 L^4} \left[ 3\mathcal{I}^2 \mathcal{J} + \mathcal{L} \left( 186 - \frac{241n}{2} + \frac{29}{2} n^2 \right) + \mathcal{J}^2 \left( \frac{n^2}{4} + \frac{3n}{4} - \frac{7}{2} \right) \right. \\ & \left. + \mathcal{I} \mathcal{K}(4n - 14) + \mathcal{U}(32n - 64) + \mathcal{V}(16n - 32) \right] + \mathcal{O}(1/L^5). \end{aligned}$$

Here  $\mathcal{I}, \mathcal{J}, \dots$  are known geometric constants  
and  $\alpha_1, \alpha_2$  are one- and two-boson current couplings

# Nonperturbative and non-relativistic

## Non-relativistic Faddeev analysis

In 2012, Polejaeva and Rusetzky derived a Lüscher-like result using  
**non-relativistic Faddeev equations**

Polejaeva and Rusetzky, *Eur. Phys. J. A*48, 67 (2012)

**Demonstrates that on-shell S-matrix determines spectrum**

**Difficult to extract scattering from the formalism**

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## Dimer formalism

In 2013, Briceño and Davoudi studied three-particles in finite-volume  
using the Dimer formalism

Briceño and Davoudi, *Phys. Rev. D*87, 094507 (2013)

**Recovered Lüscher result when two of the three become bound**

$$k \cot \delta = -k \cot \phi + \eta \frac{e^{-\gamma L}}{L}$$

**Final result involves an integral equation that one  
needs to solve numerically**

# Three-particle bound state

This year Meißner, Rios and Rusetsky determined the **finite-volume energy shift to a three-body bound state**

$$\Delta E = c \frac{\kappa^2}{m} \frac{|A|^2}{(kL)^{3/2}} \exp(-2\kappa L / \sqrt{3}) + \dots$$

Meißner, Rios and Rusektsky. *Phys. Rev. Lett.* 114, 091602 (2015)

**Assumes the unitary limit for two-particle scattering**

**Result derived using non-relativistic quantum mechanics**



# Relativistic QFT in finite volume

**based on**

MTH and Sharpe, *Phys. Rev. D* 90, 116003 (2014)

MTH and Sharpe, (2015) 1504.04248

MTH and Sharpe, *to appear*

**guided by**

Kim, Sachrajda, and Sharpe, *Nucl. Phys. B* 727, 218 (2005)

# Deriving Lüscher's Result

**Finite volume**

**Infinite volume**

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**single scalar, mass  $m$**

# Deriving Lüscher's Result

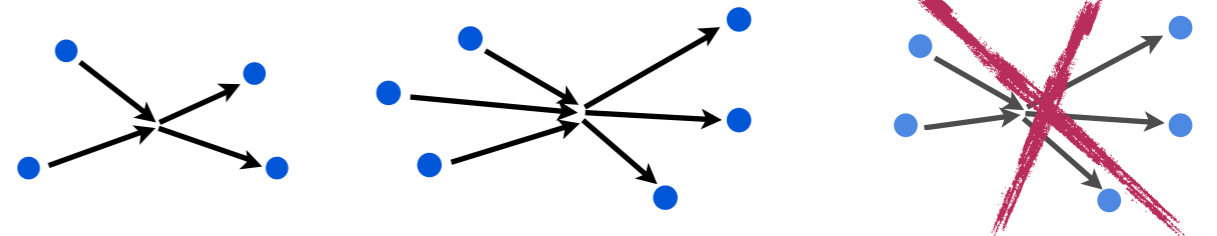
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single scalar, mass  $m$

relativistic field theory

$\mathbb{Z}_2$  symmetry



(For pions in QCD this is G-parity)

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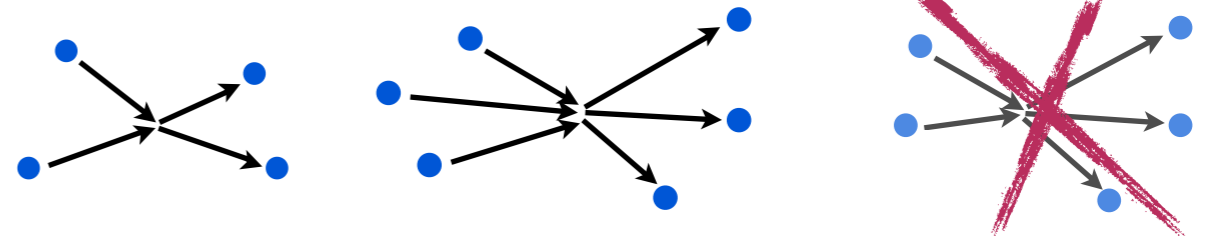
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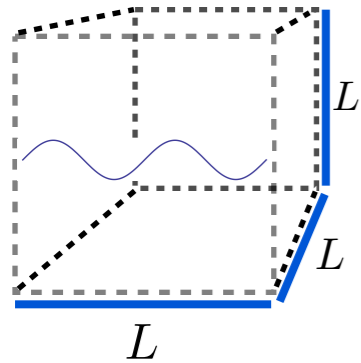


(For pions in QCD this is G-parity)

**Include all vertices**  
with even number of legs

# Deriving Lüscher's Result

## Finite volume



**cubic**, spatial volume  
(extent  $L$ )

**periodic** boundary  
conditions

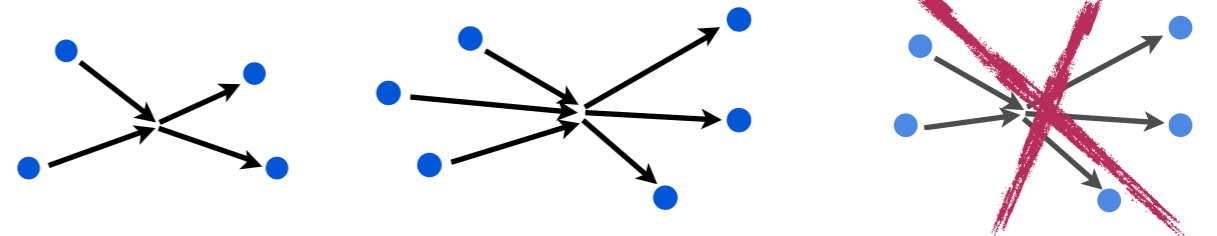
$$\vec{p} \in (2\pi/L)\mathbb{Z}^3$$

## Infinite volume

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relativistic field theory

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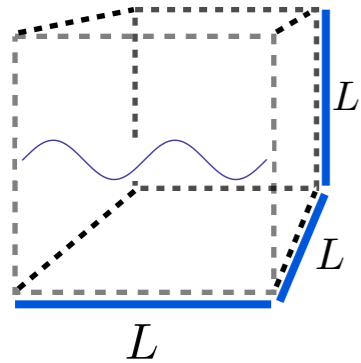


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time direction **infinite** and **Minkowski**

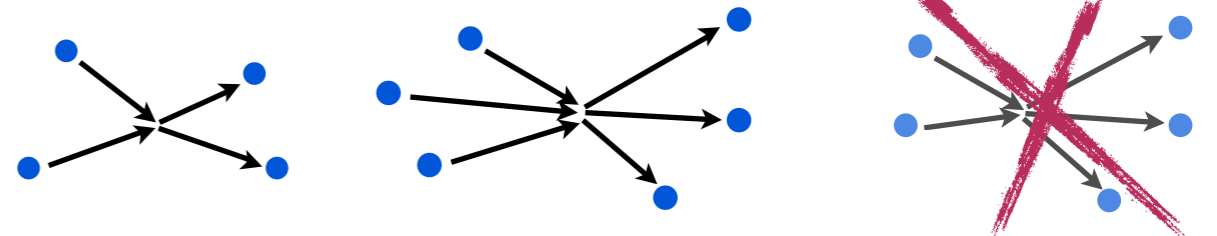


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**single scalar**, mass  $m$

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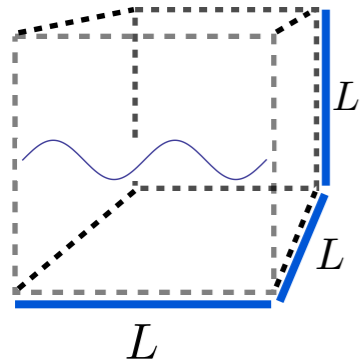


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$$\vec{p} \in (2\pi/L)\mathbb{Z}^3$$

time direction **infinite** and **Minkowski**

A horizontal double-headed arrow pointing left and right, indicating the time direction is infinite.

Take  $L$  large enough to ignore

**dropped**  
**throughout!**  $e^{-mL}$

A blue arrow points from the text "dropped throughout!" to the term  $e^{-mL}$ , which is underlined in blue.

Take space to be continuous

**lattice spacing set to**  
**zero**

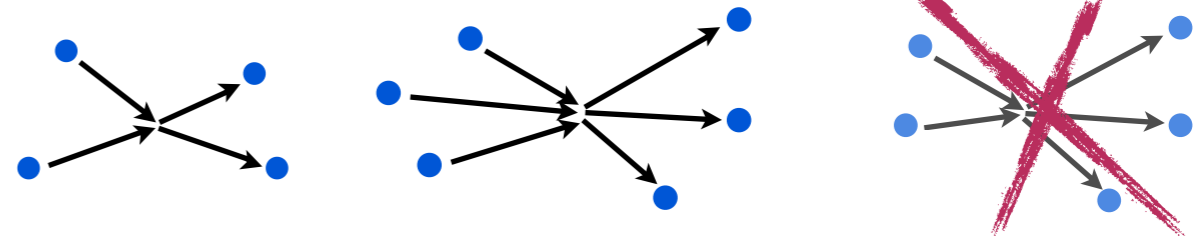
A blue arrow points from the text "lattice spacing set to zero" up towards the text "Take space to be continuous".

## Infinite volume

**single scalar**, mass  $m$

relativistic field theory

**$\mathbb{Z}_2$  symmetry**



(For pions in QCD this is G-parity)

**Include all vertices**  
with even number of legs



$$C_L(E, \vec{P}) \equiv \int_L d^4x e^{i(Ex^0 - \vec{P} \cdot \vec{x})} \langle 0 | \mathbf{T} \sigma(x) \sigma^\dagger(0) | 0 \rangle$$

$$C_L(E, \vec{P}) \equiv \int_L d^4x e^{i(Ex^0 - \vec{P} \cdot \vec{x})} \langle 0 | \mathbf{T} \sigma(x) \sigma^\dagger(0) | 0 \rangle$$

energy  $E$ , momentum  $\vec{P} = (2\pi/L)\vec{n}_P$

CM energy  $E^{*2} \equiv E^2 - \vec{P}^2$

$$C_L(E, \vec{P}) \equiv \int_L d^4x e^{i(Ex^0 - \vec{P} \cdot \vec{x})} \langle 0 | \mathbf{T} \sigma(x) \sigma^\dagger(0) | 0 \rangle$$

energy  $E$ , momentum  $\vec{P} = (2\pi/L)\vec{n}_P$

CM energy  $E^{*2} \equiv E^2 - \vec{P}^2$

interpolating field  
even particle quantum numbers

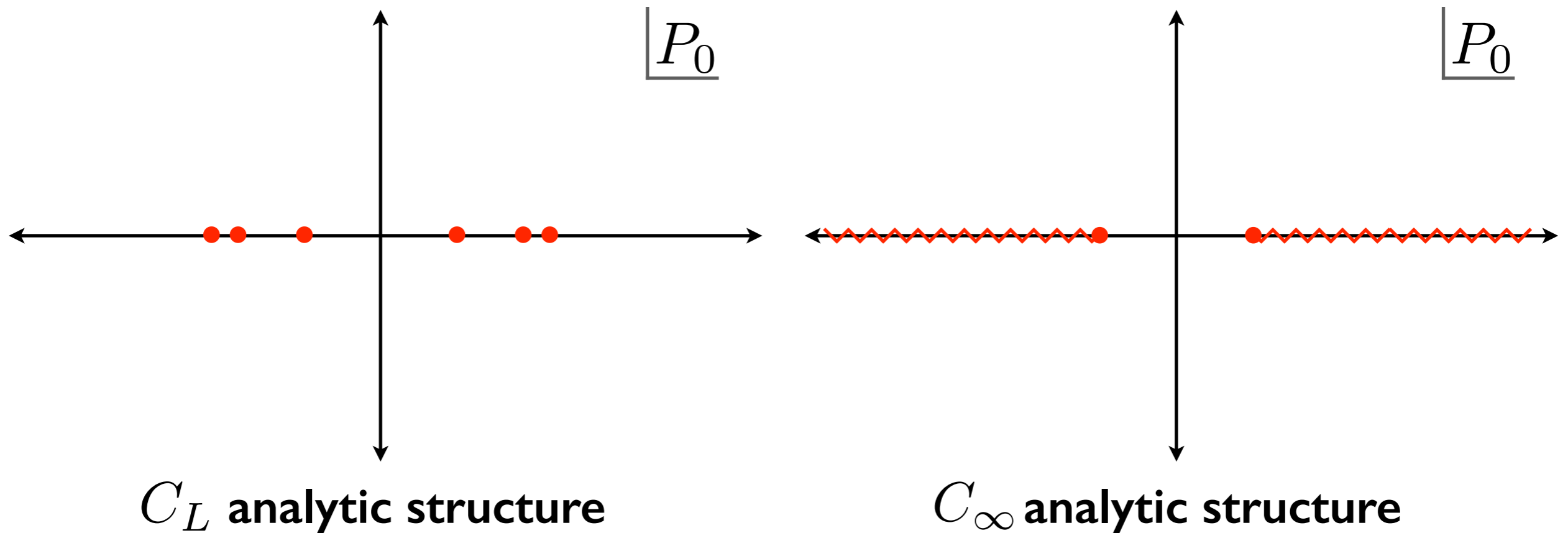
$$C_L(E, \vec{P}) \equiv \int_L d^4x e^{i(Ex^0 - \vec{P} \cdot \vec{x})} \langle 0 | T \sigma(x) \sigma^\dagger(0) | 0 \rangle$$

energy  $E$ , momentum  $\vec{P} = (2\pi/L)\vec{n}_P$

CM energy  $E^{*2} \equiv E^2 - \vec{P}^2$

interpolating field  
even particle quantum numbers

At fixed  $L, \vec{P}$ , poles in  $C_L$  give finite-volume spectrum



$$C_L(E, \vec{P}) \equiv \int_L d^4x e^{i(Ex^0 - \vec{P} \cdot \vec{x})} \langle 0 | \mathbf{T} \sigma(x) \sigma^\dagger(0) | 0 \rangle$$

energy  $E$ , momentum  $\vec{P} = (2\pi/L)\vec{n}_P$

CM energy  $E^{*2} \equiv E^2 - \vec{P}^2$

interpolating field  
even particle quantum numbers

**At fixed  $L, \vec{P}$ , poles in  $C_L$  give finite-volume spectrum**

**Calculate  $C_L(E, \vec{P})$  to all orders in perturbation theory and determine condition of divergence.**

$$C_L(E, \vec{P}) \equiv \int_L d^4x e^{i(Ex^0 - \vec{P} \cdot \vec{x})} \langle 0 | T \sigma(x) \sigma^\dagger(0) | 0 \rangle$$

energy  $E$ , momentum  $\vec{P} = (2\pi/L)\vec{n}_P$

CM energy  $E^{*2} \equiv E^2 - \vec{P}^2$

interpolating field  
even particle quantum numbers

At fixed  $L, \vec{P}$ , poles in  $C_L$  give finite-volume spectrum

Calculate  $C_L(E, \vec{P})$  to all orders in perturbation theory and determine condition of divergence.

$$C_L(E, \vec{P}) = \begin{array}{c} \text{---} \sigma^\dagger \text{---} \text{---} \sigma \text{---} \\ \text{---} \sigma^\dagger \text{---} \text{---} iK \text{---} \text{---} \sigma \text{---} \\ \text{---} \sigma^\dagger \text{---} \text{---} iK \text{---} \text{---} iK \text{---} \text{---} \sigma \text{---} + \dots \end{array}$$

$$C_L(E, \vec{P}) \equiv \int_L d^4x e^{i(Ex^0 - \vec{P} \cdot \vec{x})} \langle 0 | T \sigma(x) \sigma^\dagger(0) | 0 \rangle$$

energy  $E$ , momentum  $\vec{P} = (2\pi/L)\vec{n}_P$

CM energy  $E^{*2} \equiv E^2 - \vec{P}^2$

interpolating field  
even particle quantum numbers

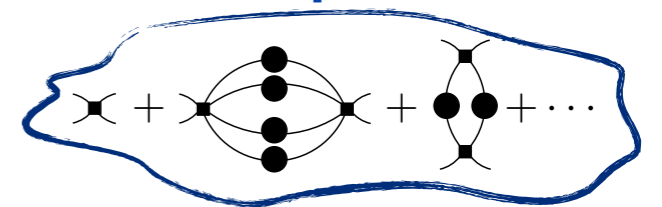
At fixed  $L, \vec{P}$ , poles in  $C_L$  give finite-volume spectrum

Calculate  $C_L(E, \vec{P})$  to all orders in perturbation theory and determine condition of divergence.

$$C_L(E, \vec{P}) = \text{diagram 1} + \text{diagram 2} + \text{diagram 3} + \dots$$

The diagrams are Feynman diagrams for the two-point function  $C_L(E, \vec{P})$ . Each diagram consists of a chain of vertices connected by lines. The first vertex is  $\sigma^\dagger$  and the last is  $\sigma$ . The internal vertices are  $iK$ . The first diagram has two internal vertices. The second has three. The third has four. Ellipses indicate higher-order terms.

Bethe Salpeter kernel



$$C_L(E, \vec{P}) \equiv \int_L d^4x e^{i(Ex^0 - \vec{P} \cdot \vec{x})} \langle 0 | T \sigma(x) \sigma^\dagger(0) | 0 \rangle$$

energy  $E$ , momentum  $\vec{P} = (2\pi/L)\vec{n}_P$

CM energy  $E^{*2} \equiv E^2 - \vec{P}^2$

interpolating field  
even particle quantum numbers

At fixed  $L, \vec{P}$ , poles in  $C_L$  give finite-volume spectrum

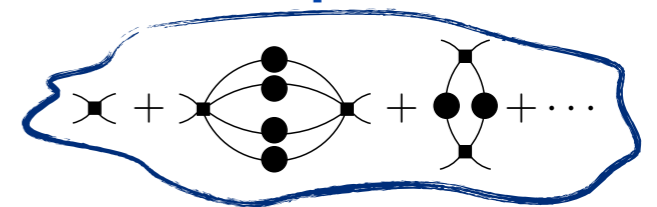
Calculate  $C_L(E, \vec{P})$  to all orders in perturbation theory and determine condition of divergence.

$$C_L(E, \vec{P}) = \text{diagram 1} + \text{diagram 2}$$

Diagram 1: A circle labeled  $\sigma^\dagger$  on the left and a circle labeled  $\sigma$  on the right. Two vertical lines connect them, each with a black dot. A dashed blue box encloses the two lines.

