Three-body observables from the lattice

Maxwell T. Hansen

Institut für Kernphysik and HIM

Johannes Gutenberg Universität Mainz

Lattice 2015, Kobe, Japan

July 15th, 2015

HIM

Helmholtz-Institut Mainz

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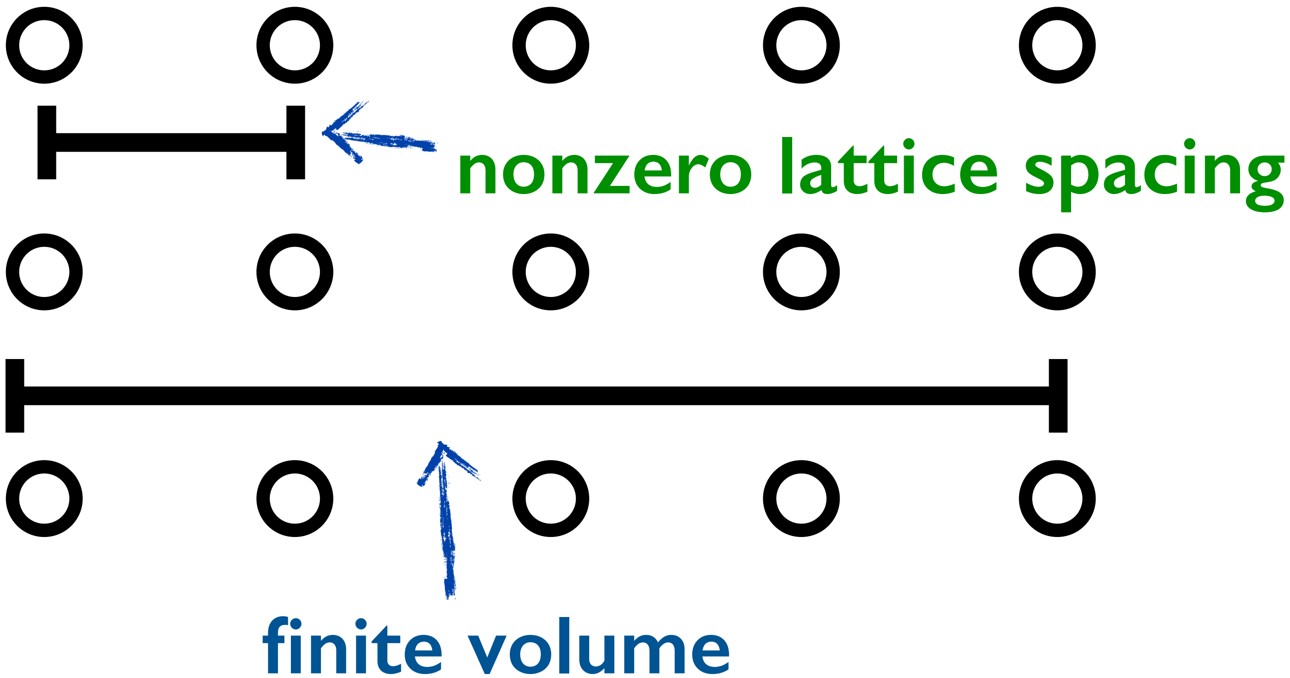
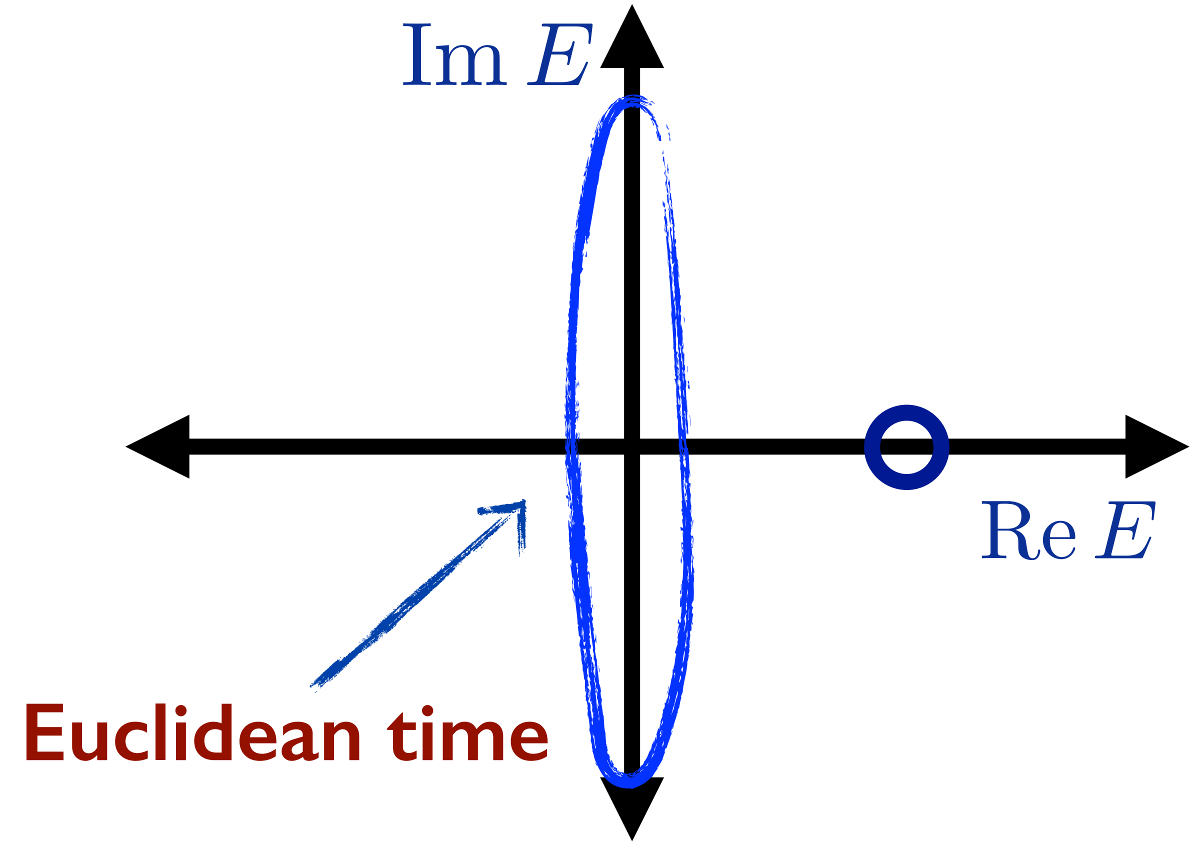
Compromises in numerical Lattice QCD