Diagram 2: A circle labeled  $\sigma^\dagger$  on the left, a circle labeled  $iK$  in the middle, and a circle labeled  $\sigma$  on the right. Two vertical lines connect  $\sigma^\dagger$  to  $iK$ , and two vertical lines connect  $iK$  to  $\sigma$ . Each line has a black dot. A dashed blue box encloses the two lines between  $\sigma^\dagger$  and  $iK$ .

Bethe Salpeter kernel



spatial loop momenta  
are summed

$$\frac{1}{L^3} \sum_{\vec{k} \in (2\pi/L)\mathbb{Z}^3} \int \frac{dk^0}{2\pi}$$

$$+ \text{diagram 3} + \dots$$

Diagram 3: A circle labeled  $\sigma^\dagger$  on the left, a circle labeled  $iK$  in the middle, another circle labeled  $iK$  in the middle, and a circle labeled  $\sigma$  on the right. Two vertical lines connect  $\sigma^\dagger$  to the first  $iK$ , and two vertical lines connect the second  $iK$  to  $\sigma$ . Each line has a black dot. A dashed blue box encloses the two lines between  $\sigma^\dagger$  and the first  $iK$ .



$$C_L(E, \vec{P}) \equiv \int_L d^4x e^{i(Ex^0 - \vec{P} \cdot \vec{x})} \langle 0 | T \sigma(x) \sigma^\dagger(0) | 0 \rangle$$

energy  $E$ , momentum  $\vec{P} = (2\pi/L)\vec{n}_P$

CM energy  $E^{*2} \equiv E^2 - \vec{P}^2$

interpolating field  
even particle quantum numbers

At fixed  $L, \vec{P}$ , poles in  $C_L$  give finite-volume spectrum

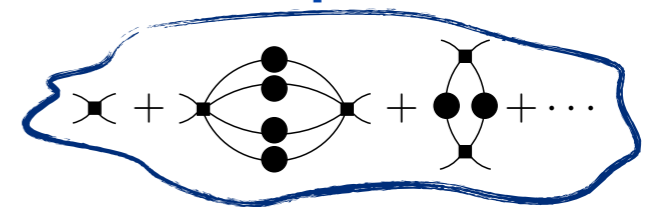
Calculate  $C_L(E, \vec{P})$  to all orders in perturbation theory and determine condition of divergence.

$$C_L(E, \vec{P}) = \text{diagram 1} + \text{diagram 2}$$

Diagram 1: A circle labeled  $\sigma^\dagger$  on the left and a circle labeled  $\sigma$  on the right. Two vertical lines connect them, each with a black dot. A dashed box encloses these two lines.

Diagram 2: A circle labeled  $\sigma^\dagger$  on the left and a circle labeled  $\sigma$  on the right. Two vertical lines connect them, each with a black dot. A dashed box encloses these lines. A circle labeled  $iK$  is inserted between the lines. Another dashed box encloses the  $iK$  circle and the lines it connects.

Bethe Salpeter kernel



spatial loop momenta  
are summed

$$\frac{1}{L^3} \sum_{\vec{k} \in (2\pi/L)\mathbb{Z}^3} \int \frac{dk^0}{2\pi}$$

$$+ \text{diagram 3} + \dots$$

Diagram 3: Similar to diagram 2, but with two  $iK$  circles in series between the lines.

If  $E^* < 4m$  then  $K_L = K_\infty + \mathcal{O}(e^{-mL})$

$$C_L(E, \vec{P}) = \text{diagram}_1 + \text{diagram}_2 + \text{diagram}_3 + \dots$$

The equation shows a series of Feynman diagrams for the correlation function  $C_L(E, \vec{P})$ . The first diagram consists of two vertices,  $\sigma^\dagger$  and  $\sigma$ , connected by two arcs. A dashed box encloses the two internal vertices. The second diagram is similar but includes a vertex labeled  $iK$  between the two internal vertices. The third diagram includes two  $iK$  vertices. The series continues with an ellipsis.

Next we introduce an important identity

**off-shell** **on-shell**

The identity is shown as a diagrammatic equation. On the left, a diagram with two vertices  $\sigma^\dagger$  and  $\sigma$  connected by two arcs and two internal vertices (enclosed in a dashed box) is subtracted from a similar diagram with two vertices  $\sigma^\dagger$  and  $\sigma$  connected by two arcs and two internal vertices. This is equal to a diagram with two vertices  $\sigma^\dagger$  and  $\sigma$  connected by two horizontal lines and a dashed box labeled  $F$ .

$\frac{1}{L^3} \sum_{\vec{k}}$   $\int_{\vec{k}}$

**contains all power-law corrections**

$$C_L(E, \vec{P}) = \sigma^\dagger \text{---} \text{---} \sigma + \sigma^\dagger \text{---} \text{---} iK \text{---} \text{---} \sigma + \sigma^\dagger \text{---} \text{---} iK \text{---} \text{---} iK \text{---} \text{---} \sigma + \dots$$

$$\frac{1}{L^3} \sum_{\vec{k}} \text{---} = \int_{\vec{k}} \text{---} + \text{---} F$$

**Can be applied in all two-particle loops**

$$C_L(E, \vec{P}) = \sigma^\dagger \begin{array}{c} \bullet \\ \bullet \end{array} \sigma + \sigma^\dagger \begin{array}{c} \bullet \\ \bullet \end{array} iK \begin{array}{c} \bullet \\ \bullet \end{array} \sigma + \sigma^\dagger \begin{array}{c} \bullet \\ \bullet \end{array} iK \begin{array}{c} \bullet \\ \bullet \end{array} iK \begin{array}{c} \bullet \\ \bullet \end{array} \sigma + \dots$$

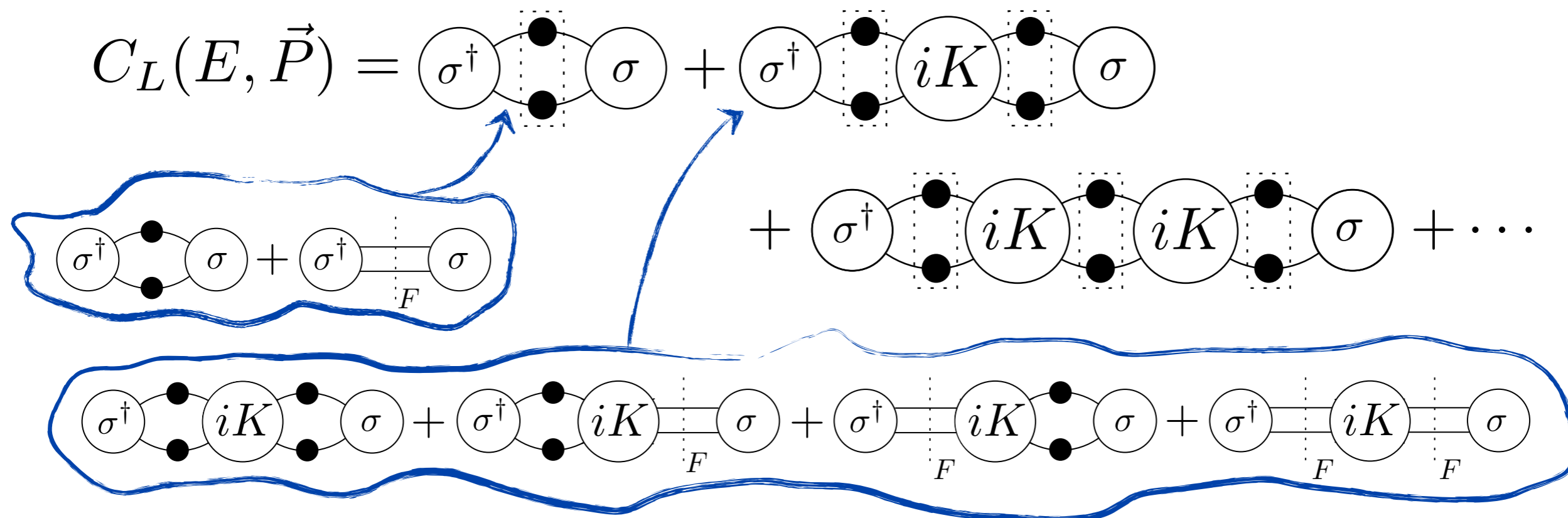
The diagram illustrates the expansion of the Green's function  $C_L(E, \vec{P})$ . The main equation shows a series of terms:
 

- Term 1:  $\sigma^\dagger$  connected to  $\sigma$  via two vertices (black dots).
- Term 2:  $\sigma^\dagger$  connected to  $iK$  via two vertices, which is then connected to  $\sigma$  via two vertices.
- Term 3:  $\sigma^\dagger$  connected to  $iK$  via two vertices, which is connected to another  $iK$  via two vertices, which is then connected to  $\sigma$  via two vertices.
- Ellipsis:  $+\dots$

A blue box highlights the first two terms, with a blue arrow pointing from the second term to the first term of the main equation. Another blue box highlights the first three terms, with a blue arrow pointing from the third term to the second term of the main equation. A large blue box at the bottom highlights the first four terms, with blue arrows pointing from the fourth, third, second, and first terms to their respective positions in the main equation.

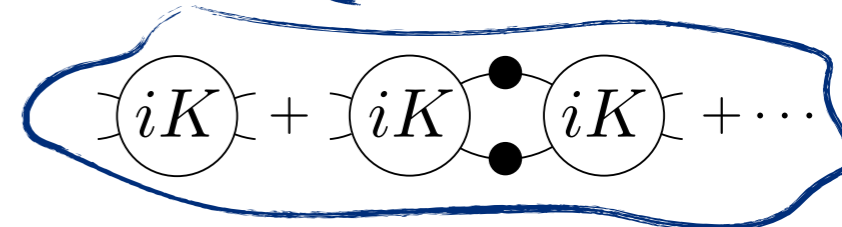
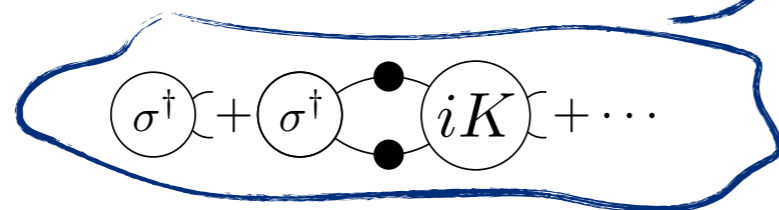
The terms in the blue boxes are:
 

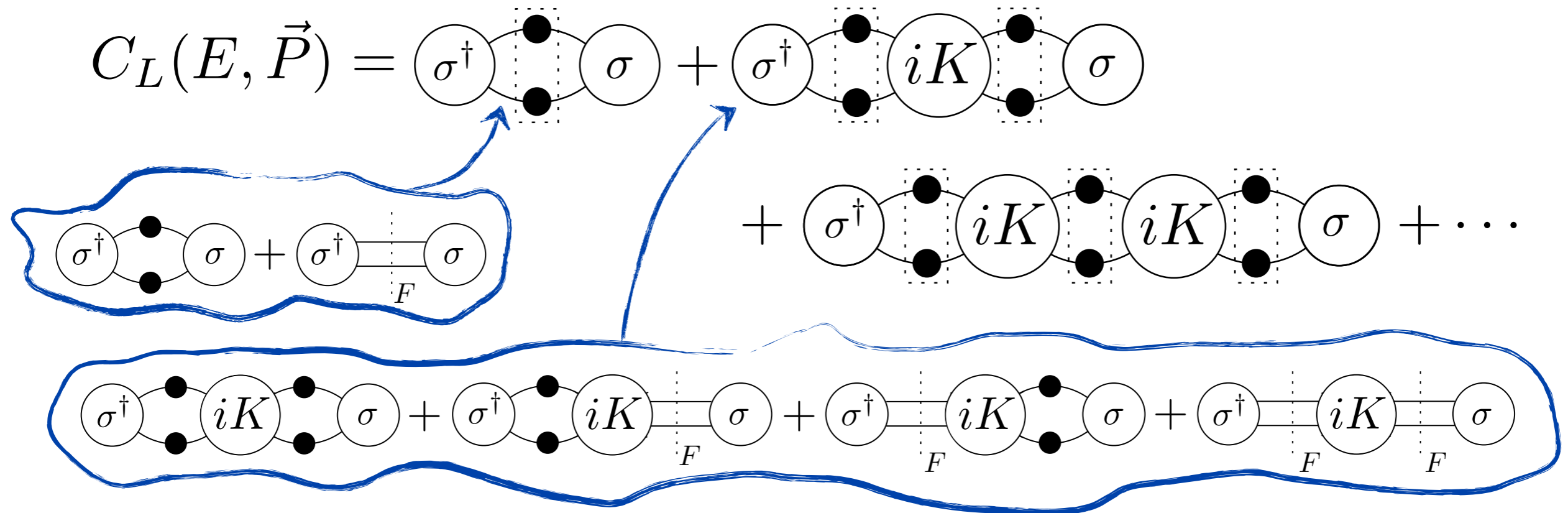
- Box 1:  $\sigma^\dagger \begin{array}{c} \bullet \\ \bullet \end{array} \sigma + \sigma^\dagger \begin{array}{c} \bullet \\ \bullet \end{array} \sigma$  (with a vertical dashed line labeled  $F$  between the two  $\sigma$  nodes).
- Box 2:  $\sigma^\dagger \begin{array}{c} \bullet \\ \bullet \end{array} iK \begin{array}{c} \bullet \\ \bullet \end{array} \sigma + \sigma^\dagger \begin{array}{c} \bullet \\ \bullet \end{array} iK \begin{array}{c} \bullet \\ \bullet \end{array} \sigma$  (with a vertical dashed line labeled  $F$  between the  $iK$  and  $\sigma$  nodes).
- Box 3:  $\sigma^\dagger \begin{array}{c} \bullet \\ \bullet \end{array} iK \begin{array}{c} \bullet \\ \bullet \end{array} iK \begin{array}{c} \bullet \\ \bullet \end{array} \sigma + \sigma^\dagger \begin{array}{c} \bullet \\ \bullet \end{array} iK \begin{array}{c} \bullet \\ \bullet \end{array} \sigma + \sigma^\dagger \begin{array}{c} \bullet \\ \bullet \end{array} iK \begin{array}{c} \bullet \\ \bullet \end{array} \sigma + \sigma^\dagger \begin{array}{c} \bullet \\ \bullet \end{array} iK \begin{array}{c} \bullet \\ \bullet \end{array} \sigma$  (with vertical dashed lines labeled  $F$  between the  $iK$  and  $\sigma$  nodes in each of the four terms).



**Now regroup by number of  $F$  cuts**

$$C_L(E, \vec{P}) = C_\infty(E, \vec{P}) + \begin{array}{c} \text{---} \\ | \\ A \text{---} A' \\ | \\ F \end{array} + \begin{array}{c} \text{---} \\ | \\ A \text{---} i\mathcal{M} \text{---} A' \\ | \quad | \\ F \quad F \end{array} + \dots$$

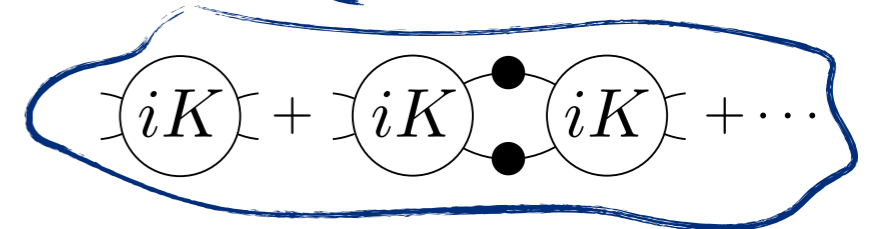
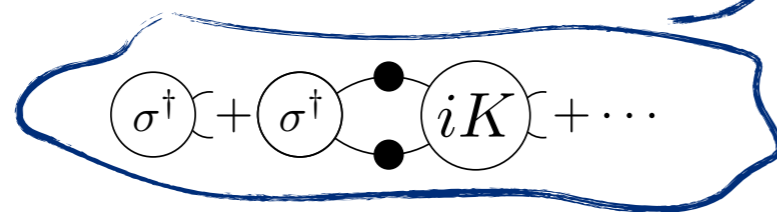




**Now regroup by number of  $F$  cuts**

$$C_L(E, \vec{P}) = C_\infty(E, \vec{P}) + \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} \begin{array}{c} \bullet \\ \bullet \end{array} \begin{array}{c} \bullet \\ \bullet \end{array} \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} + \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} \begin{array}{c} \bullet \\ \bullet \end{array} i\mathcal{M} \begin{array}{c} \bullet \\ \bullet \end{array} \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} + \dots$$

The regrouped equation shows  $C_L(E, \vec{P}) = C_\infty(E, \vec{P}) + \dots$ . The first term is  $C_\infty(E, \vec{P})$ , which is the sum of terms with 0  $F$  cuts. The subsequent terms are grouped by the number of  $F$  cuts: the first term has 1  $F$  cut and contains  $A$  and  $A'$ ; the second term has 1  $F$  cut and contains  $A$ ,  $i\mathcal{M}$ , and  $A'$ .



**As Promised!**

**Infinite-volume on-shell two-to-two scattering amplitude**

$$C_L(E, \vec{P}) = C_\infty(E, \vec{P})$$

$$\begin{aligned}
 &+ \begin{array}{c} \textcircled{A} \text{---} \textcircled{A'} \\ \vdots \quad \quad \quad \vdots \\ F \end{array} + \begin{array}{c} \textcircled{A} \text{---} \textcircled{i\mathcal{M}} \text{---} \textcircled{A'} \\ \vdots \quad \quad \quad \vdots \quad \quad \quad \vdots \\ F \quad \quad \quad \quad \quad \quad F \end{array} \\
 &+ \begin{array}{c} \textcircled{A} \text{---} \textcircled{i\mathcal{M}} \text{---} \textcircled{i\mathcal{M}} \text{---} \textcircled{A'} \\ \vdots \quad \quad \quad \vdots \quad \quad \quad \vdots \quad \quad \quad \vdots \\ F \quad \quad \quad \quad \quad \quad F \quad \quad \quad \quad \quad \quad F \end{array} + \dots
 \end{aligned}$$

$$C_L(E, \vec{P}) = C_\infty(E, \vec{P}) + \sum_{n=0}^{\infty} A' iF [i\mathcal{M}_{2 \rightarrow 2} iF]^n A$$

$$C_L(E, \vec{P}) = C_\infty(E, \vec{P}) + A' iF \frac{1}{1 - i\mathcal{M}_{2 \rightarrow 2} iF} A$$

# Two-particle review

1

$$C_L(E, \vec{P}) = \begin{array}{c} \sigma^\dagger \text{---} \text{---} \sigma + \sigma^\dagger \text{---} iK \text{---} \sigma \\ + \sigma^\dagger \text{---} iK \text{---} iK \text{---} \sigma + \dots \end{array}$$

The diagrammatic expansion of the two-particle Green's function  $C_L(E, \vec{P})$  is shown. The first row contains the bare propagator  $\sigma^\dagger \text{---} \text{---} \sigma$  and the first-order correction  $\sigma^\dagger \text{---} iK \text{---} \sigma$ . The second row shows the second-order correction  $\sigma^\dagger \text{---} iK \text{---} iK \text{---} \sigma$  and an ellipsis. A blue callout box highlights the expansion of the  $iK$  self-energy vertex, showing a sum of diagrams: a tadpole diagram, a diagram with a loop and a self-energy insertion, and a diagram with a loop and a self-energy insertion on the other line, followed by an ellipsis.