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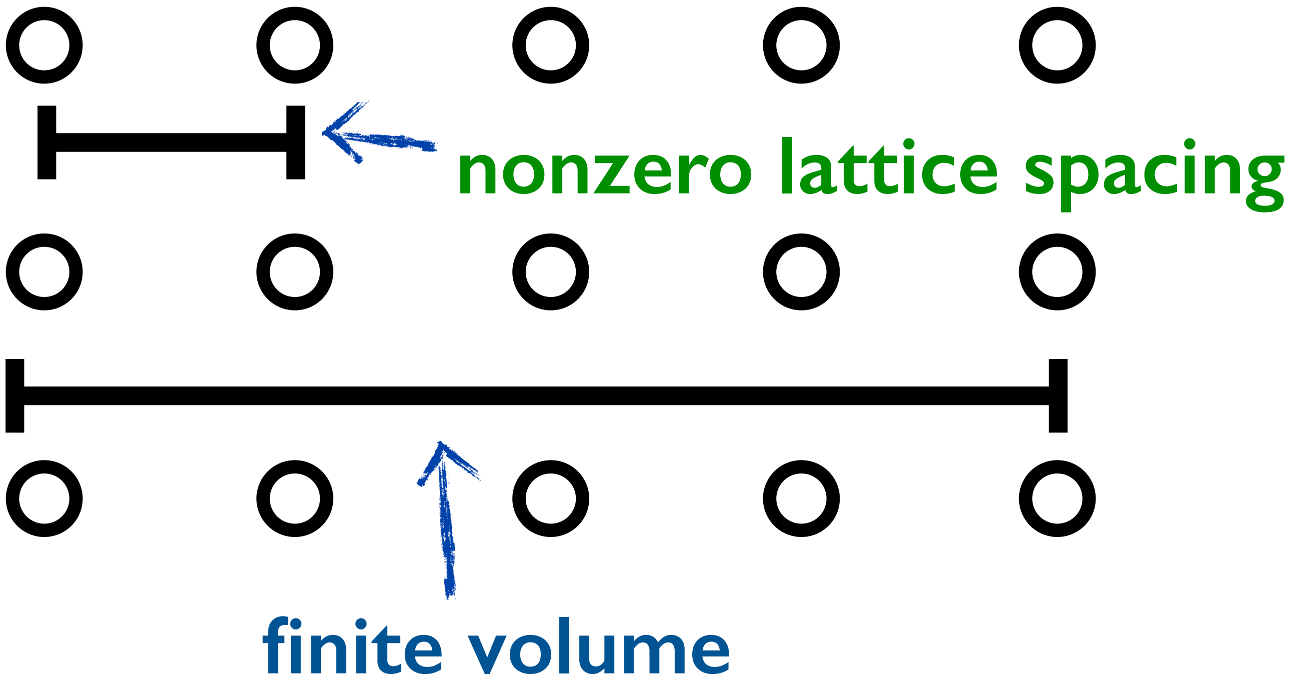
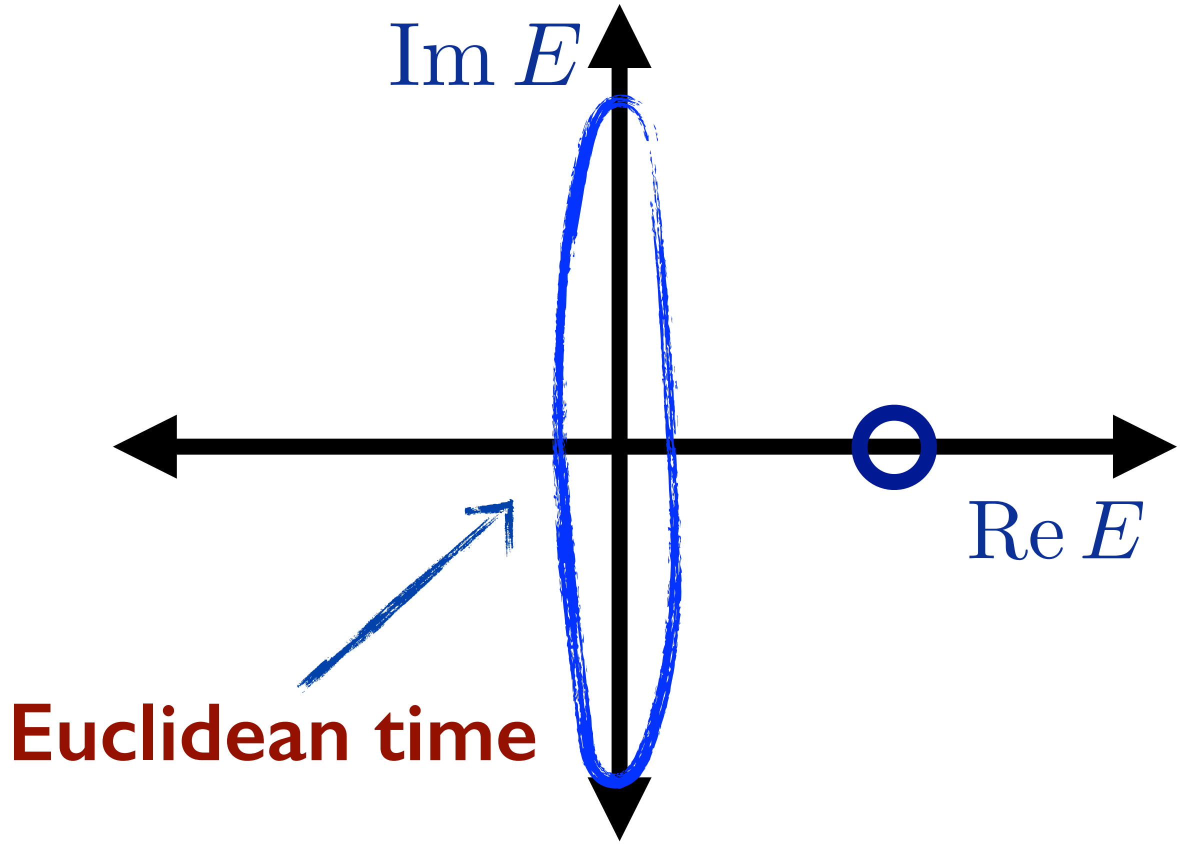
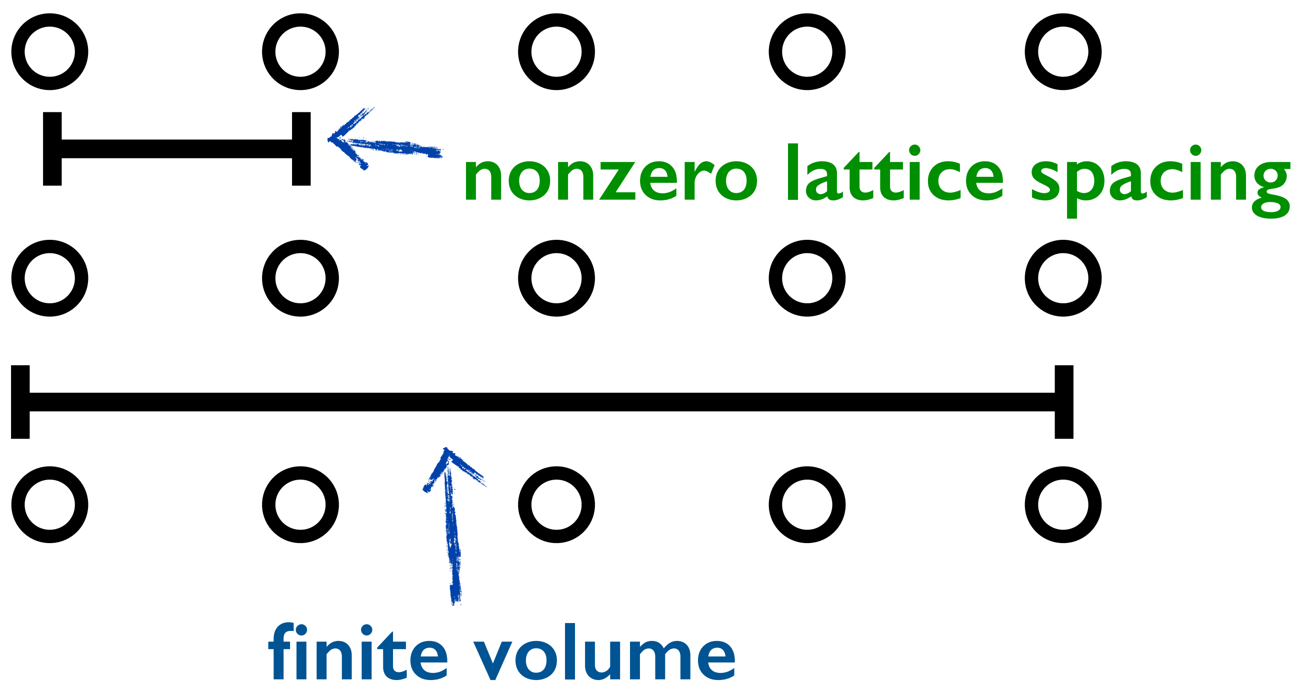
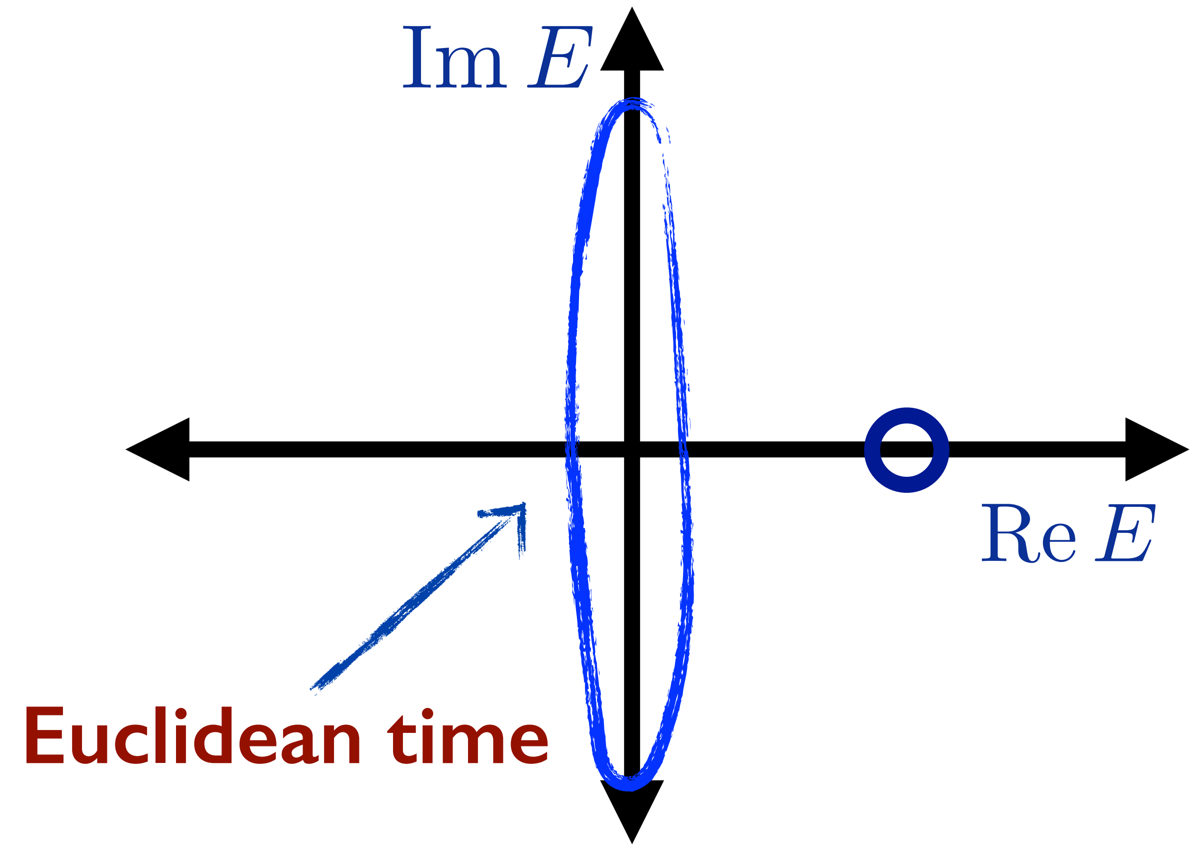




Not possible to directly calculate scattering amplitudes

Compromises in numerical Lattice QCD





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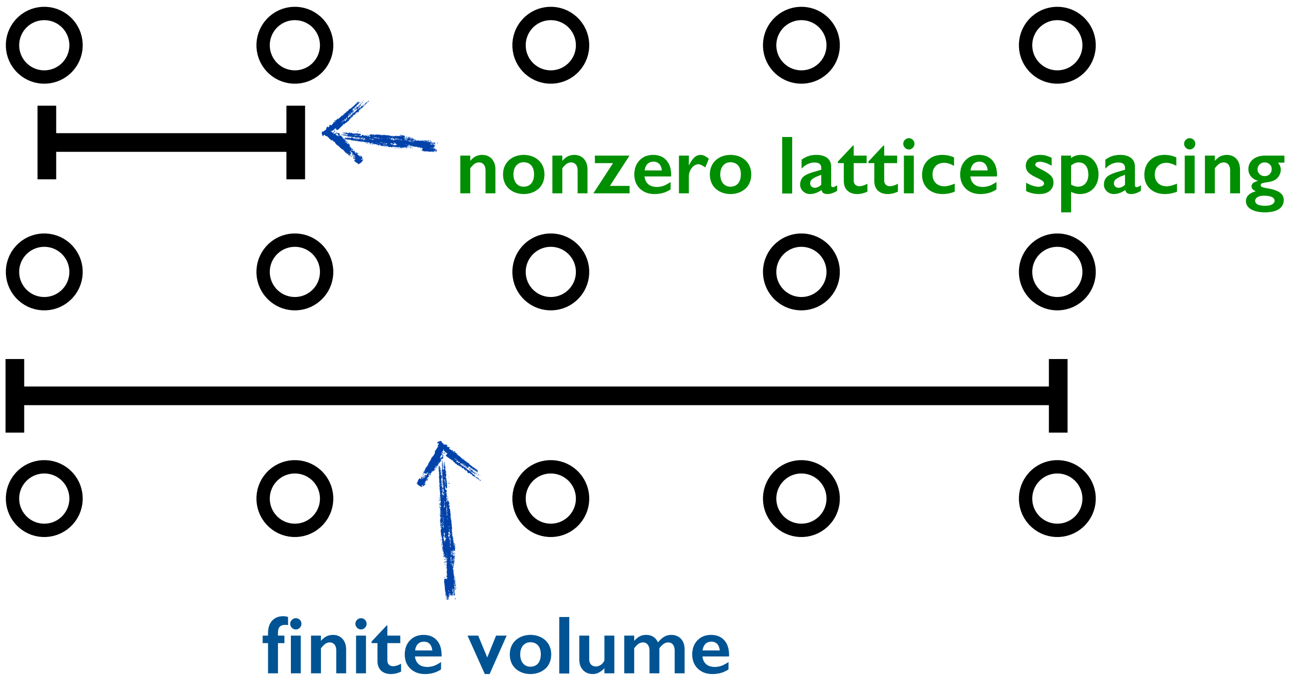
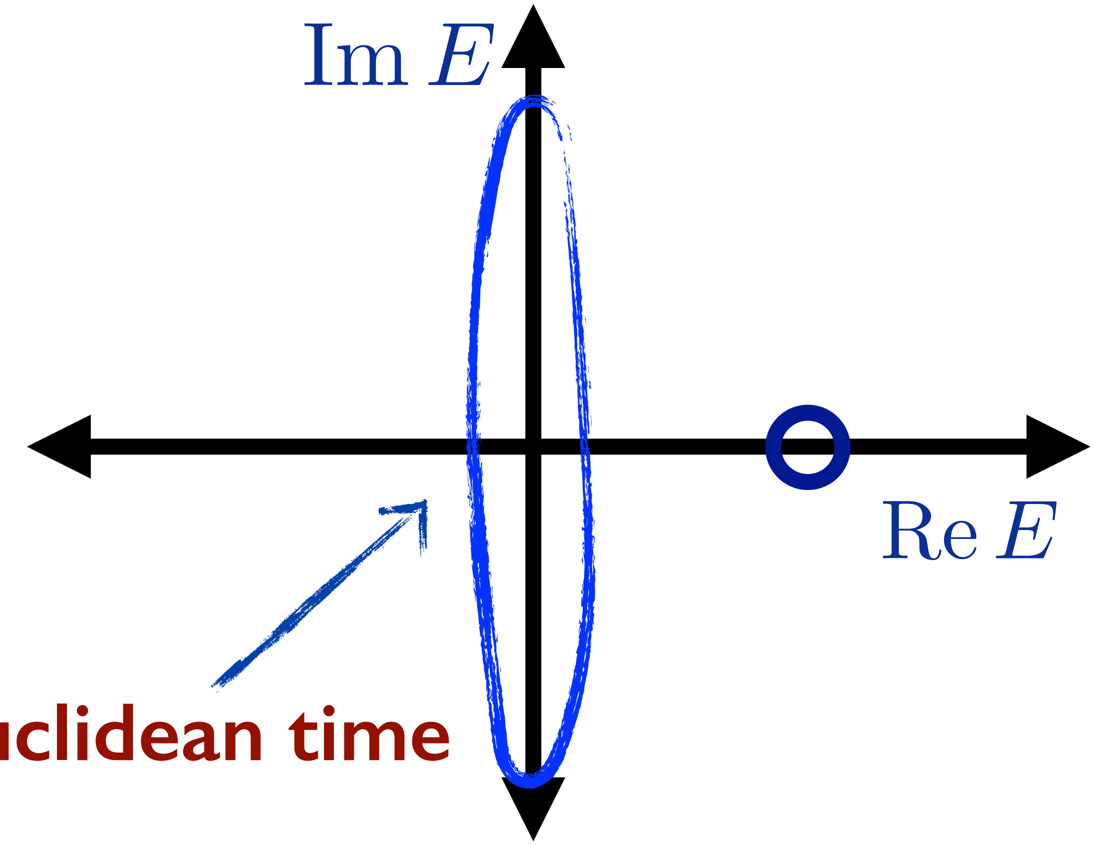
Large Euclidean time limit is dominated by either threshold or off-shell states