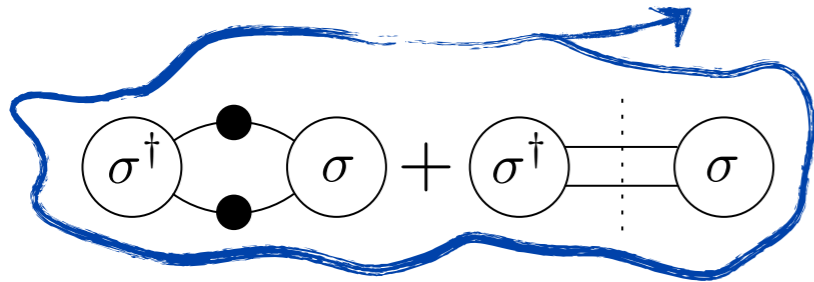


# Two-particle review

1

$$C_L(E, \vec{P}) = \sigma^\dagger \begin{array}{c} \bullet \\ \bullet \end{array} \sigma + \sigma^\dagger \begin{array}{c} \bullet \\ \bullet \end{array} iK \begin{array}{c} \bullet \\ \bullet \end{array} \sigma$$

2



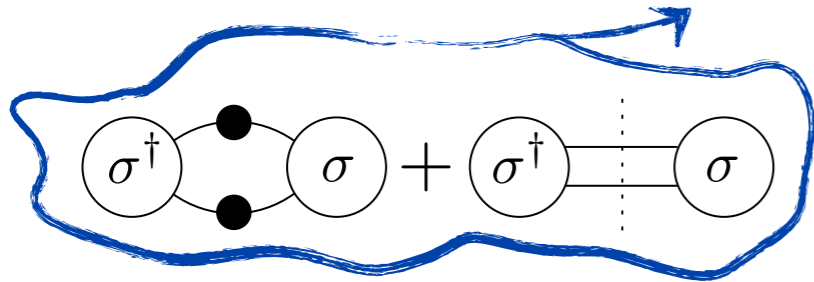
$$+ \sigma^\dagger \begin{array}{c} \bullet \\ \bullet \end{array} iK \begin{array}{c} \bullet \\ \bullet \end{array} iK \begin{array}{c} \bullet \\ \bullet \end{array} \sigma + \dots$$

# Two-particle review

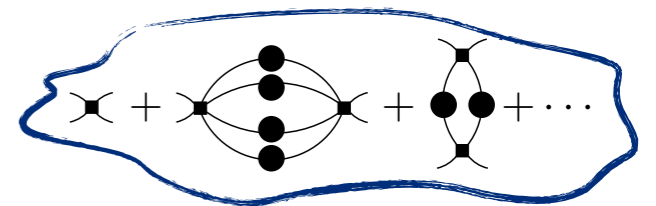
1

$$C_L(E, \vec{P}) = \sigma^\dagger \begin{array}{c} \bullet \\ \bullet \end{array} \sigma + \sigma^\dagger \begin{array}{c} \bullet \\ \bullet \end{array} iK \begin{array}{c} \bullet \\ \bullet \end{array} \sigma + \dots$$

2



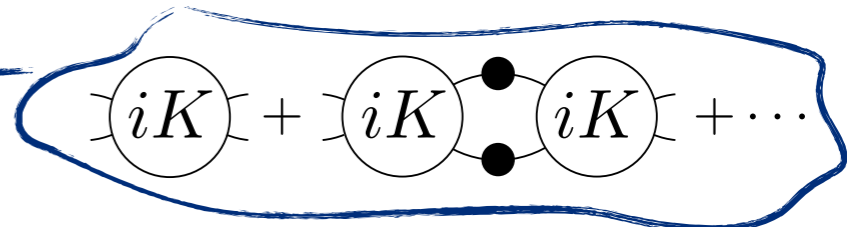
$$+ \sigma^\dagger \begin{array}{c} \bullet \\ \bullet \end{array} iK \begin{array}{c} \bullet \\ \bullet \end{array} iK \begin{array}{c} \bullet \\ \bullet \end{array} \sigma + \dots$$



$$C_L(E, \vec{P}) = C_\infty(E, \vec{P})$$

$$+ \begin{array}{c} A \\ \hline F \\ A' \end{array} + \begin{array}{c} A \\ \hline F \\ i\mathcal{M} \\ \hline F \\ A' \end{array}$$

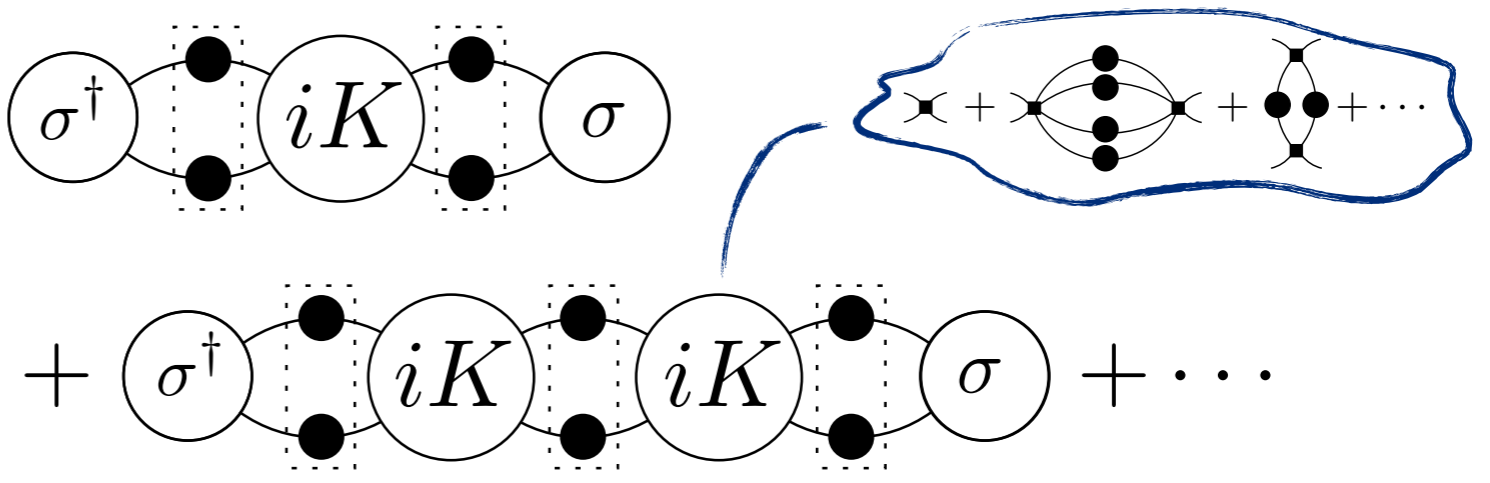
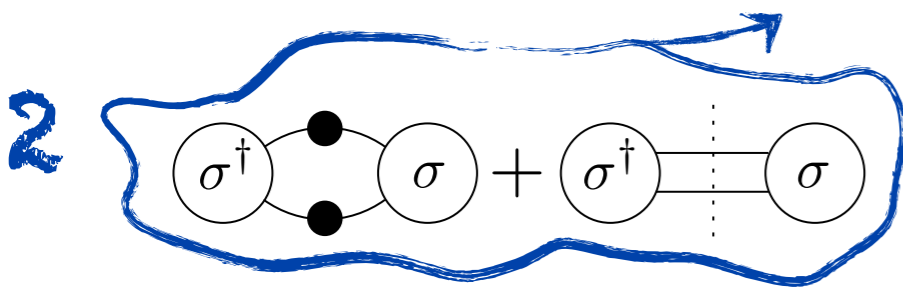
$$+ \begin{array}{c} A \\ \hline F \\ i\mathcal{M} \\ \hline F \\ i\mathcal{M} \\ \hline F \\ A' \end{array} + \dots$$



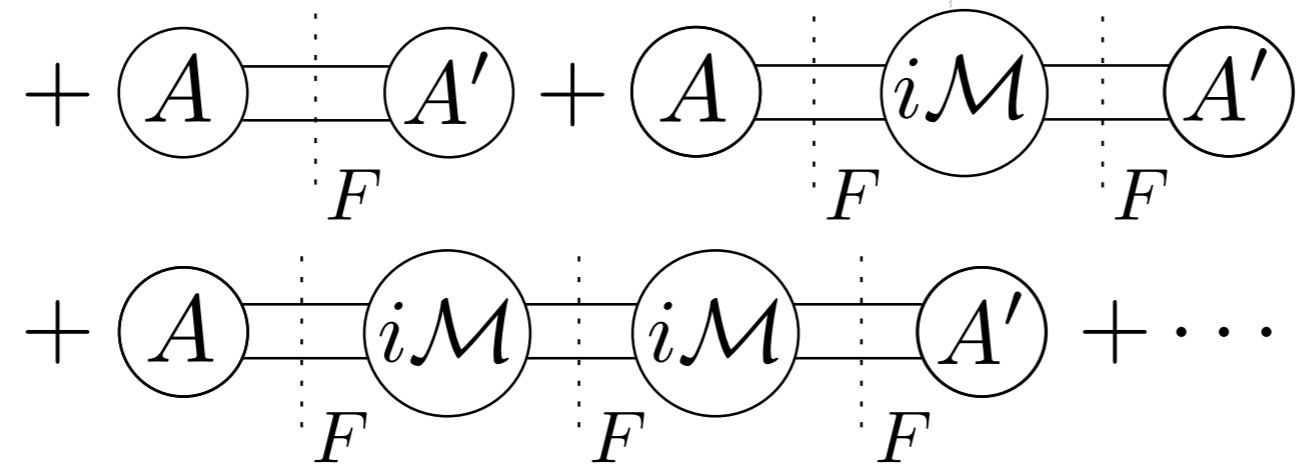
3

# Two-particle review

$$C_L(E, \vec{P}) = \sigma^\dagger \begin{array}{c} \bullet \\ \bullet \end{array} \sigma + \sigma^\dagger \begin{array}{c} \bullet \\ \bullet \end{array} iK \begin{array}{c} \bullet \\ \bullet \end{array} \sigma$$



$$C_L(E, \vec{P}) = C_\infty(E, \vec{P})$$



$$C_L(E, \vec{P}) = C_\infty(E, \vec{P}) + A' i F \frac{1}{1 - i \mathcal{M}_{2 \rightarrow 2} i F} A$$

no poles

no poles

no poles

4

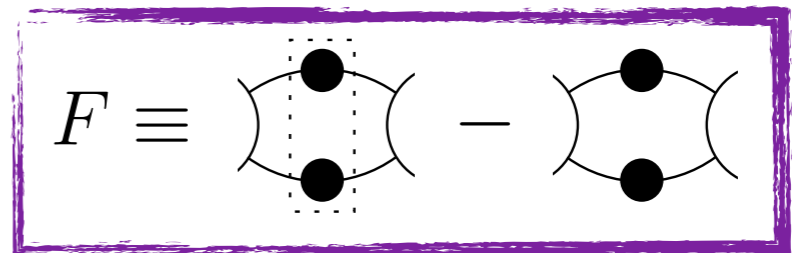
# Two-particle result

At fixed  $(L, \vec{P})$ , finite-volume spectrum is all solutions to

$$\det[1 - i\mathcal{M}_{2 \rightarrow 2} iF] = 0$$

diagonal matrix in  
angular momentum space

kinematic, not diagonal  
(related to Lüscher Zeta function)

$$F \equiv \text{[diagram 1]} - \text{[diagram 2]}$$


# Two-particle result

At fixed  $(L, \vec{P})$ , finite-volume spectrum is all solutions to

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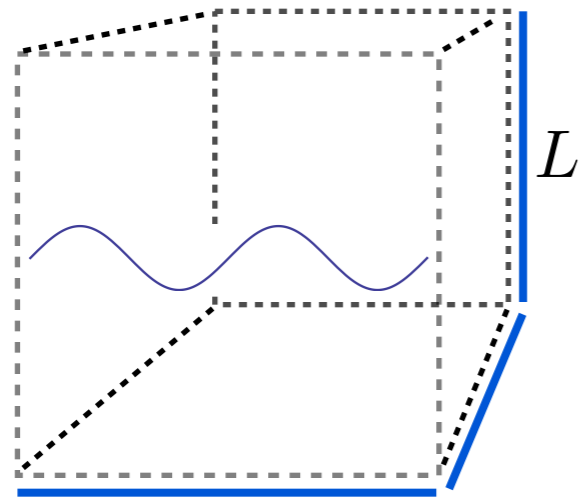
kinematic, not diagonal  
(related to Lüscher Zeta function)

$$F \equiv \text{[diagram: two vertices connected by two lines, with a dashed box around the top vertex]} - \text{[diagram: two vertices connected by two lines, with a dashed box around the bottom vertex]}$$

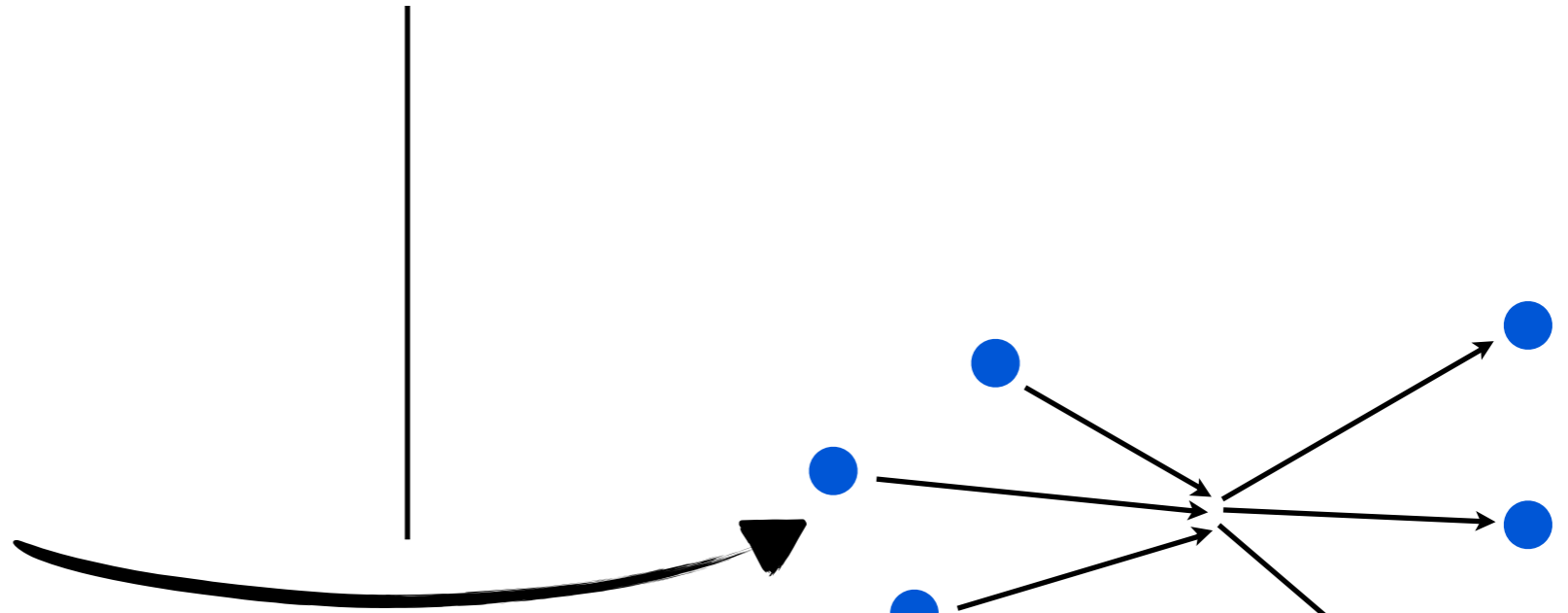
Can be rewritten as

$$\det \left[ \cot \delta(E_n^*) + \cot \phi(E_n, \vec{P}, L) \right] = 0$$

# Now, three particles in a box

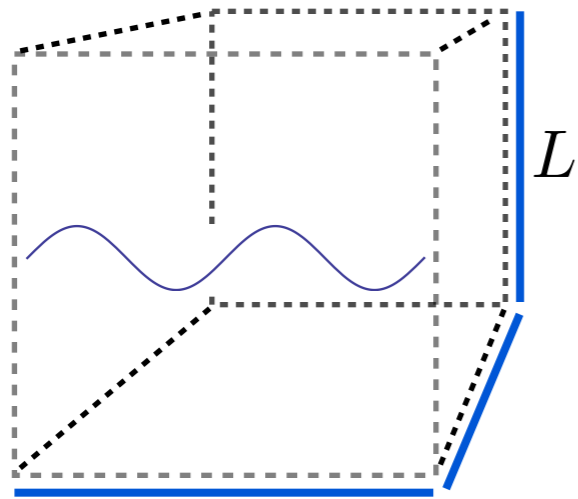


**Finite volume**

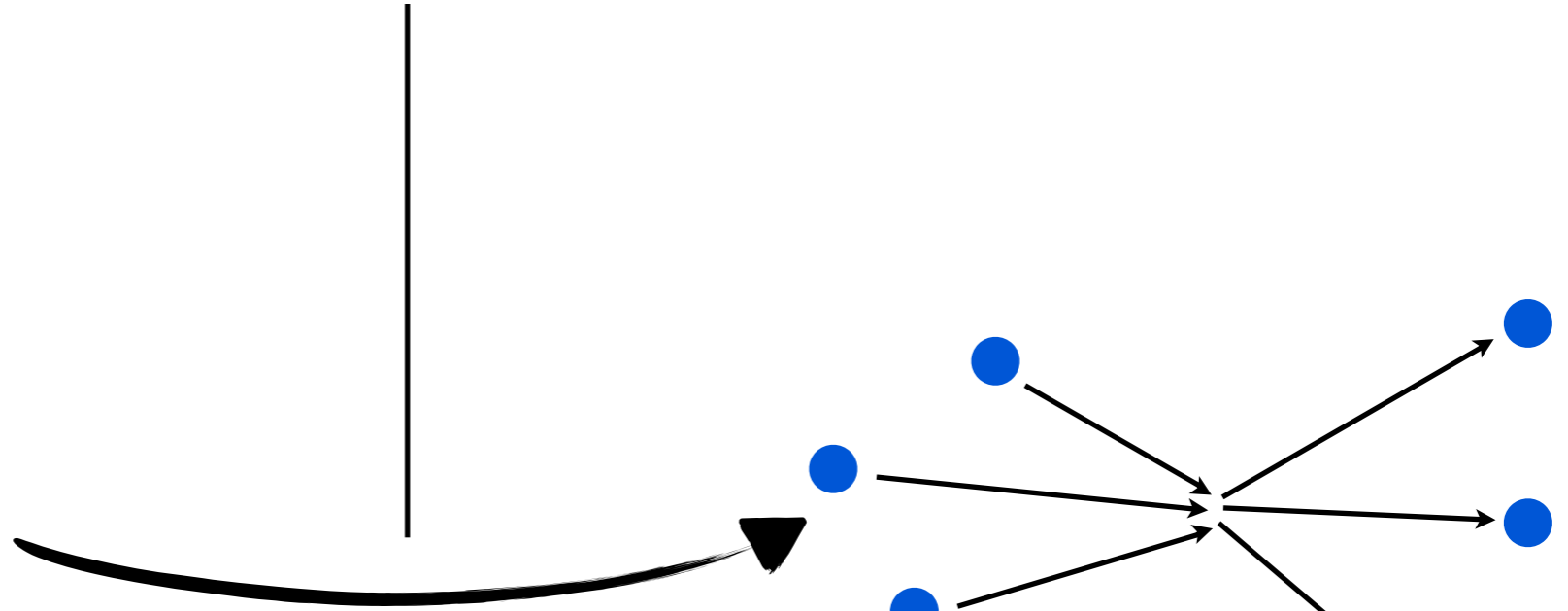


**Infinite volume**

# Now, three particles in a box

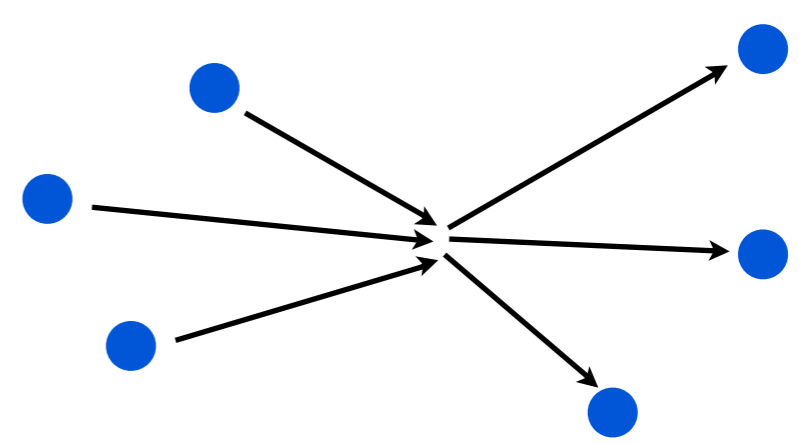


**Finite volume**



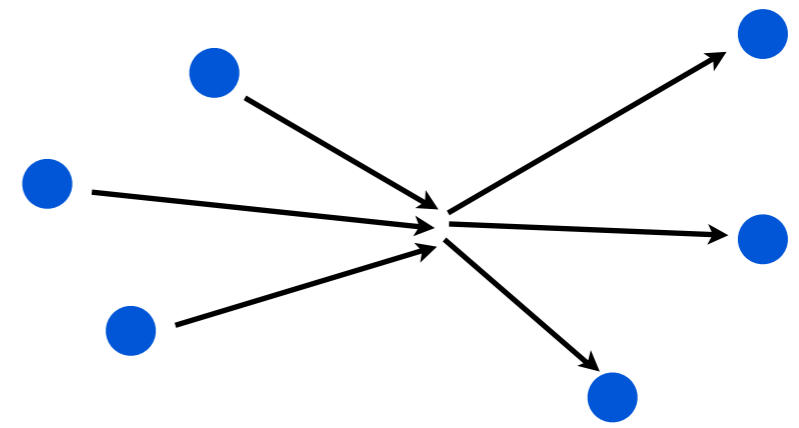
**Infinite volume**

**Infinite volume**



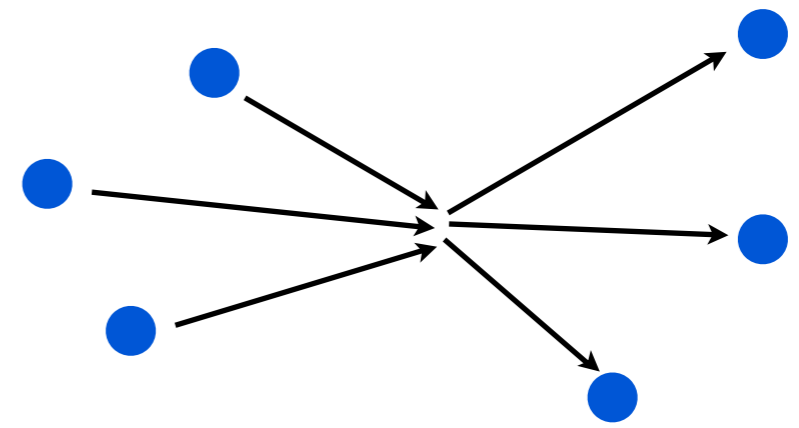


**Infinite volume**



**Degrees of freedom for three on-shell particles with  $(E, \vec{P})$**

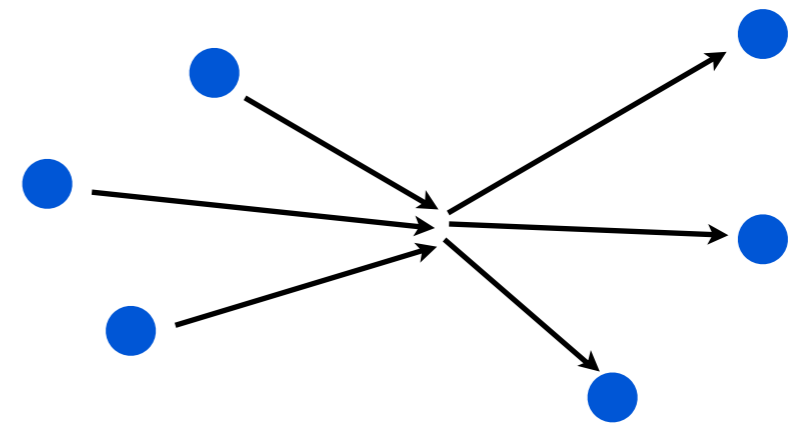
# Infinite volume



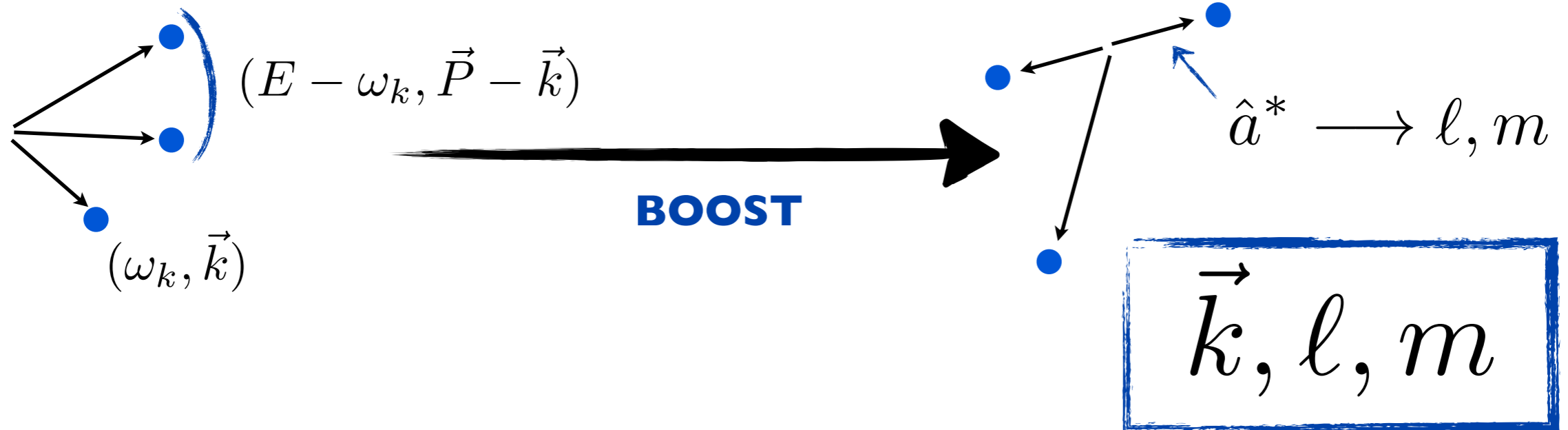
Degrees of freedom for three on-shell particles with  $(E, \vec{P})$



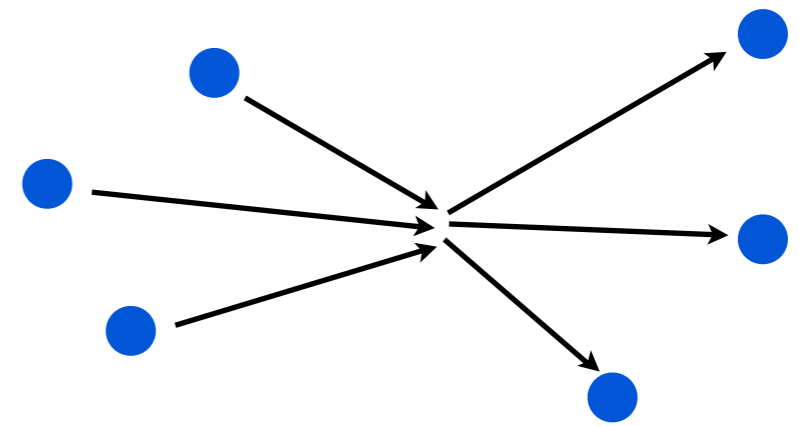
# Infinite volume



Degrees of freedom for three on-shell particles with  $(E, \vec{P})$



# Infinite volume



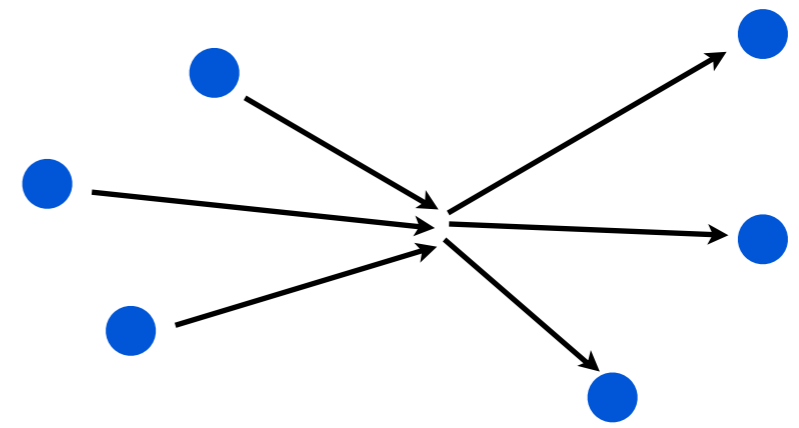
Degrees of freedom for three on-shell particles with  $(E, \vec{P})$



Three particle divergences

$$\vec{k}, l, m$$

# Infinite volume

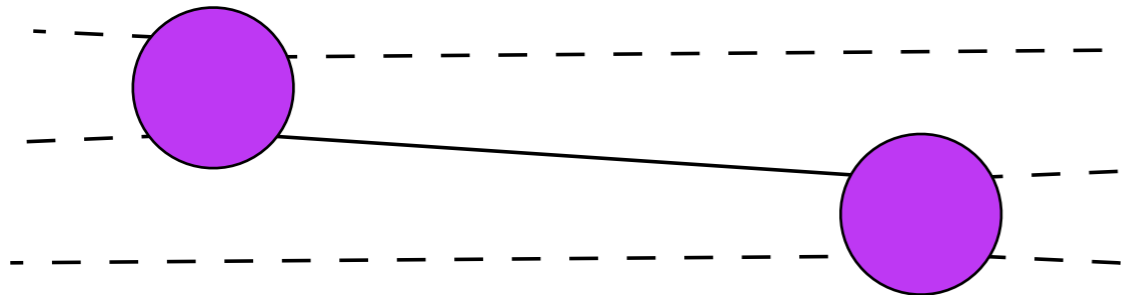


Degrees of freedom for three on-shell particles with  $(E, \vec{P})$

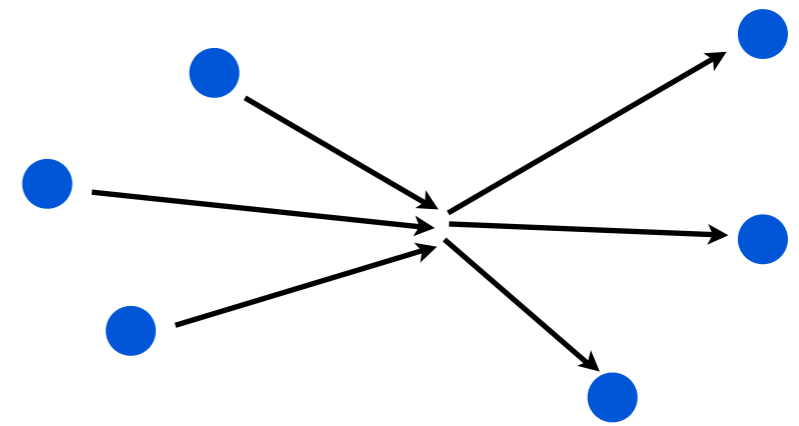


Three particle divergences

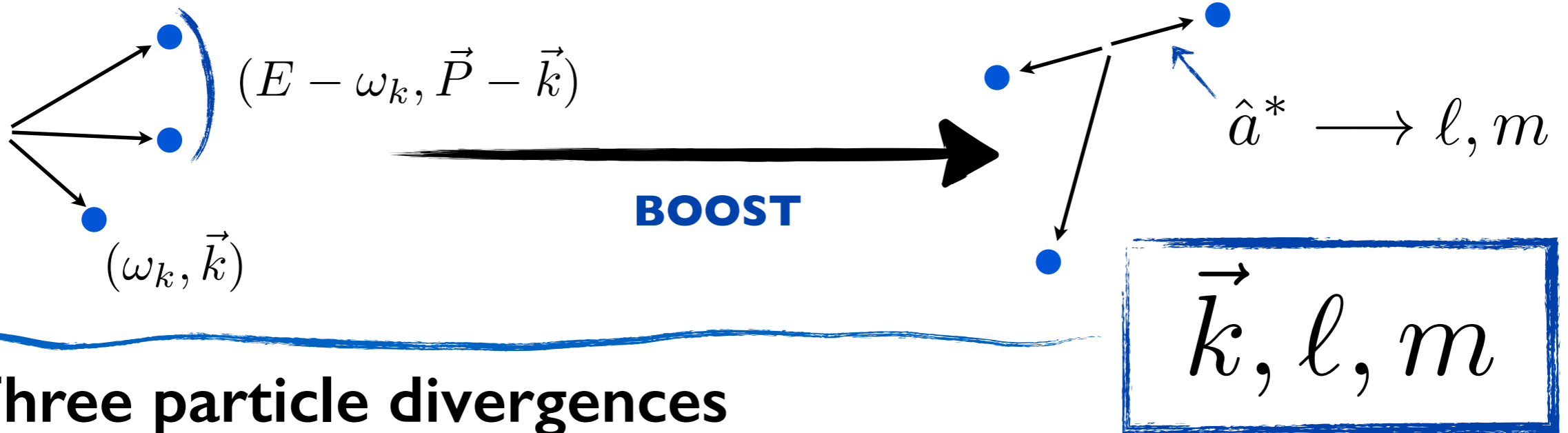
$\mathcal{M}_{3 \rightarrow 3}$  contains the diagram



# Infinite volume

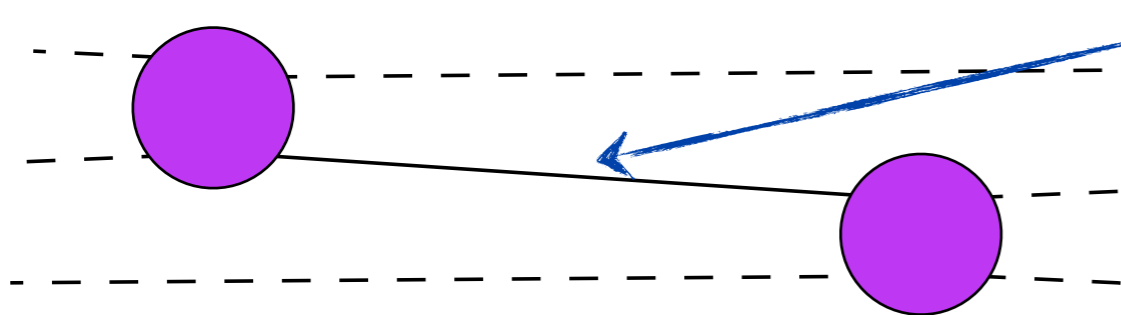


Degrees of freedom for three on-shell particles with  $(E, \vec{P})$



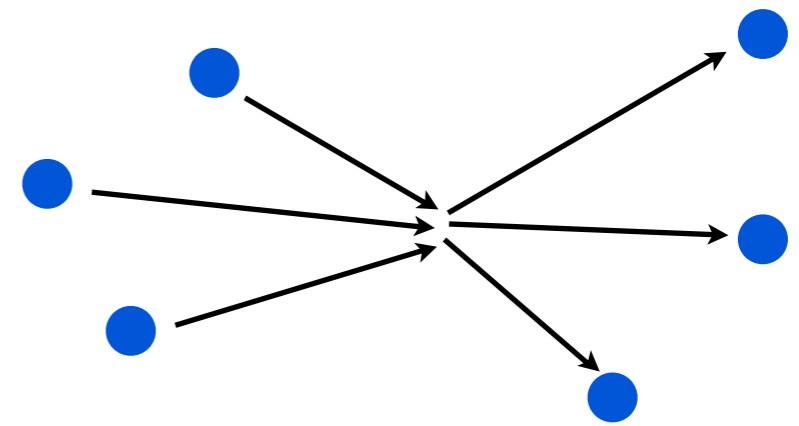
## Three particle divergences

$\mathcal{M}_{3 \rightarrow 3}$  contains the diagram

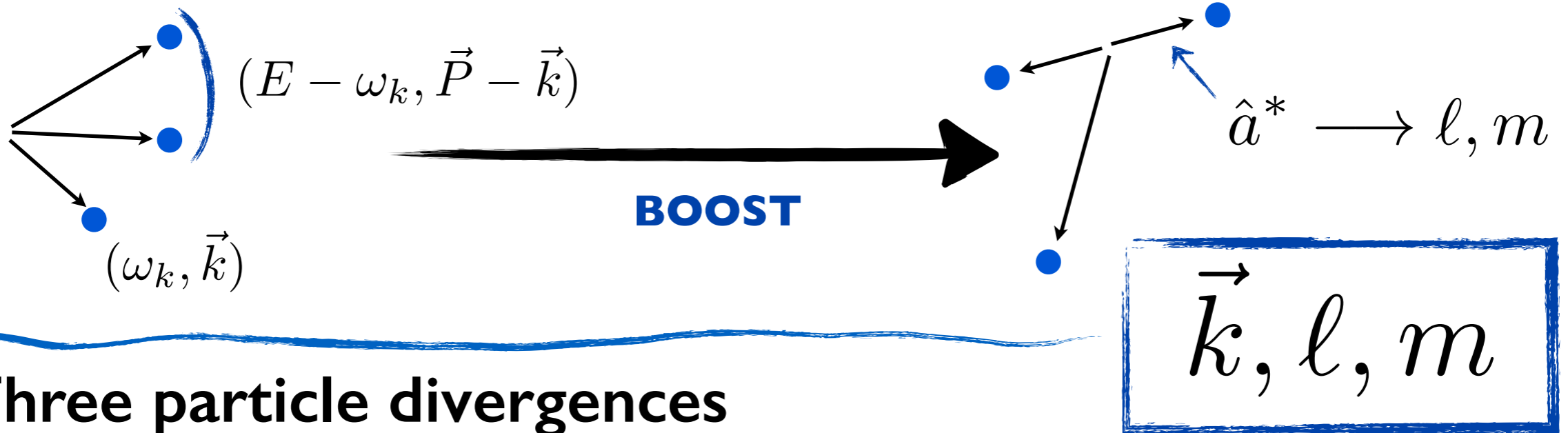


Certain external momenta  $\rightarrow$   $\mathcal{M}_{3 \rightarrow 3}$  has singularities  
 put this on-shell!