L. Maiani and M. Testa, *Phys.Lett.* B245 (1990) 585–590

Compromises in numerical Lattice QCD





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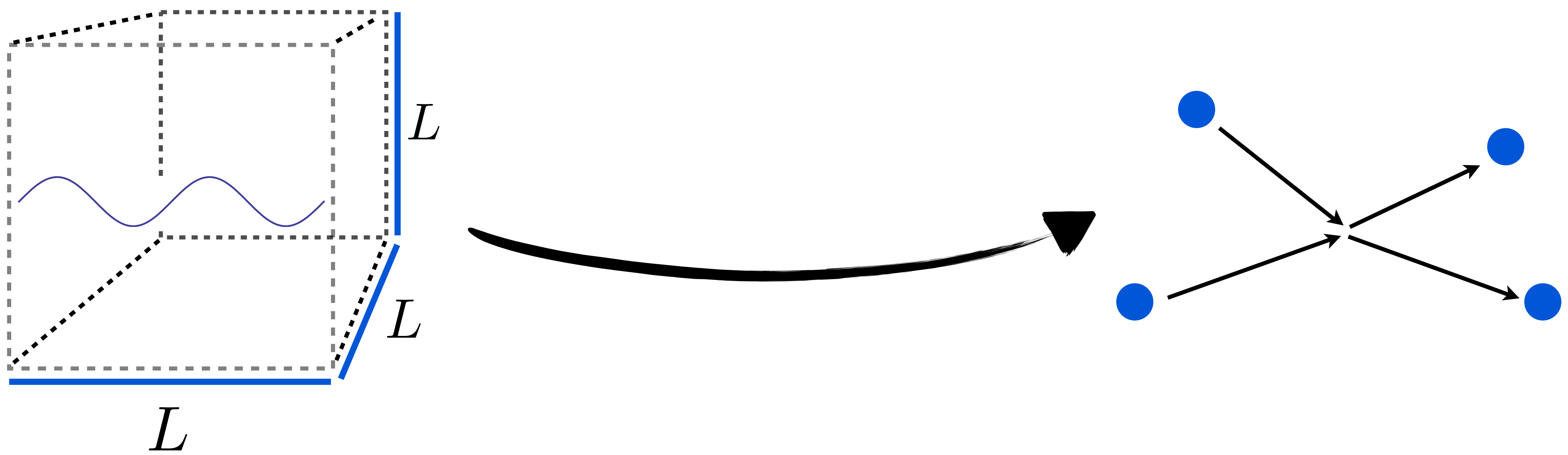
L. Maiani and M. Testa, *Phys.Lett.* B245 (1990) 585–590

Analytic continuation of numerical Euclidean correlators is an ill-posed problem

Martin Lüscher found a method to circumvent this issue and extract $\pi\pi \to \pi\pi$ scattering from LQCD.

Lüscher, M. *Nucl. Phys* B354, 531-578 (1991)



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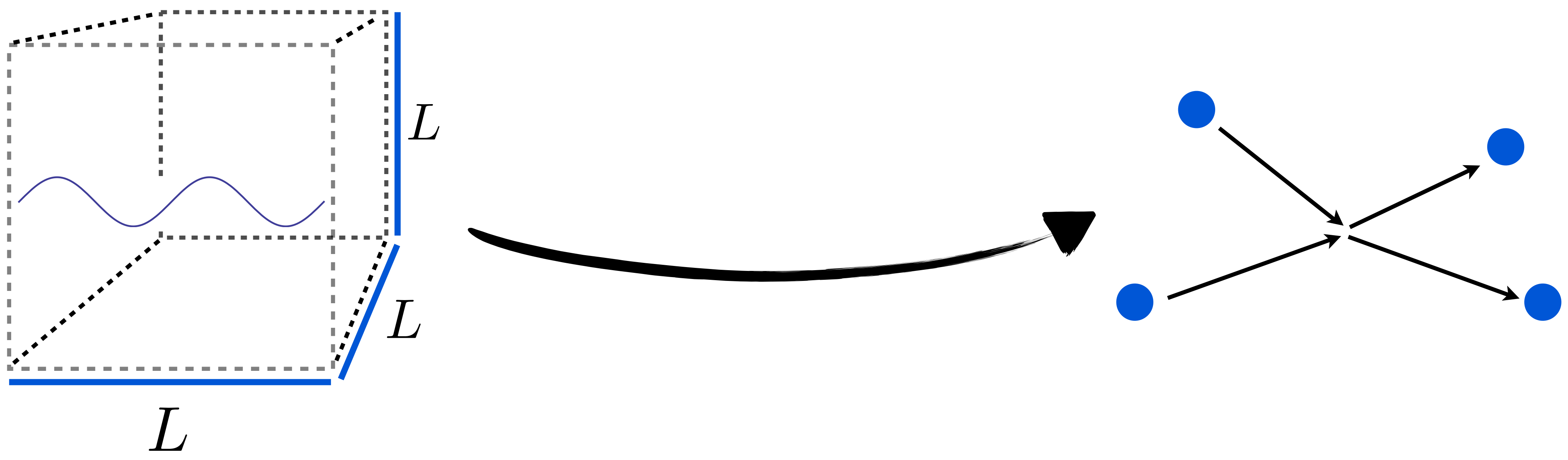