# Infinite volume

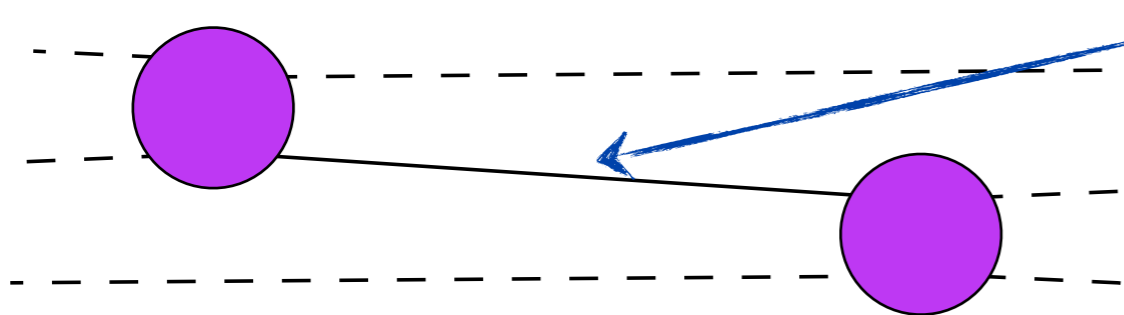


Degrees of freedom for three on-shell particles with  $(E, \vec{P})$



## Three particle divergences

$\mathcal{M}_{3 \rightarrow 3}$  contains the diagram



Certain external momenta  $\longrightarrow$   $\mathcal{M}_{3 \rightarrow 3}$  has singularities  
 put this on-shell!

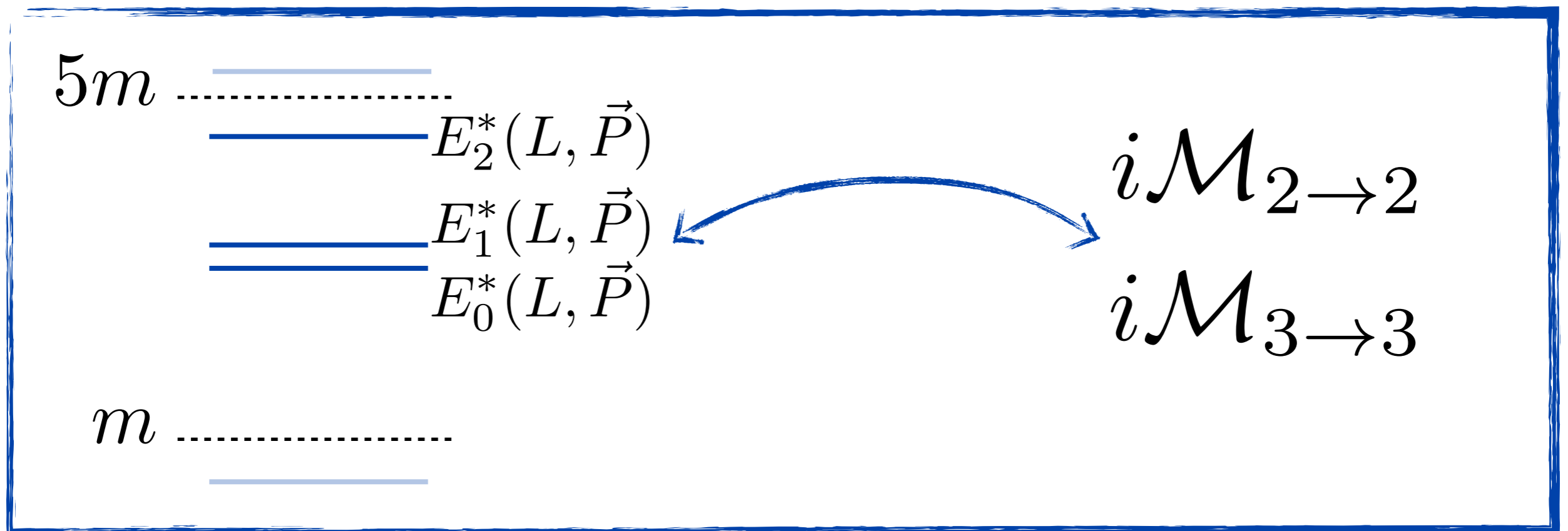
**No dominance of lowest partial waves**

# Three particles in a box

$$C_L(E, \vec{P}) \equiv \int_L d^4x e^{i(Ex^0 - \vec{P} \cdot \vec{x})} \langle 0 | T \sigma(x) \sigma^\dagger(0) | 0 \rangle$$

Require  $m < E^* < 5m$

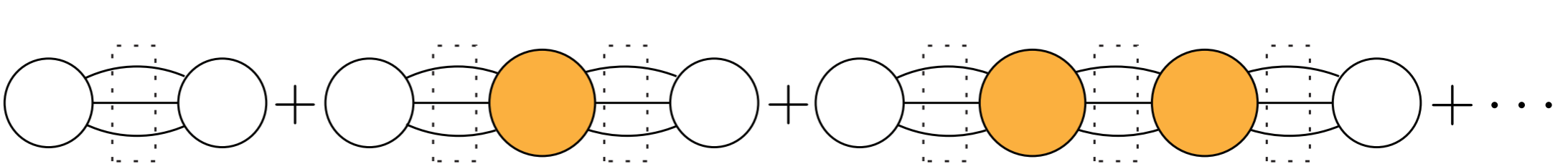
**odd-particle quantum numbers**



**Assume no two-particle bound state or resonance**

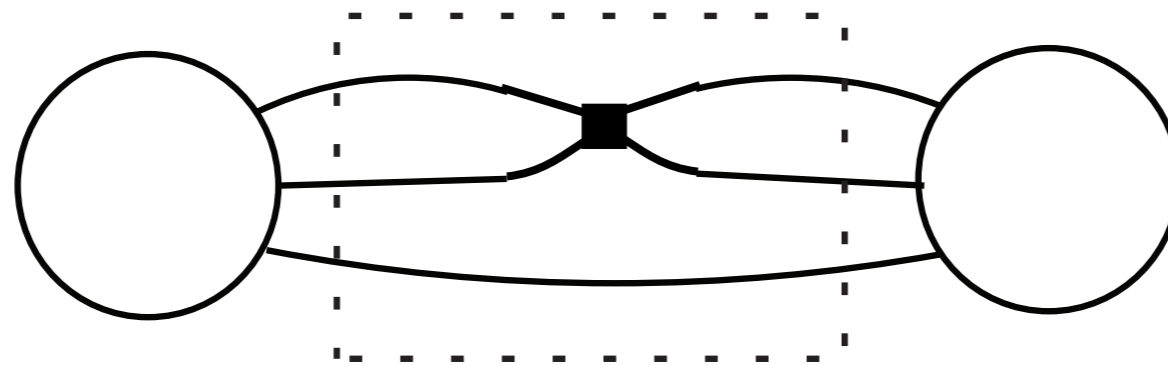


# New skeleton expansion

$$C_L(E, \vec{P}) \stackrel{?}{=} \text{Diagram 1} + \text{Diagram 2} + \text{Diagram 3} + \dots$$


(propagators still fully dressed)

**No! We also need diagrams like**



(  **should only contain connected diagrams** )

# New skeleton expansion

$$C_L(E, \vec{P}) = \text{Diagram 1} + \text{Diagram 2} + \text{Diagram 3} + \dots$$

$$+ \text{Diagram 4} + \text{Diagram 5} + \text{Diagram 6} + \dots$$

The diagrams in the expansion are:

- Diagram 1: Two white circles connected by two arcs, enclosed in a dashed box.
- Diagram 2: A white circle connected to an orange circle, which is connected to another white circle, all enclosed in a dashed box.
- Diagram 3: A white circle connected to two orange circles, which are connected to another white circle, all enclosed in a dashed box.
- Diagram 4: A white circle connected to a purple circle, which is connected to another white circle, all enclosed in a dashed box.
- Diagram 5: A white circle connected to two purple circles, which are connected to another white circle, all enclosed in a dashed box.
- Diagram 6: A white circle connected to three purple circles, which are connected to another white circle, all enclosed in a dashed box.

## Kernel definitions:

$$\text{Purple circle} \equiv \text{Diagram A} + \text{Diagram B} + \text{Diagram C} + \dots$$

The diagrams in the kernel definition for the purple circle are:

- Diagram A: A vertex with four external lines.
- Diagram B: A vertex with four external lines and two internal arcs.
- Diagram C: A vertex with four external lines and two internal arcs forming a figure-eight shape.

$$\text{Orange circle} \equiv \text{Diagram D} + \text{Diagram E} + \text{Diagram F} + \dots$$

The diagrams in the kernel definition for the orange circle are:

- Diagram D: A vertex with four external lines.
- Diagram E: A vertex with four external lines and a single internal line.
- Diagram F: A vertex with four external lines and two internal arcs.

# New skeleton expansion

$$\begin{aligned}
 C_L(E, \vec{P}) = & \text{Diagram 1} + \text{Diagram 2} + \text{Diagram 3} + \dots \\
 & + \text{Diagram 4} + \text{Diagram 5} + \text{Diagram 6} + \dots \\
 & + \text{Diagram 7} + \text{Diagram 8} + \text{Diagram 9} + \dots \\
 & + \text{Diagram 10} + \text{Diagram 11} + \dots
 \end{aligned}$$

The diagrams in the expansion are arranged in four rows. The first row contains three diagrams with two white circles connected by two lines, with one, two, or three orange circles inserted between them. The second row contains three diagrams with two white circles and one, two, or three purple circles inserted between them. The third row contains three diagrams with two white circles and two, three, or four purple circles inserted between them. The fourth row contains two diagrams with two white circles and three purple circles inserted between them. Dashed boxes in each diagram indicate the skeleton structure.

## Kernel definitions:

$$\text{Purple circle} \equiv \text{Diagram 1} + \text{Diagram 2} + \text{Diagram 3} + \dots$$

The purple circle kernel is defined as the sum of three diagrams: a single vertex, a vertex with two arcs, and a vertex with two vertices connected by two arcs.

$$\text{Orange circle} \equiv \text{Diagram 1} + \text{Diagram 2} + \text{Diagram 3} + \dots$$

The orange circle kernel is defined as the sum of three diagrams: a single vertex, a vertex with two vertices connected by a straight line, and a vertex with two vertices connected by two arcs.

# New skeleton expansion

$$\begin{aligned}
 C_L(E, \vec{P}) = & \text{Diagram 1} + \text{Diagram 2} + \text{Diagram 3} + \dots \\
 & + \text{Diagram 4} + \text{Diagram 5} + \text{Diagram 6} + \dots \\
 & + \text{Diagram 7} + \text{Diagram 8} + \text{Diagram 9} + \dots \\
 & + \text{Diagram 10} + \text{Diagram 11} + \dots \\
 & + \dots \\
 & + \text{Diagram 12} + \text{Diagram 13} + \dots
 \end{aligned}$$

The diagrams in the expansion are Feynman-like diagrams with external lines and internal vertices. The first row shows diagrams with orange vertices. The second and third rows show diagrams with purple vertices. The fourth row shows diagrams with a mix of purple and orange vertices. Dashed boxes in the diagrams indicate the skeleton structure.

## Kernel definitions:

$$\begin{aligned}
 \text{Purple Vertex} & \equiv \text{Diagram A} + \text{Diagram B} + \text{Diagram C} + \dots \\
 \text{Orange Vertex} & \equiv \text{Diagram D} + \text{Diagram E} + \text{Diagram F} + \dots
 \end{aligned}$$

The kernel definitions show how the purple and orange vertices are expanded into series of diagrams. The purple vertex expansion includes diagrams with two external lines and one internal vertex, and diagrams with two external lines and two internal vertices. The orange vertex expansion includes diagrams with two external lines and one internal vertex, and diagrams with two external lines and two internal vertices.

# New skeleton expansion

$$\begin{aligned}
 C_L(E, \vec{P}) = & \text{Diagram 1} + \text{Diagram 2} + \text{Diagram 3} + \dots \\
 & + \text{Diagram 4} + \text{Diagram 5} + \text{Diagram 6} + \dots \\
 & + \text{Diagram 7} + \text{Diagram 8} + \text{Diagram 9} + \dots \\
 & + \text{Diagram 10} + \text{Diagram 11} + \dots \\
 & + \dots \\
 & + \text{Diagram 12} + \text{Diagram 13} + \dots
 \end{aligned}$$

The diagrams in the expansion are:

- Diagram 1: Two white circles connected by two lines, with a dashed box around the lines.
- Diagram 2: Two white circles connected by two lines, with an orange circle in the middle. Dashed boxes are around the lines and the orange circle.
- Diagram 3: Two white circles connected by two lines, with two orange circles in the middle. Dashed boxes are around the lines and each orange circle.
- Diagram 4: Two white circles connected by two lines, with one purple circle in the middle. Dashed boxes are around the lines and the purple circle.
- Diagram 5: Two white circles connected by two lines, with two purple circles in the middle. Dashed boxes are around the lines and each purple circle.
- Diagram 6: Two white circles connected by two lines, with three purple circles in the middle. Dashed boxes are around the lines and each purple circle.
- Diagram 7: Two white circles connected by two lines, with two purple circles in the middle. Dashed boxes are around the lines and the pair of purple circles.
- Diagram 8: Two white circles connected by two lines, with three purple circles in the middle. Dashed boxes are around the lines and the pair of purple circles.
- Diagram 9: Two white circles connected by two lines, with four purple circles in the middle. Dashed boxes are around the lines and the pair of purple circles.
- Diagram 10: Two white circles connected by two lines, with three purple circles in the middle. Dashed boxes are around the lines and the pair of purple circles.
- Diagram 11: Two white circles connected by two lines, with four purple circles in the middle. Dashed boxes are around the lines and the pair of purple circles.
- Diagram 12: Two white circles connected by two lines, with one purple circle and one orange circle in the middle. Dashed boxes are around the lines, the purple circle, and the orange circle.
- Diagram 13: Two white circles connected by two lines, with one orange circle and one purple circle in the middle. Dashed boxes are around the lines, the orange circle, and the purple circle.

Compare to two-particle skeleton expansion

$$C_L(E, \vec{P}) = \text{Diagram 1} + \text{Diagram 2} + \text{Diagram 3} + \dots$$

The diagrams in the two-particle skeleton expansion are:

- Diagram 1: Two white circles connected by two lines, with a dashed box around the lines.
- Diagram 2: Two white circles connected by two lines, with a single purple circle in the middle. Dashed boxes are around the lines and the purple circle.
- Diagram 3: Two white circles connected by two lines, with two purple circles in the middle. Dashed boxes are around the lines and each purple circle.

# What is new here?

## 1. Degrees of freedom are different

two particles

two-particle angular momentum

three particles

$\vec{k}$  + two-particle angular momentum



Our result only depends on finite-volume momentum

$$\vec{k} = \frac{2\pi}{L} \vec{n}$$

# What is new here?

## 1. Degrees of freedom are different

two particles

two-particle angular momentum

three particles

$\vec{k}$  + two-particle angular momentum



Our result only depends on finite-volume momentum  $\vec{k} = \frac{2\pi}{L} \vec{n}$

Quantization condition expressed using matrices with indices

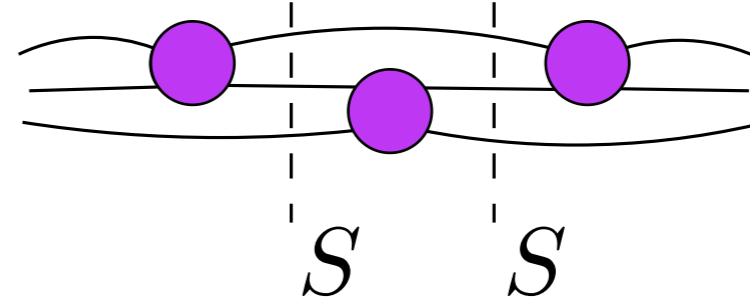
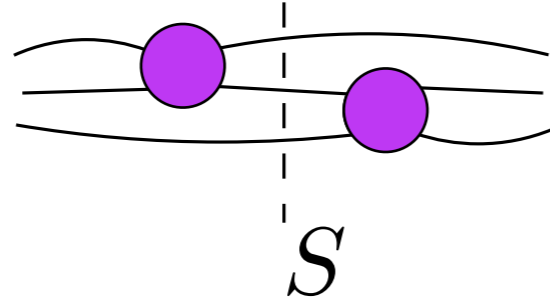
$$\vec{k}, l, m$$

# What is new here?

## 2. Three particle divergences

Define  $i\mathcal{M}_{\text{df},3\rightarrow 3}$

$$\equiv i\mathcal{M}_{3\rightarrow 3} - \left[ i\mathcal{M}_{2\rightarrow 2} S i\mathcal{M}_{2\rightarrow 2} + \int i\mathcal{M}_{2\rightarrow 2} S i\mathcal{M}_{2\rightarrow 2} S i\mathcal{M}_{2\rightarrow 2} + \dots \right]$$



only on-shell  
amplitudes here

infinite series  
built with factors of  $S i\mathcal{M}_{2\rightarrow 2}$

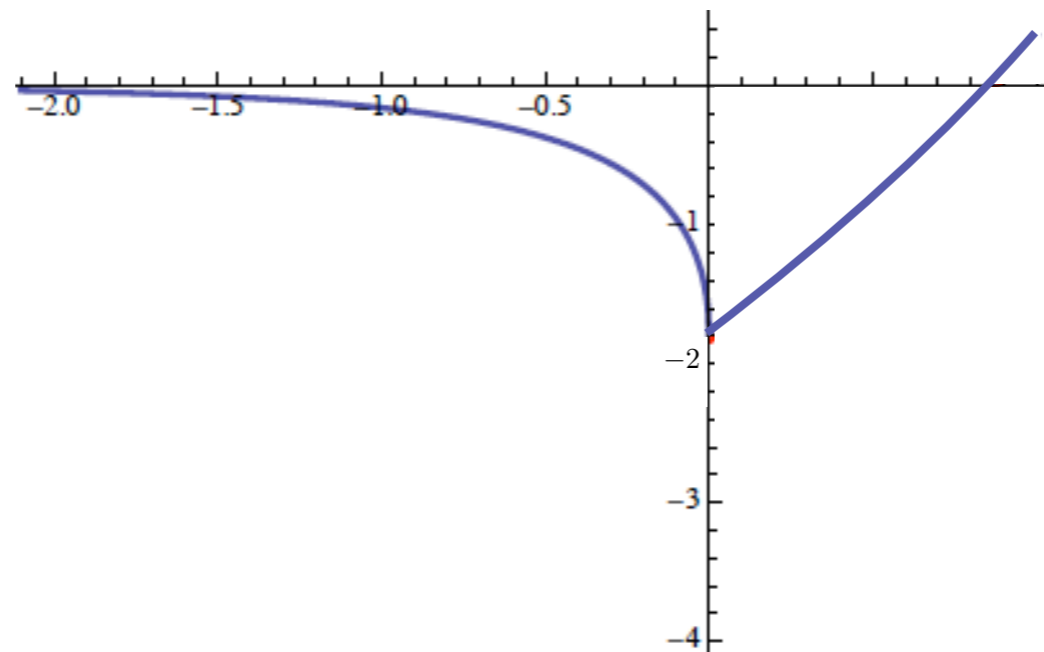
**This subtraction emerges naturally in our  
finite-volume analysis**



# What is new here?

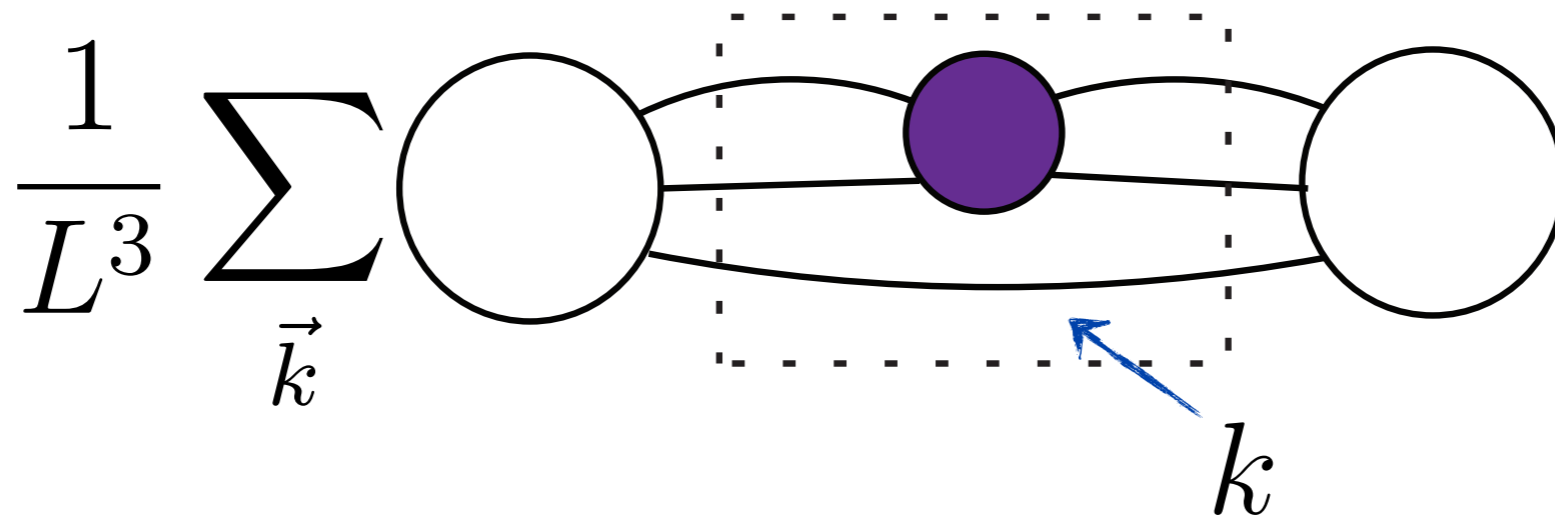
3. Must now worry about sum crossing  
two-particle unitary cusp

two-particle  
scattering  
(real part)



depends on  $k$

two particle energy



# What is new here?

3. Must now worry about sum crossing  
two-particle unitary cusp

## To remove cusp

$i\epsilon$  prescription  principal value  $\widetilde{PV}$

Analytically continue principal value below threshold  
then interpolate to prescription-free subthreshold form

# What is new here?

3. Must now worry about sum crossing  
two-particle unitary cusp

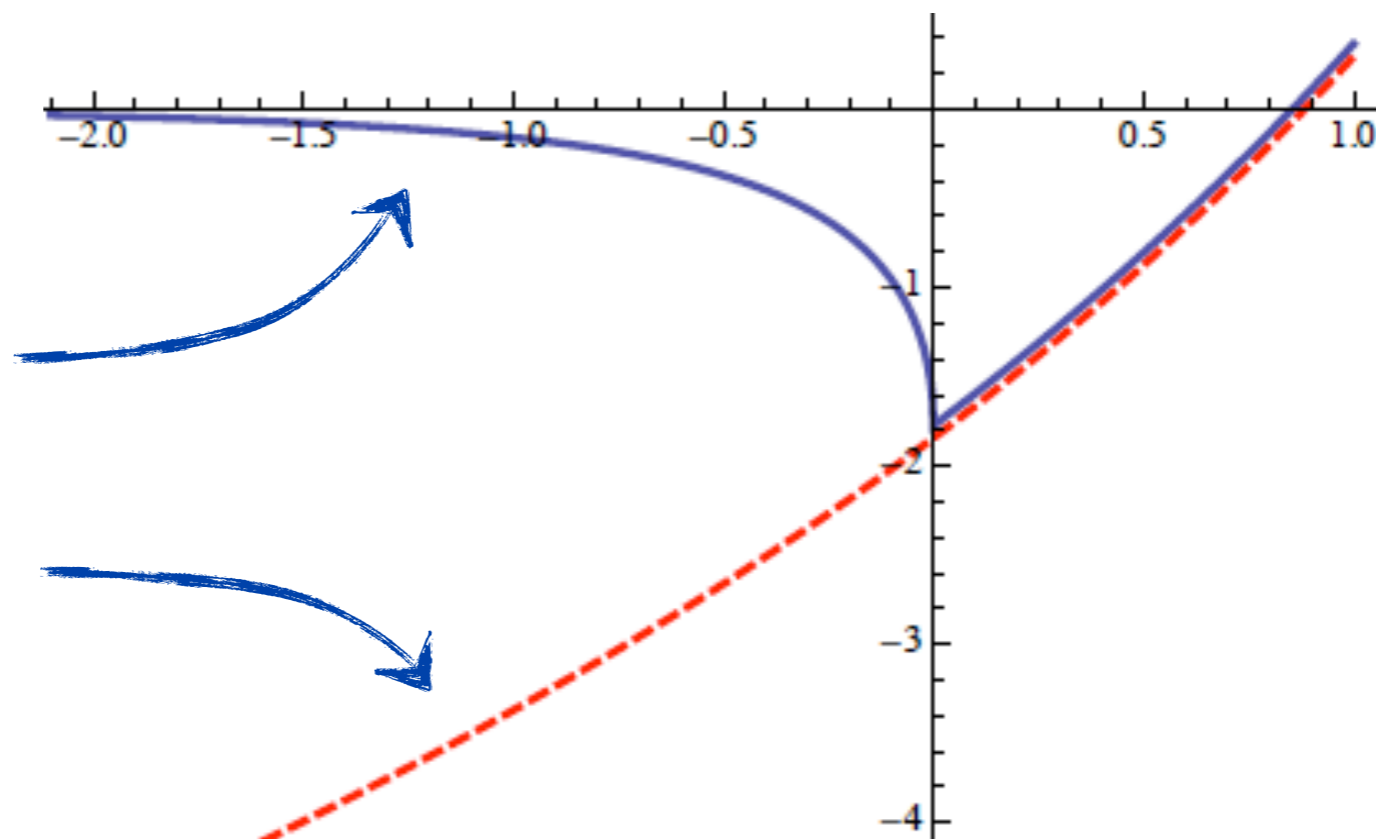
## To remove cusp

$i\epsilon$  prescription

principal value  $\widetilde{PV}$

standard definition

modification  $\widetilde{PV}$



# What is new here?

3. Must now worry about sum crossing  
two-particle unitary cusp

has a cusp

$$i\mathcal{M}_{2\rightarrow 2} = \text{---} \bullet \text{---} + \text{---} \bullet \text{---} \bullet \text{---} + \text{---} \bullet \text{---} \bullet \text{---} \bullet \text{---} + \dots$$

$$i\tilde{\mathcal{K}}_{2\rightarrow 2} = \text{---} \bullet \text{---} =$$

$$\text{---} \bullet \text{---} + \text{---} \bullet \text{---} \overset{\sim}{\text{PV}} \bullet \text{---} + \text{---} \bullet \text{---} \overset{\sim}{\text{PV}} \bullet \overset{\sim}{\text{PV}} \bullet \text{---} + \dots$$

What is new here?

3. Must now worry about sum crossing  
two-particle unitary cusp

$$\begin{array}{ccc} i\mathcal{M}_{2\rightarrow 2} & \xrightarrow{\quad} & i\mathcal{K}_{2\rightarrow 2} \\ i\mathcal{M}_{\text{df},3\rightarrow 3} & & i\mathcal{K}_{\text{df},3\rightarrow 3} \end{array}$$

**We relate these infinite-volume quantities  
to the finite-volume spectrum**

# Three-particle result

$$C_L(E, \vec{P}) = C_\infty(E, \vec{P}) + A'_3 iF_3 \frac{1}{1 - i\mathcal{K}_{\text{df},3 \rightarrow 3} iF_3} A_3$$

$$iF_3 \equiv \frac{iF}{2\omega L^3} \left[ \frac{1}{3} + \frac{1}{1 - i\mathcal{M}_{L,2 \rightarrow 2} iG} i\mathcal{M}_{L,2 \rightarrow 2} iF \right]$$

$$i\mathcal{M}_{L,2 \rightarrow 2} \equiv i\mathcal{K}_{2 \rightarrow 2} \frac{1}{1 - iF i\mathcal{K}_{2 \rightarrow 2}}$$

All factors are matrices with indices  $\vec{k}, \ell, m$

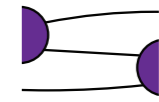
# Three-particle result

$$C_L(E, \vec{P}) = C_\infty(E, \vec{P}) + A'_3 iF_3 \frac{1}{1 - i\mathcal{K}_{\text{df},3\rightarrow3} iF_3} A_3$$

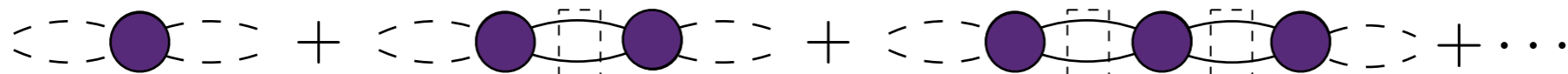
**sum-integral difference** 

$$iF_3 \equiv \frac{iF}{2\omega L^3} \left[ \frac{1}{3} + \frac{1}{1 - i\mathcal{M}_{L,2\rightarrow2} iG} i\mathcal{M}_{L,2\rightarrow2} iF \right]$$

$$i\mathcal{M}_{L,2\rightarrow2} \equiv i\mathcal{K}_{2\rightarrow2} \frac{1}{1 - iF i\mathcal{K}_{2\rightarrow2}}$$

**encodes switches** 

**sum of all two-particle loops (with summed momenta)**



**All factors are matrices with indices  $\vec{k}, \ell, m$**

# Three-particle result

At fixed  $(L, \vec{P})$ , finite-volume spectrum is all solutions to

$$\det [1 - i\mathcal{K}_{\text{df},3 \rightarrow 3} iF_3] = 0$$

$$iF_3 \equiv \frac{iF}{2\omega L^3} \left[ \frac{1}{3} + \frac{1}{1 - i\mathcal{M}_{L,2 \rightarrow 2} iG} i\mathcal{M}_{L,2 \rightarrow 2} iF \right] \quad i\mathcal{M}_{L,2 \rightarrow 2} \equiv i\mathcal{K}_{2 \rightarrow 2} \frac{1}{1 - iF i\mathcal{K}_{2 \rightarrow 2}}$$

MTH and Sharpe, *Phys. Rev. D* 90, 116003 (2014)



# Three-particle result

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MTH and Sharpe, *Phys. Rev. D* 90, 116003 (2014)

**Model independent general result** of relativistic scalar field theory

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MTH and Sharpe, *Phys. Rev. D* 90, 116003 (2014)

**Model independent general result** of relativistic scalar field theory

**Assumes no two-body bound states or resonances**

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**Infinite matrices truncate** if we truncate in angular momentum

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At fixed  $(L, \vec{P})$ , finite-volume spectrum is all solutions to

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MTH and Sharpe, *Phys. Rev. D* 90, 116003 (2014)

**Model independent general result** of relativistic scalar field theory

**Assumes no two-body bound states or resonances**

**Infinite matrices truncate** if we truncate in angular momentum

**Strongest truncation is the isotropic limit, gives simple result**

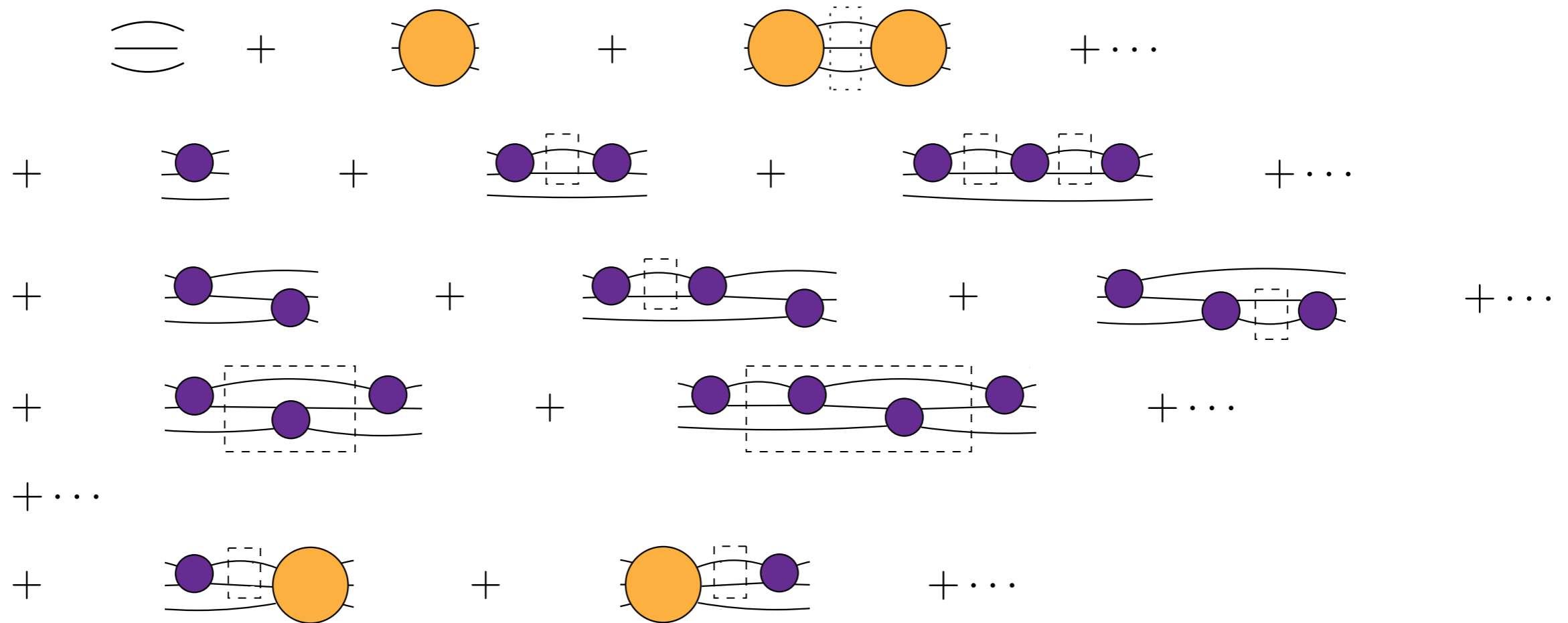
$$\mathcal{K}_{\text{df},3\rightarrow 3}(E_n^*) = -[F_{3,\text{iso}}(E_n, \vec{P}, L)]^{-1}$$

# Relating $i\mathcal{K}_{\text{df},3\rightarrow 3}$ to $i\mathcal{M}_{3\rightarrow 3}$

$$\begin{aligned}
 C_L(E, \vec{P}) = & \text{Diagram 1} + \text{Diagram 2} + \text{Diagram 3} + \dots \\
 & + \text{Diagram 4} + \text{Diagram 5} + \text{Diagram 6} + \dots \\
 & + \text{Diagram 7} + \text{Diagram 8} + \text{Diagram 9} + \dots \\
 & + \text{Diagram 10} + \text{Diagram 11} + \dots \\
 & + \dots \\
 & + \text{Diagram 12} + \text{Diagram 13} + \dots
 \end{aligned}$$

**First we modify  $C_L(E, \vec{P})$  to define  $i\mathcal{M}_{L,3\rightarrow 3}$**

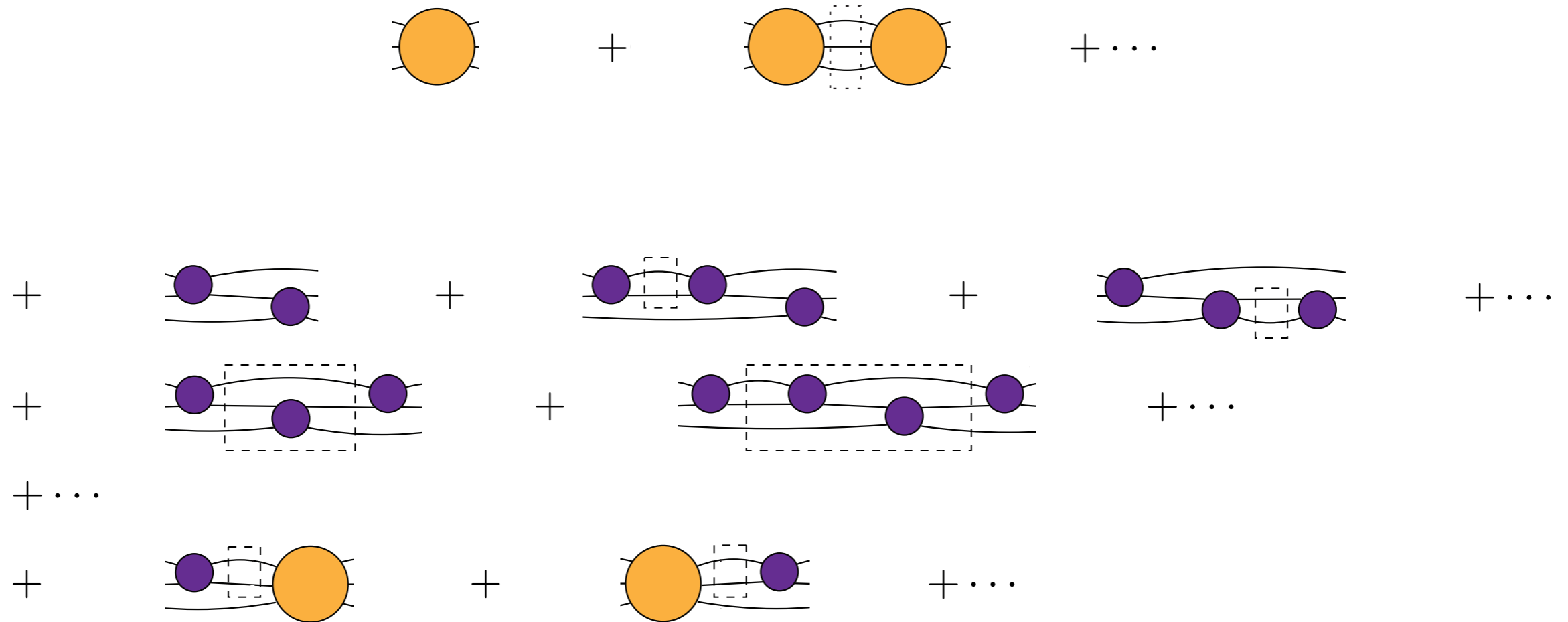
# Relating $i\mathcal{K}_{\text{df},3\rightarrow 3}$ to $i\mathcal{M}_{3\rightarrow 3}$