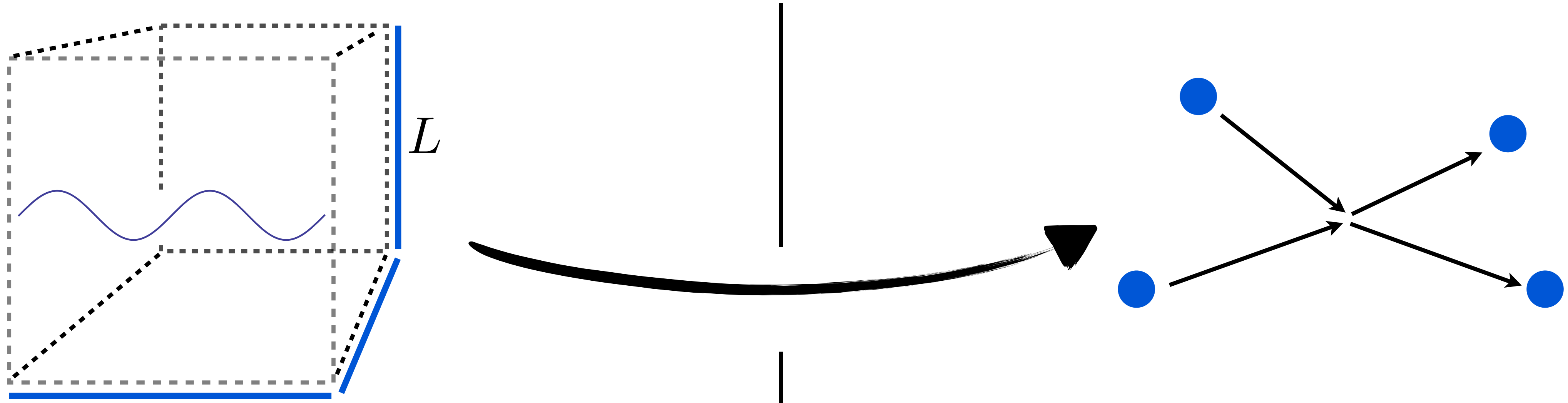
His key insight was to **use finite volume as a tool.**

He gave a mapping between the **finite-volume energy spectrum** and **elastic pion scattering amplitude.**

The same problem was addressed earlier in perturbative non-relativistic quantum mechanics

K. Huang and C. Yang, *Phys. Rev.* 105 (1957) 767-775

Understanding Lüscher's Result



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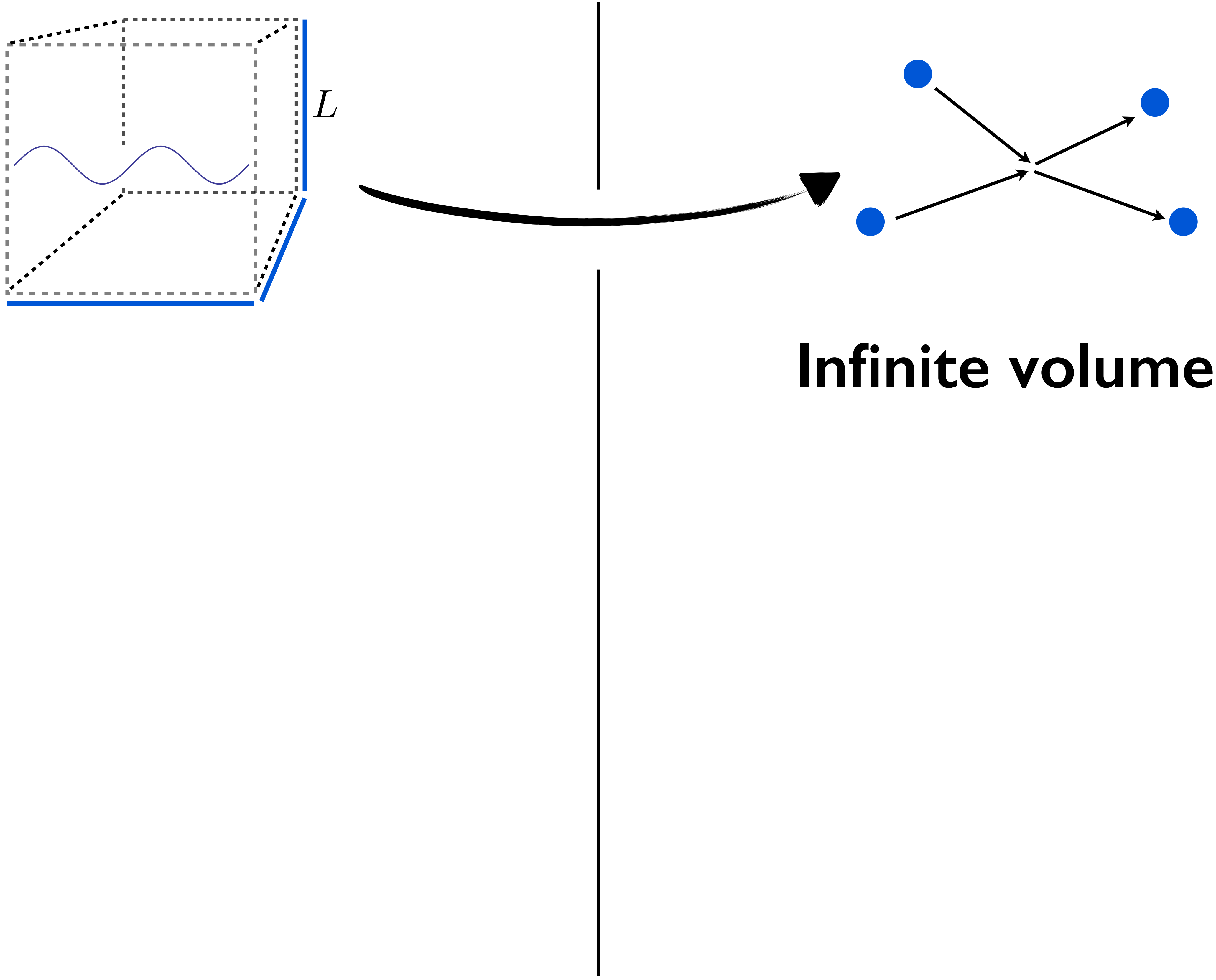
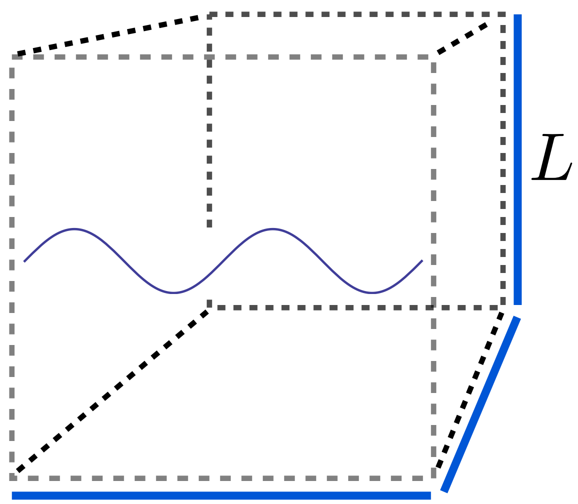
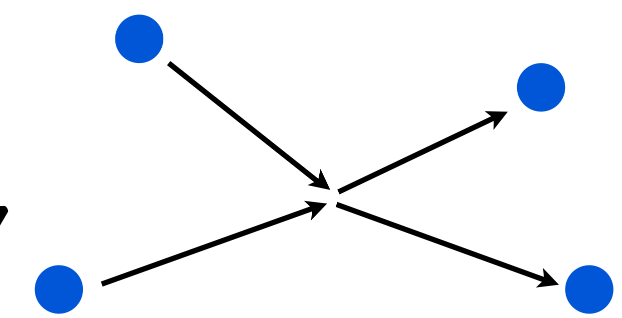