**First we modify  $C_L(E, \vec{P})$  to define  $i\mathcal{M}_{L,3\rightarrow 3}$**

1. Amputate interpolating fields

# Relating $i\mathcal{K}_{\text{df},3\rightarrow 3}$ to $i\mathcal{M}_{3\rightarrow 3}$



**First we modify  $C_L(E, \vec{P})$  to define  $i\mathcal{M}_{L,3\rightarrow 3}$**

2. Drop disconnected diagrams

# Relating $i\mathcal{K}_{\text{df},3\rightarrow 3}$ to $i\mathcal{M}_{3\rightarrow 3}$

$$i\mathcal{M}_{L,3\rightarrow 3} \equiv \mathcal{S} \left\{ \begin{array}{l} \text{Orange circle} + \text{Two orange circles connected by a dashed box} + \dots \\ + \text{Three purple circles on a line with a dashed box between the first two} + \text{Three purple circles on a line with a dashed box between the last two} + \dots \\ + \text{Three purple circles on a line with a dashed box between the first and last} + \dots \\ + \dots \\ + \text{Purple circle connected to an orange circle with a dashed box} + \text{Orange circle connected to a purple circle with a dashed box} + \dots \end{array} \right\}$$

**First we modify  $C_L(E, \vec{P})$  to define  $i\mathcal{M}_{L,3\rightarrow 3}$**

**3. Symmetrize**



# Relating $i\mathcal{K}_{\text{df},3\rightarrow 3}$ to $i\mathcal{M}_{3\rightarrow 3}$

$$i\mathcal{M}_{L,3\rightarrow 3} \equiv \mathcal{S} \left\{ \begin{array}{l} \text{Orange circle} + \text{Orange circle} \text{ with dashed box} + \dots \\ \text{Purple circles on lines} + \text{Purple circles on lines with dashed box} + \dots \\ \text{Purple circles on lines with dashed box} + \dots \\ \dots \\ \text{Purple circles on lines} + \text{Orange circle} + \text{Orange circle} + \dots \end{array} \right\}$$

Replacing all loop momentum sums with  $i$ -epsilon prescription integrals gives physical three-to-three scattering amplitude

$$i\mathcal{M}_{3\rightarrow 3} = \lim_{L \rightarrow \infty} \Big|_{i\epsilon} i\mathcal{M}_{L,3\rightarrow 3}$$

# Relating $i\mathcal{K}_{\text{df},3\rightarrow 3}$ to $i\mathcal{M}_{3\rightarrow 3}$

$$i\mathcal{M}_{L,3\rightarrow 3} = i\mathcal{D}_L + \mathcal{S} \left[ \mathcal{L}_L \ i\mathcal{K}_{\text{df},3\rightarrow 3} \frac{1}{1 - iF_3 \ i\mathcal{K}_{\text{df},3\rightarrow 3}} \ \mathcal{R}_L \right]$$

$$i\mathcal{M}_{3\rightarrow 3} = \lim_{L \rightarrow \infty} \left. i\mathcal{M}_{L,3\rightarrow 3} \right|_{i\epsilon}$$

MTH and Sharpe, (2015) 1504.04248

**Gives integral equation relating  $i\mathcal{K}_{\text{df},3\rightarrow 3}$  to  $i\mathcal{M}_{3\rightarrow 3}$**

**Completes formal story (for the setup considered!)**

**Relation only depends on on-shell scattering quantities**

# Connecting to other work

## Reproducing Beane, Detmold and Savage threshold expansion

$$E = 3m + \frac{12\pi a}{mL^3} \left[ 1 + A \frac{a}{L} + B \frac{a^2}{L^2} \right] + C_1 \frac{1}{L^6} - \frac{\mathcal{M}_{\text{df},3 \rightarrow 3,\text{thr}}}{48m^3 L^6} + C_2 \frac{\log(mL)}{L^6}$$

We agree unambiguously  $A, B, C_2$  and relate  $C_1 - \frac{\mathcal{M}_{\text{df},3 \rightarrow 3,\text{thr}}}{48m^3}$   
to a non-relativistic contact interaction

Beane, S., Detmold, W. & Savage, M. *Phys. Rev. D* 76 (2007) 074507

Tan, S. *Phys. Rev. A* 78 (2008) 013636

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## Meißner, Rios and Rusetsky three-body bound state

$$\Delta E = c(\kappa^2/m)(\kappa L)^{-3/2} |A|^2 \exp(-2\kappa L/\sqrt{3})$$

would be interesting to check agreement

Meißner, Rios and Rusektsky. *Phys. Rev. Lett.* 114, 091602 (2015)

# Summary

**Lüscher formalism for *simplest* three-to-three system is complete**

- relates on-shell scattering to finite-volume spectrum
- derived in general relativistic quantum field theory
- passes non-trivial checks
- no two-particle bound state or resonance
- identical bosons
- no even-odd coupling

# Future work

Include two-particle bound states and resonances

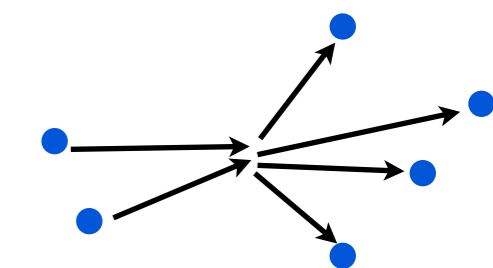
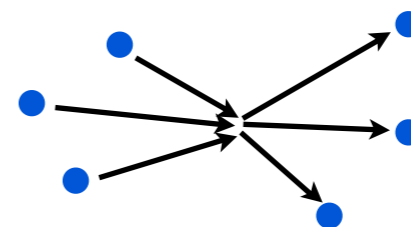
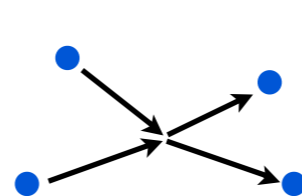
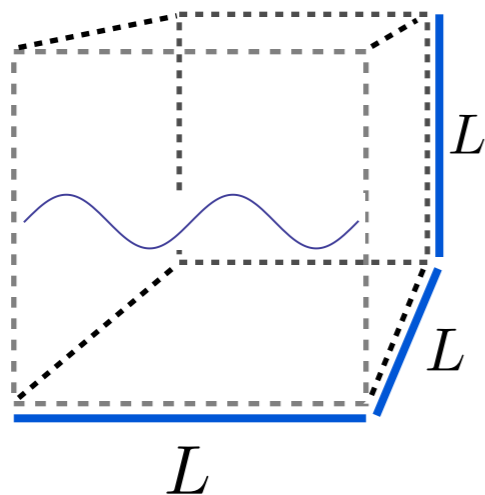
Include two-to-three coupling

Generalize Lellouch-Lüscher method to extract  
three-particle weak decays

$$K \longrightarrow \pi\pi\pi$$

Include non-identical, non-degenerate and spin-half particles

Extend mapping to four-particle states

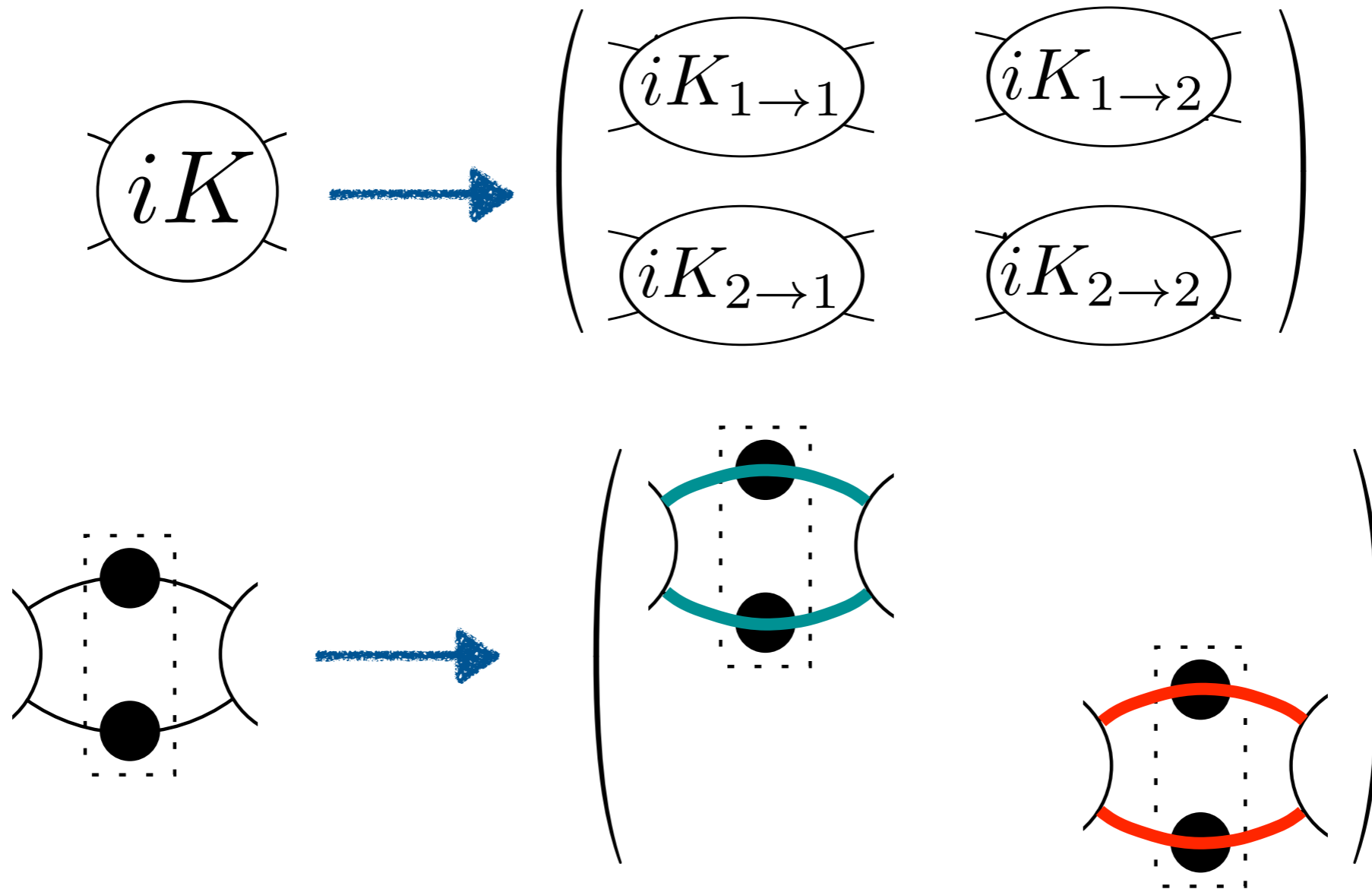


# Backup Slides

# Scattering of multiple two-particle channels

$$\pi\pi \rightarrow \bar{K}K \quad \pi K \rightarrow \eta K$$

Make following replacements



# Scattering of multiple two-particle channels

$$\pi\pi \rightarrow \bar{K}K \quad \pi K \rightarrow \eta K$$

One finds

$$\det \left[ 1 - \begin{pmatrix} i\mathcal{M}_{1\rightarrow 1} & i\mathcal{M}_{1\rightarrow 2} \\ i\mathcal{M}_{2\rightarrow 1} & i\mathcal{M}_{2\rightarrow 2} \end{pmatrix} \begin{pmatrix} iF_1 & 0 \\ 0 & iF_2 \end{pmatrix} \right] = 0$$

M. Lage, U.-G. Meißner, and A. Rusetsky, *Phys.Lett.*, B681, 439 (2009)

V. Bernard, M. Lage, U.-G. Meißner, and A. Rusetsky, *JHEP*, 1101, 019 (2011)

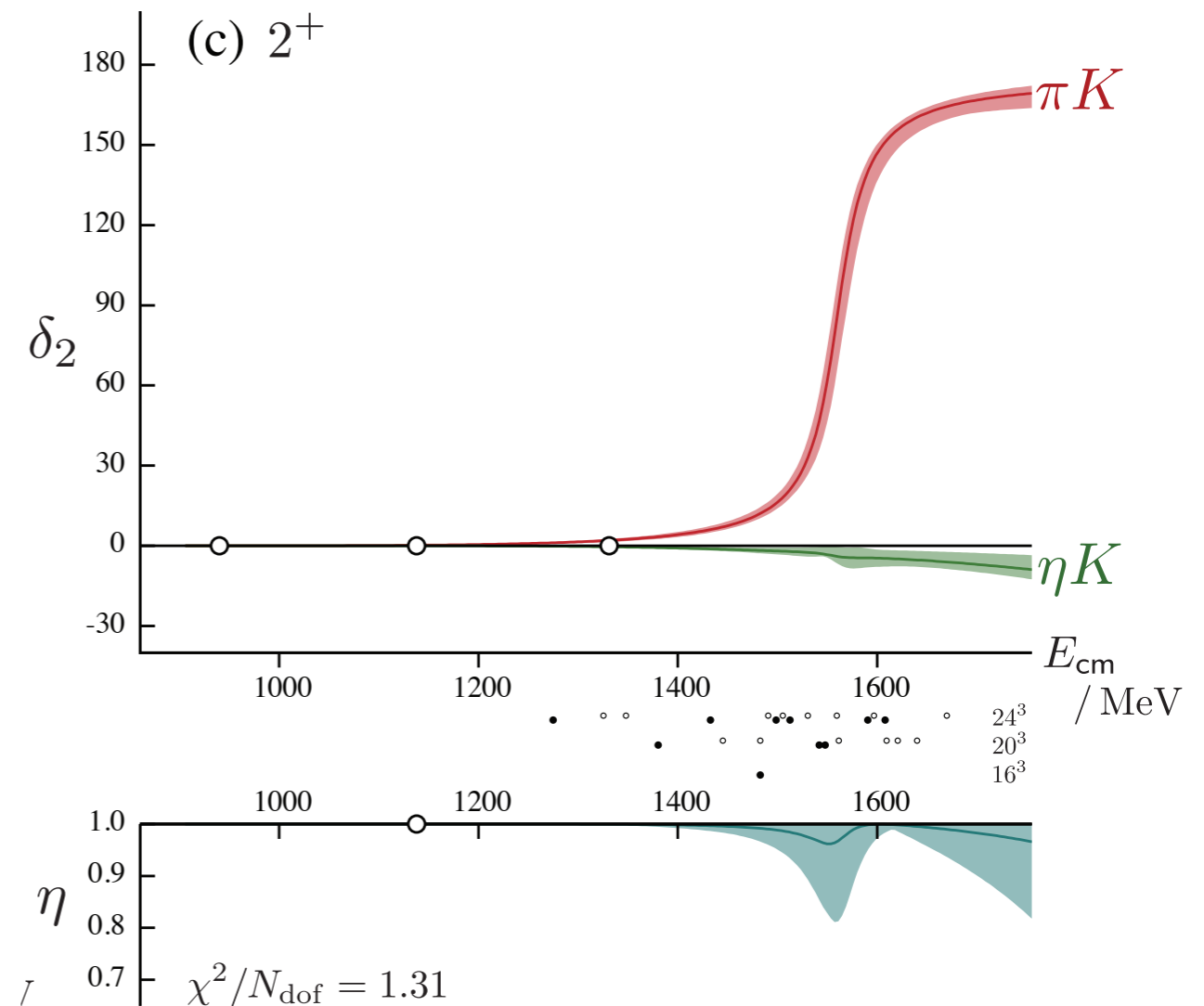
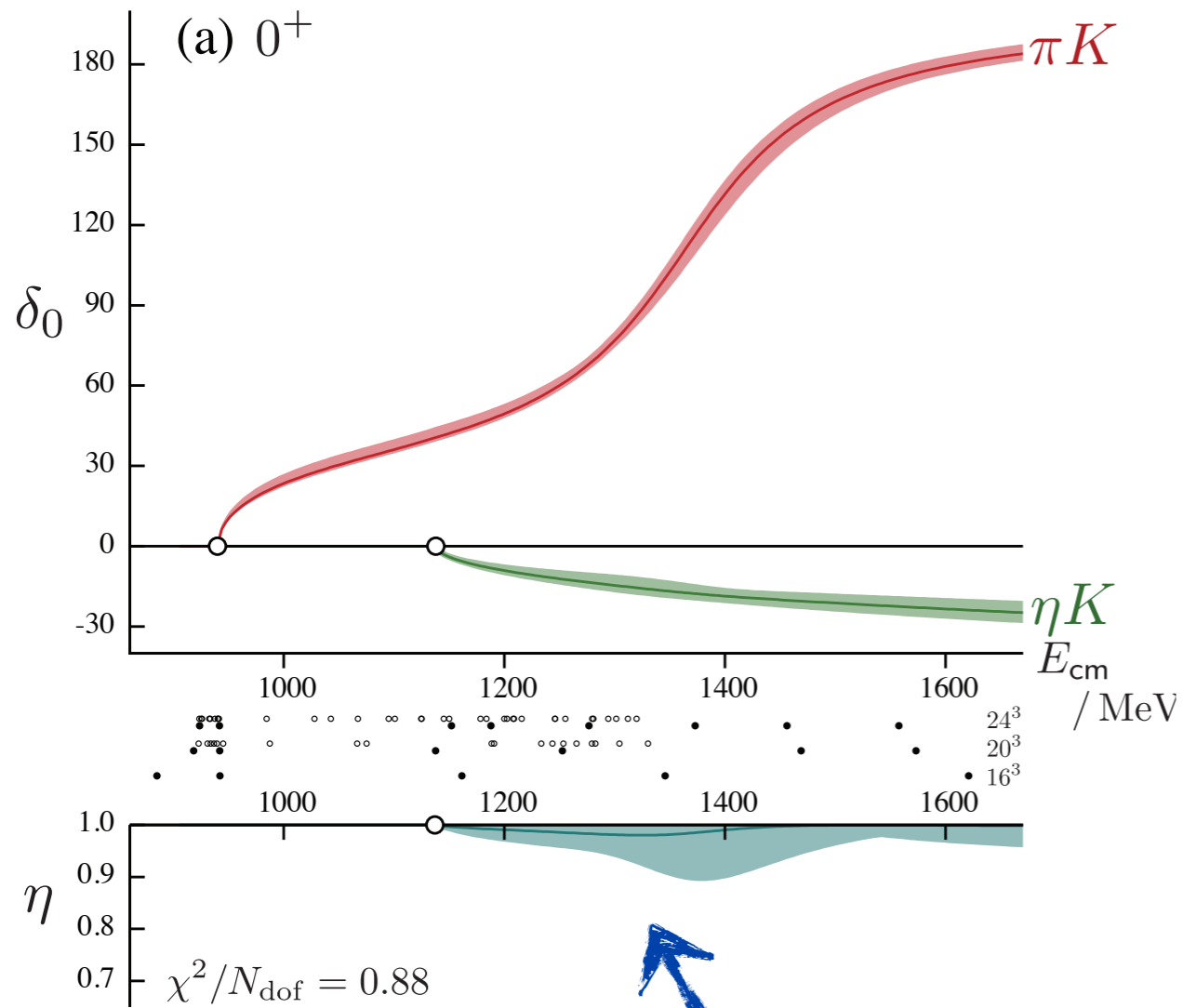
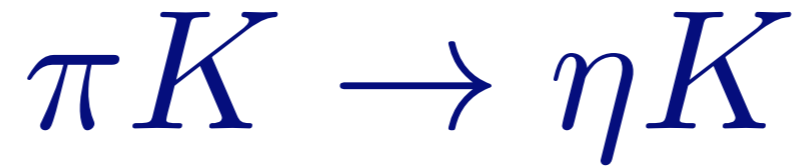
M. Döring, U.-G. Meißner, E. Oset, and A. Rusetsky, *Eur.Phys.J.*, A47, 139 (2011)

MTH, S. R. Sharpe, *Phys.Rev. D86* (2012) 016007

R. A. Briceño, Z. Davoudi, *Phys.Rev. D88* (2013) 094507



# Already implemented in LQCD calculation



$$\mathcal{M}(\pi K \rightarrow \eta K) \sim \sqrt{1 - \eta^2}$$

from Dudek, Edwards, Thomas, Wilson in arXiv:1406:4158

# Relating $i\mathcal{K}_{\text{df},3\rightarrow 3}$ to $i\mathcal{M}_{3\rightarrow 3}$

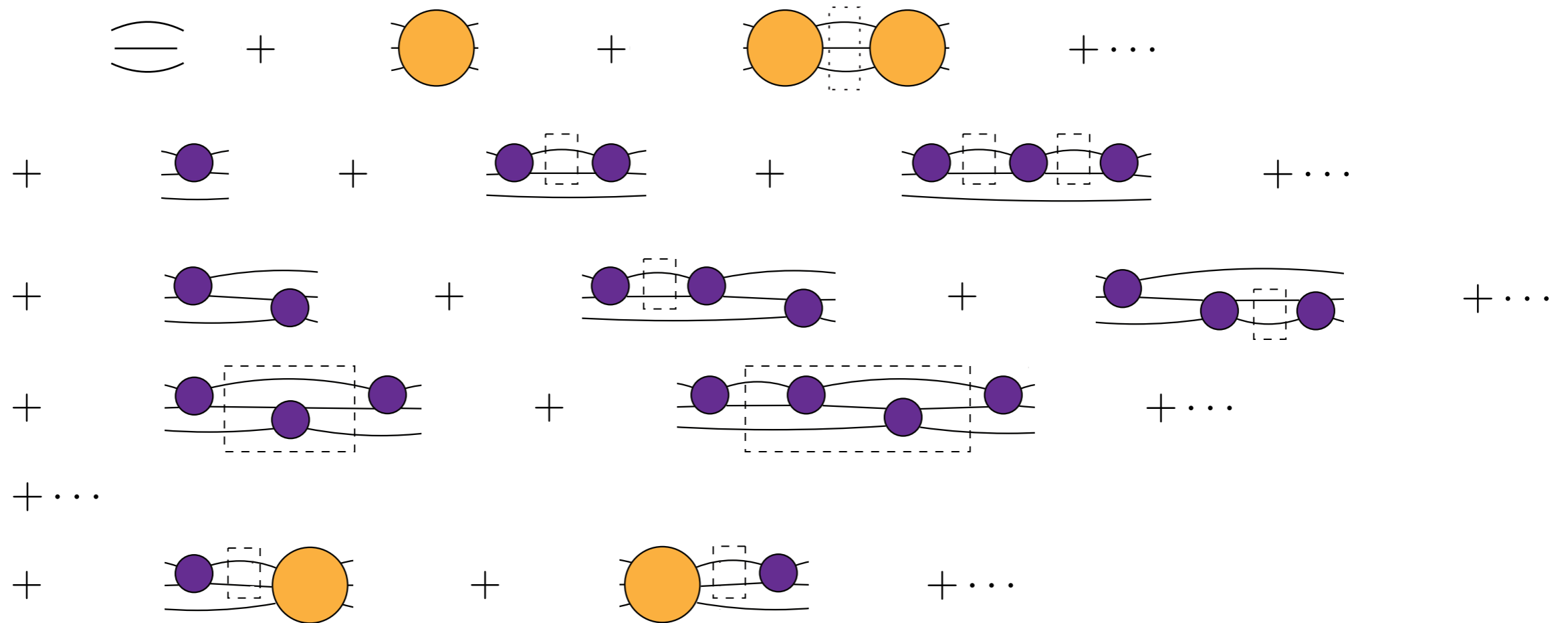
$$\begin{aligned}
 C_L(E, \vec{P}) = & \text{Diagram 1} + \text{Diagram 2} + \text{Diagram 3} + \dots \\
 & + \text{Diagram 4} + \text{Diagram 5} + \text{Diagram 6} + \dots \\
 & + \text{Diagram 7} + \text{Diagram 8} + \text{Diagram 9} + \dots \\
 & + \text{Diagram 10} + \text{Diagram 11} + \dots \\
 & + \dots \\
 & + \text{Diagram 12} + \text{Diagram 13} + \dots
 \end{aligned}$$

The diagrams in the equation represent Feynman diagrams for the transition amplitude  $C_L(E, \vec{P})$ . Each diagram consists of two external white circles connected by two lines. The internal structure varies:
 

- Diagram 1: Two white circles connected by two lines, with a dashed box around the entire structure.
- Diagram 2: A white circle connected to an orange circle, which is then connected to another white circle. The orange circle is enclosed in a dashed box.
- Diagram 3: A white circle connected to two orange circles in series, which are then connected to another white circle. The two orange circles are enclosed in a dashed box.
- Diagram 4: A white circle connected to a purple circle, which is then connected to another white circle. The purple circle is enclosed in a dashed box.
- Diagram 5: A white circle connected to two purple circles in series, which are then connected to another white circle. The two purple circles are enclosed in a dashed box.
- Diagram 6: A white circle connected to three purple circles in series, which are then connected to another white circle. The three purple circles are enclosed in a dashed box.
- Diagram 7: A white circle connected to two purple circles in series, which are then connected to another white circle. The two purple circles are enclosed in a dashed box.
- Diagram 8: A white circle connected to three purple circles in series, which are then connected to another white circle. The three purple circles are enclosed in a dashed box.
- Diagram 9: A white circle connected to four purple circles in series, which are then connected to another white circle. The four purple circles are enclosed in a dashed box.
- Diagram 10: A white circle connected to three purple circles in series, which are then connected to another white circle. The three purple circles are enclosed in a dashed box.
- Diagram 11: A white circle connected to four purple circles in series, which are then connected to another white circle. The four purple circles are enclosed in a dashed box.
- Diagram 12: A white circle connected to a purple circle, which is then connected to an orange circle, which is finally connected to another white circle. The purple circle is enclosed in a dashed box.
- Diagram 13: A white circle connected to an orange circle, which is then connected to a purple circle, which is finally connected to another white circle. The orange circle is enclosed in a dashed box.

**First we modify  $C_L(E, \vec{P})$  to define  $i\mathcal{M}_{L,3\rightarrow 3}$**

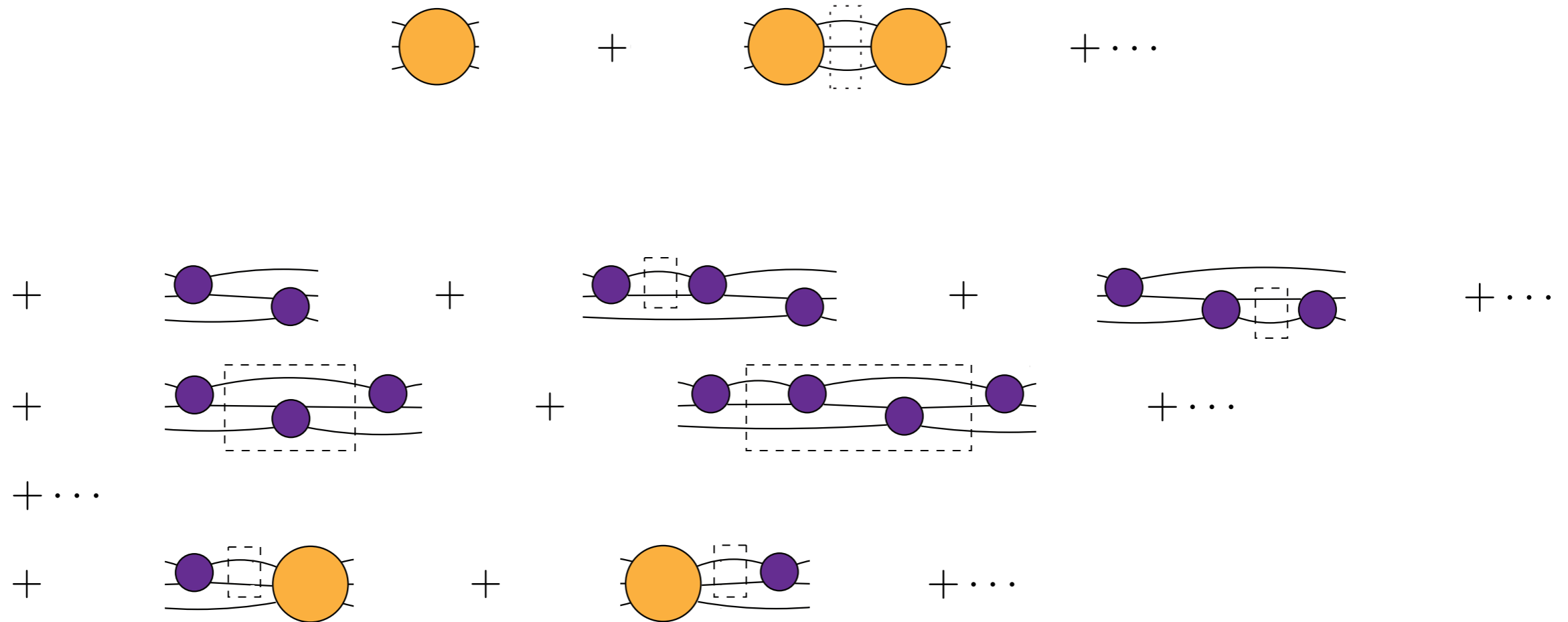
# Relating $i\mathcal{K}_{\text{df},3\rightarrow 3}$ to $i\mathcal{M}_{3\rightarrow 3}$



**First we modify  $C_L(E, \vec{P})$  to define  $i\mathcal{M}_{L,3\rightarrow 3}$**

1. Amputate interpolating fields

# Relating $i\mathcal{K}_{\text{df},3\rightarrow 3}$ to $i\mathcal{M}_{3\rightarrow 3}$



**First we modify  $C_L(E, \vec{P})$  to define  $i\mathcal{M}_{L,3\rightarrow 3}$**

2. Drop disconnected diagrams

# Relating $i\mathcal{K}_{\text{df},3\rightarrow 3}$ to $i\mathcal{M}_{3\rightarrow 3}$

$$i\mathcal{M}_{L,3\rightarrow 3} \equiv \mathcal{S} \left\{ \begin{array}{l} \text{Orange circle} + \text{Two orange circles connected by a dashed box} + \dots \\ + \text{Three purple circles on a line with a dashed box between the first two} + \text{Three purple circles on a line with a dashed box between the last two} + \dots \\ + \text{Three purple circles on a line with a dashed box between the first and last} + \dots \\ + \dots \\ + \text{Purple circle connected to an orange circle with a dashed box} + \text{Orange circle connected to a purple circle with a dashed box} + \dots \end{array} \right\}$$

**First we modify  $C_L(E, \vec{P})$  to define  $i\mathcal{M}_{L,3\rightarrow 3}$**

**3. Symmetrize**

# Relating $i\mathcal{K}_{\text{df},3\rightarrow 3}$ to $i\mathcal{M}_{3\rightarrow 3}$

$$i\mathcal{M}_{L,3\rightarrow 3} \equiv \mathcal{S} \left\{ \begin{array}{l} \text{Orange circle} + \text{Two orange circles connected by a dashed loop} + \dots \\ + \text{Three purple circles in a line} + \text{Three purple circles with a dashed loop between the first two} + \text{Three purple circles with a dashed loop between the last two} + \dots \\ + \text{Three purple circles with a dashed loop between the first and last} + \dots \\ + \dots \\ + \text{Purple circle connected to an orange circle} + \text{Orange circle connected to a purple circle} + \dots \end{array} \right\}$$

Replacing all loop momentum sums with  $i$ -epsilon prescription integrals would give physical three-to-three scattering amplitude

# Relating $i\mathcal{K}_{\text{df},3\rightarrow 3}$ to $i\mathcal{M}_{3\rightarrow 3}$

**We find a simple form for  $i\mathcal{M}_{L,3\rightarrow 3}$**

$$i\mathcal{M}_{L,3\rightarrow 3} = i\mathcal{D}_L + \mathcal{S} \left[ \mathcal{L}_L \ i\mathcal{K}_{\text{df},3\rightarrow 3} \frac{1}{1 - iF_3 \ i\mathcal{K}_{\text{df},3\rightarrow 3}} \ \mathcal{R}_L \right]$$

$$i\mathcal{D}_L \equiv \mathcal{S} \left[ \frac{1}{1 - i\mathcal{M}_{L,2\rightarrow 2} \ iG} \ i\mathcal{M}_{L,2\rightarrow 2} \ iG \ i\mathcal{M}_{L,2\rightarrow 2} [2\omega L^3] \right]$$

$$iF_3 \equiv \frac{iF}{2\omega L^3} \left[ \frac{1}{3} + \frac{1}{1 - i\mathcal{M}_{L,2\rightarrow 2} \ iG} \ i\mathcal{M}_{L,2\rightarrow 2} \ iF \right]$$

$$\equiv \frac{iF}{2\omega L^3} \mathcal{L}_L \equiv \mathcal{R}_L \frac{iF}{2\omega L^3}$$

Relating  $i\mathcal{K}_{\text{df},3\rightarrow 3}$  to  $i\mathcal{M}_{3\rightarrow 3}$

**We find a simple form for  $i\mathcal{M}_{L,3\rightarrow 3}$**

$$i\mathcal{M}_{L,3\rightarrow 3} = i\mathcal{D}_L + \mathcal{S} \left[ \mathcal{L}_L \ i\mathcal{K}_{\text{df},3\rightarrow 3} \frac{1}{1 - iF_3 \ i\mathcal{K}_{\text{df},3\rightarrow 3}} \ \mathcal{R}_L \right]$$

**Complete analysis with infinite volume limit**

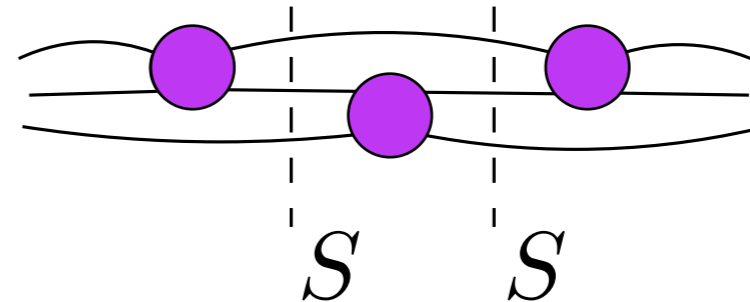
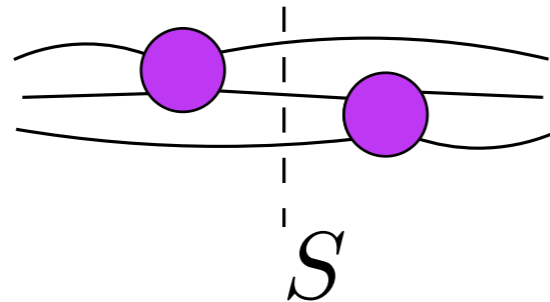
$$i\mathcal{M}_{3\rightarrow 3} = \lim_{L\rightarrow\infty} \left|_{i\epsilon} \ i\mathcal{M}_{L,3\rightarrow 3} \right.$$



$$i\mathcal{M}_{L,3\rightarrow 3} = i\mathcal{D}_L + \mathcal{S} \left[ \mathcal{L}_L \ i\mathcal{K}_{df,3\rightarrow 3} \frac{1}{1 - iF_3 \ i\mathcal{K}_{df,3\rightarrow 3}} \ \mathcal{R}_L \right]$$

**Recall**  $i\mathcal{M}_{df,3\rightarrow 3}$

$$\equiv i\mathcal{M}_{3\rightarrow 3} - \left[ i\mathcal{M}_{2\rightarrow 2} \mathcal{S} i\mathcal{M}_{2\rightarrow 2} + \int i\mathcal{M}_{2\rightarrow 2} \mathcal{S} i\mathcal{M}_{2\rightarrow 2} \mathcal{S} i\mathcal{M}_{2\rightarrow 2} + \dots \right]$$



**It reappears here...**  $i\mathcal{M}_{df,3\rightarrow 3} \equiv \lim_{L \rightarrow \infty} \Big|_{i\epsilon} [i\mathcal{M}_{L,3\rightarrow 3} - i\mathcal{D}_L]$

$$i\mathcal{D}_L \equiv \mathcal{S} \left[ \frac{1}{1 - i\mathcal{M}_{L,2\rightarrow 2} \ iG} \ i\mathcal{M}_{L,2\rightarrow 2} \ iG \ i\mathcal{M}_{L,2\rightarrow 2} [2\omega L^3] \right]$$

**encodes switches**

# Relating $i\mathcal{K}_{\text{df},3\rightarrow 3}$ to $i\mathcal{M}_{3\rightarrow 3}$

$$i\mathcal{M}_{L,3\rightarrow 3} = i\mathcal{D}_L + \mathcal{S} \left[ \mathcal{L}_L \ i\mathcal{K}_{\text{df},3\rightarrow 3} \frac{1}{1 - iF_3 \ i\mathcal{K}_{\text{df},3\rightarrow 3}} \ \mathcal{R}_L \right]$$

$$i\mathcal{M}_{3\rightarrow 3} = \lim_{L \rightarrow \infty} \left. i\mathcal{M}_{L,3\rightarrow 3} \right|_{i\epsilon}$$

**Gives integral equation relating  $i\mathcal{K}_{\text{df},3\rightarrow 3}$  to  $i\mathcal{M}_{3\rightarrow 3}$**

**Completes formal story (for the setup considered!)**

**Relation only depends on on-shell scattering quantities**