Infinite volume

Understanding Lüscher's Result





Infinite volume

Decompose scattering amplitude in partial waves

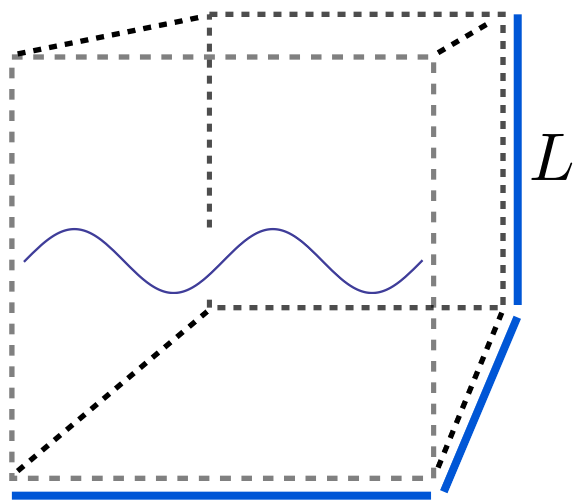
One real observable…

$$\delta_\ell(E^*)$$

in each partial wave

at each CM energy

Understanding Lüscher's Result



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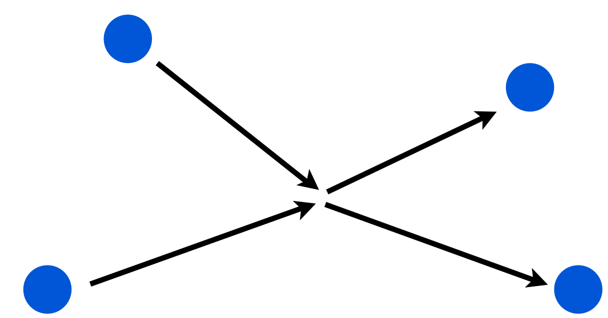

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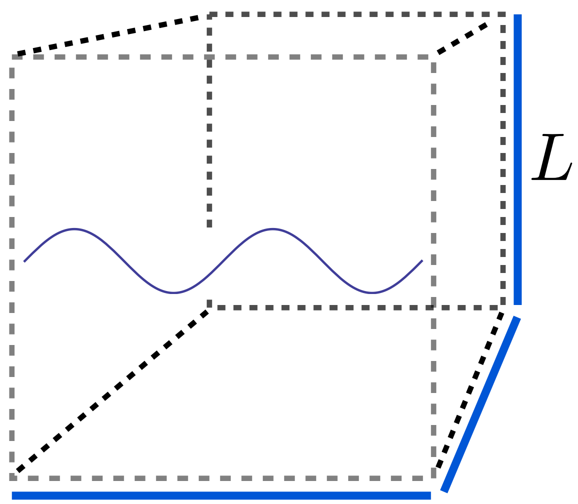
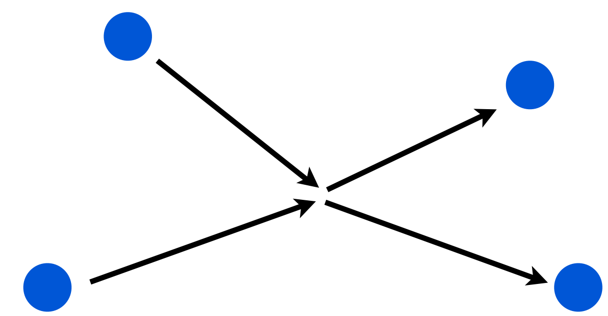
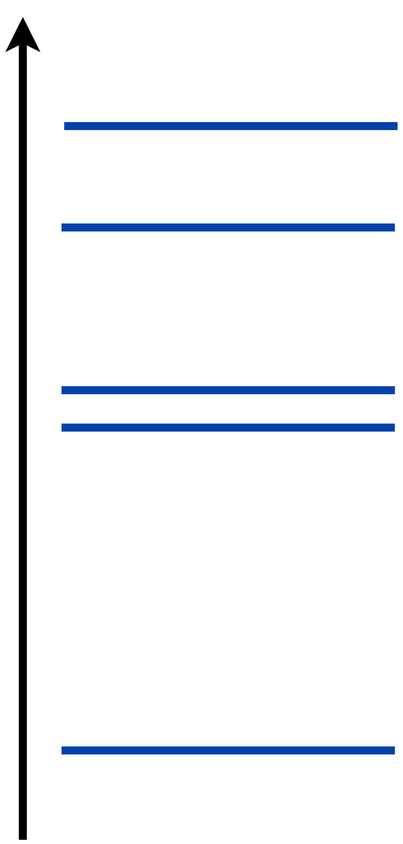




Finite volume

Discrete tower of energy levels



$$E_n(L, \vec{P})$$

depends on
finite-volume size
total momentum

Infinite volume

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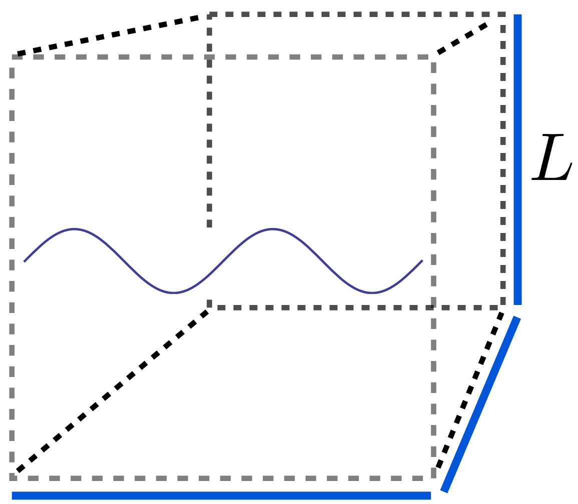

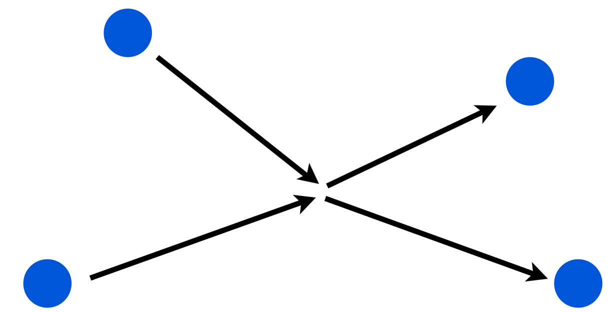

Finite volume

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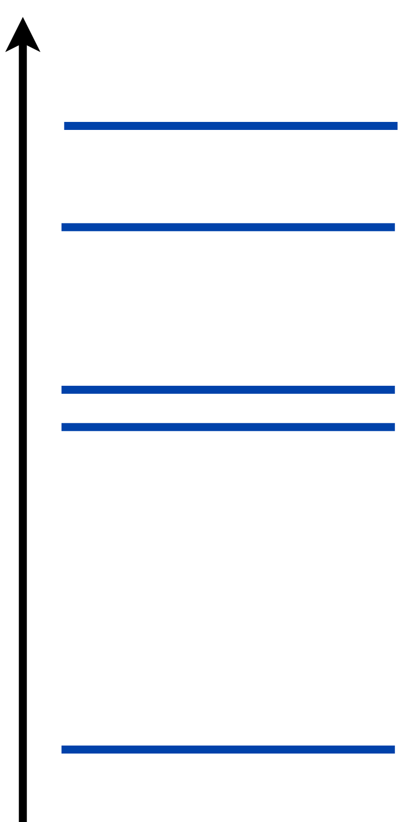

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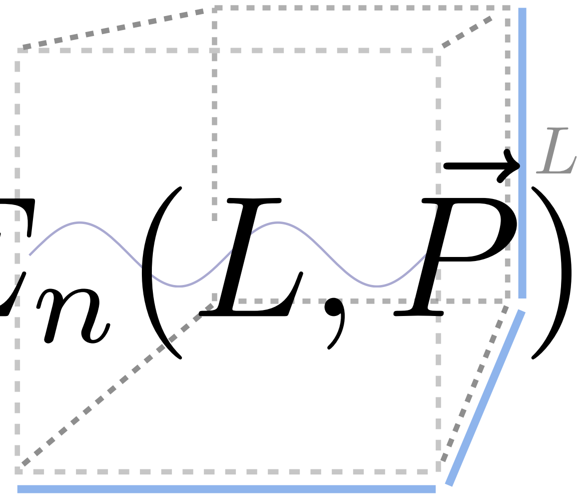
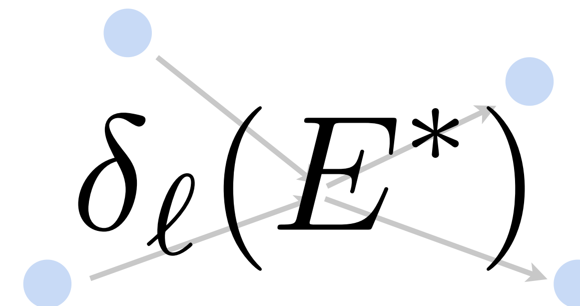
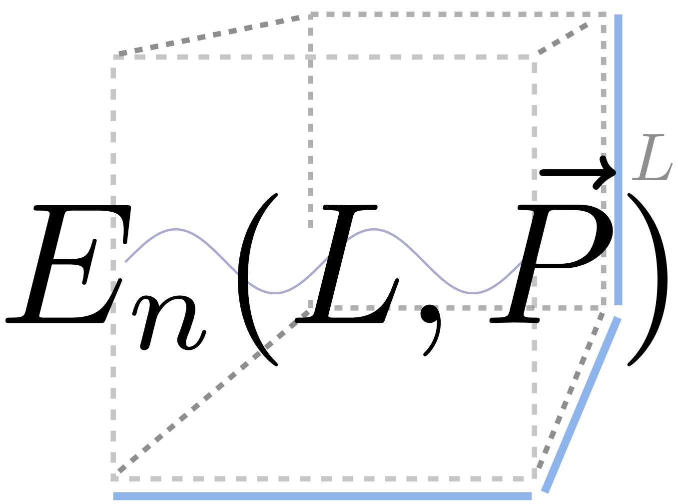




$$E_n(L, \vec{P}) \longrightarrow \delta_\ell(E^*)$$

Finite volume | **Infinite volume**

Understanding Lüscher's Result



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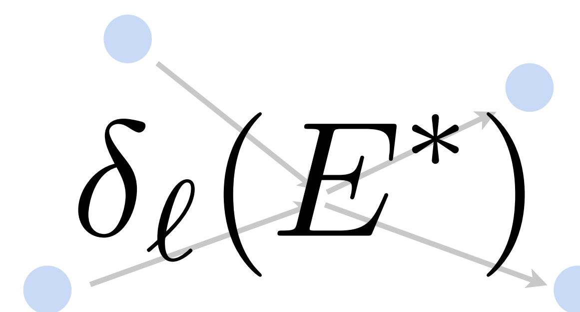


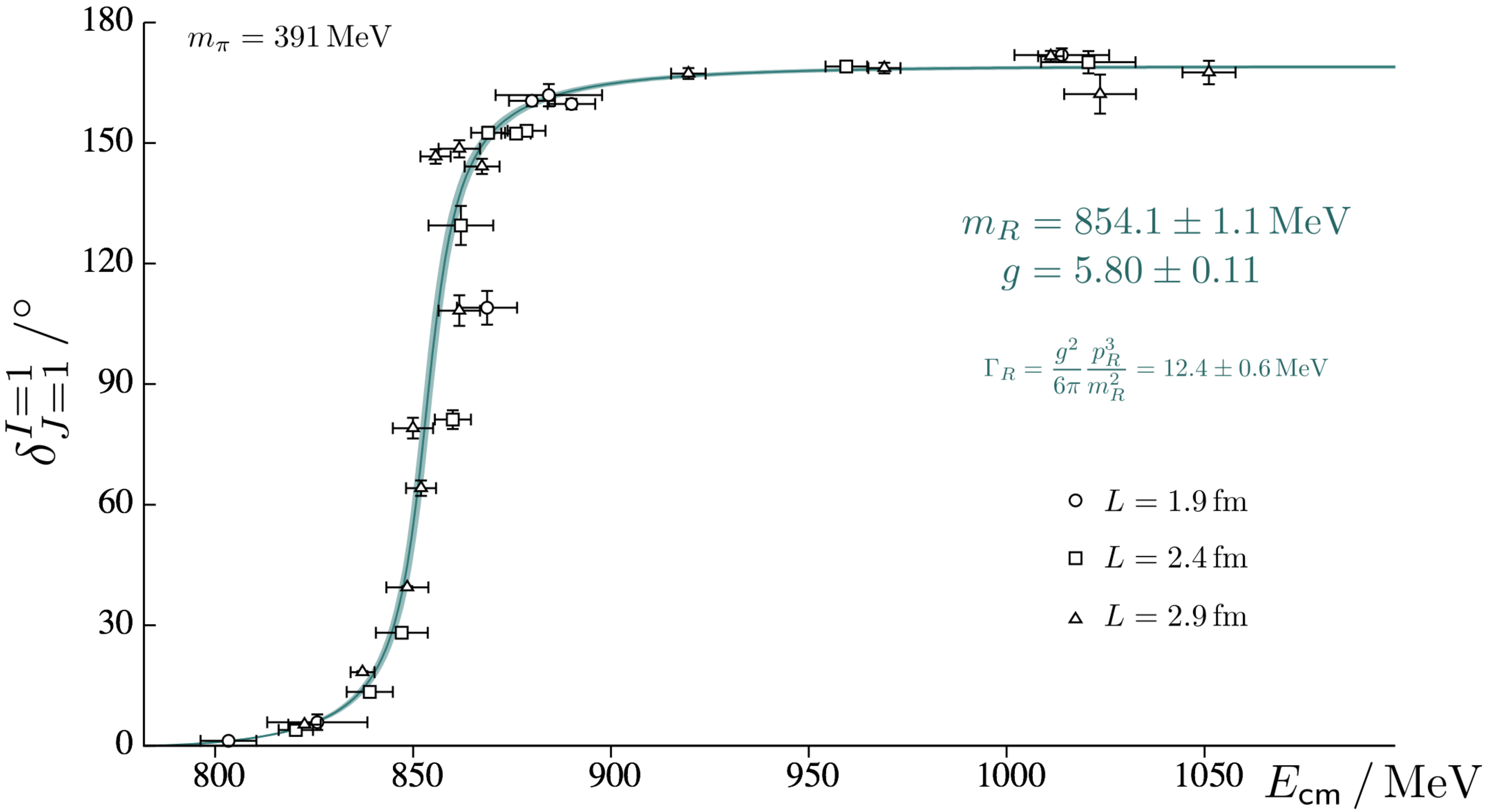
Infinite volume

$$\boxed{\det\left[\cot\delta(E_n^*) + \cot\phi(E_n, \vec{P}, L)\right] = 0}$$

$$\left\{\cot\phi = -\frac{2}{\pi\gamma}\sum_{\ell m}\frac{1}{q^{*(\ell+1)}}\mathcal{Z}^{P}_{\ell m}[1, q^{*2}]\int d\Omega\, Y^*_{\ell_1 m_1} Y^*_{\ell m} Y_{\ell_2 m_2}\right\}$$

Lüscher's method has led to a large body of work extracting phase shifts from Lattice QCD.

$\pi\pi \to \pi\pi$

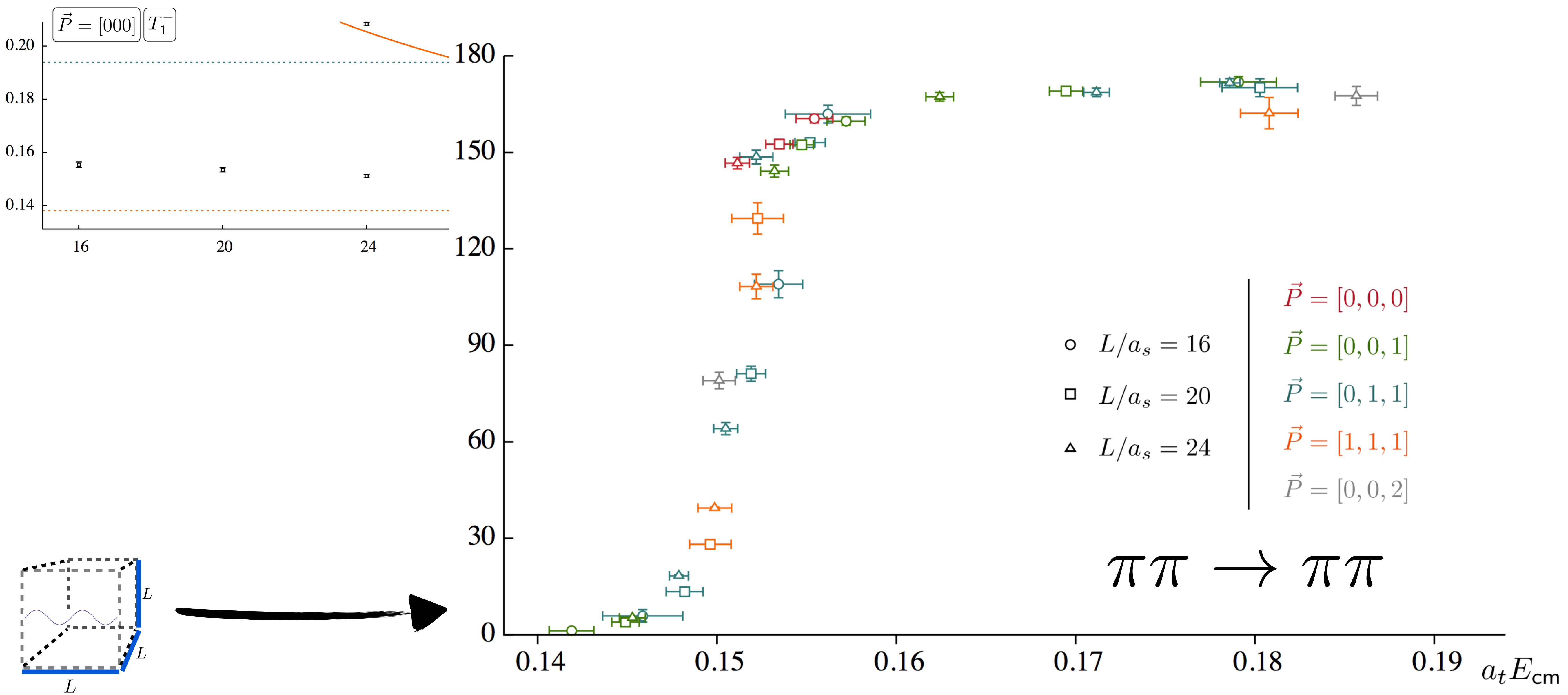


$$m_\pi = 391\,\mathrm{MeV}$$

from Dudek, Edwards, Thomas in *Phys.Rev*. D87 (2013) 034505

Using Lüscher's Result (p-wave)

$$\cot \delta_{\ell=1}(E_n^*) = \frac{1}{\pi^{3/2}\gamma q_n^*} \mathcal{Z}_{00}^{P}[1, q_n^{*2}]$$



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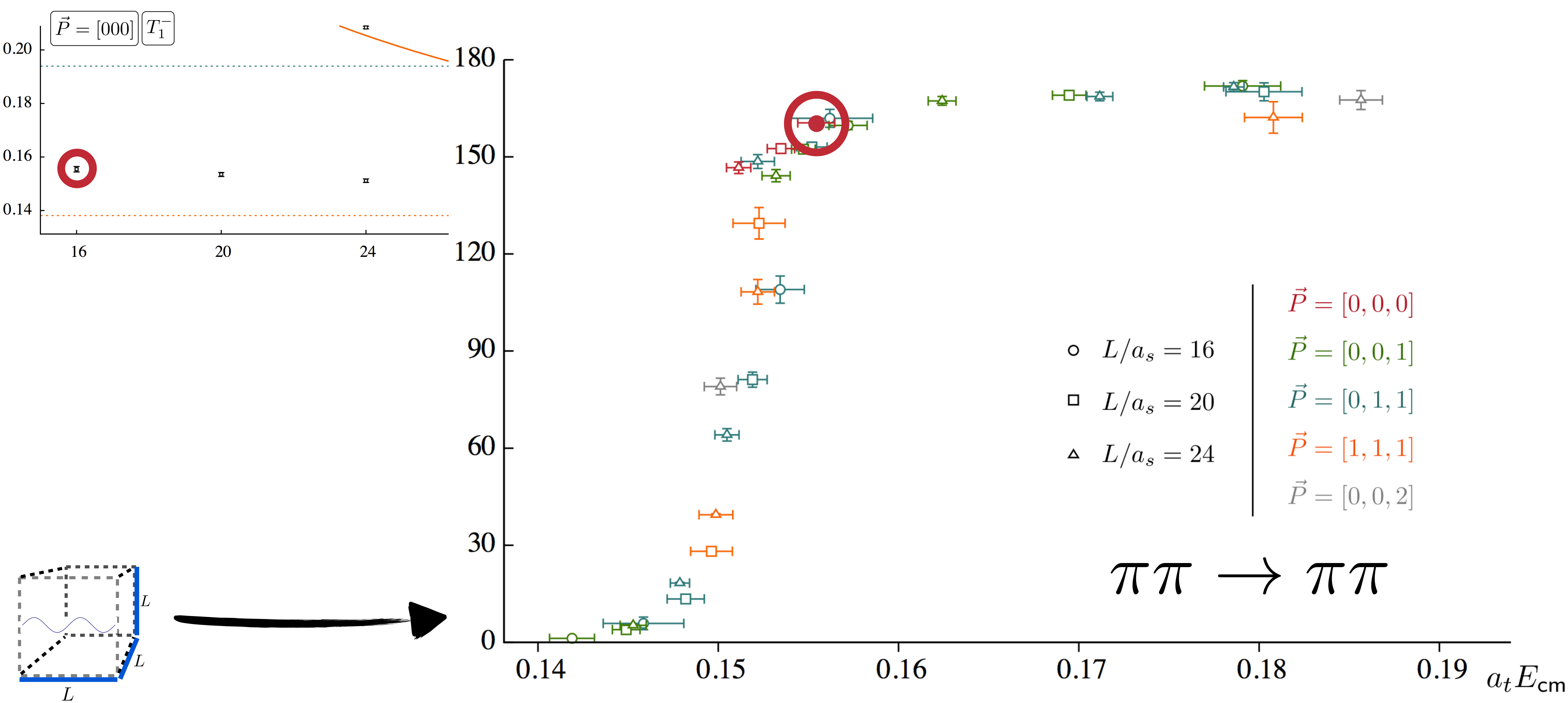

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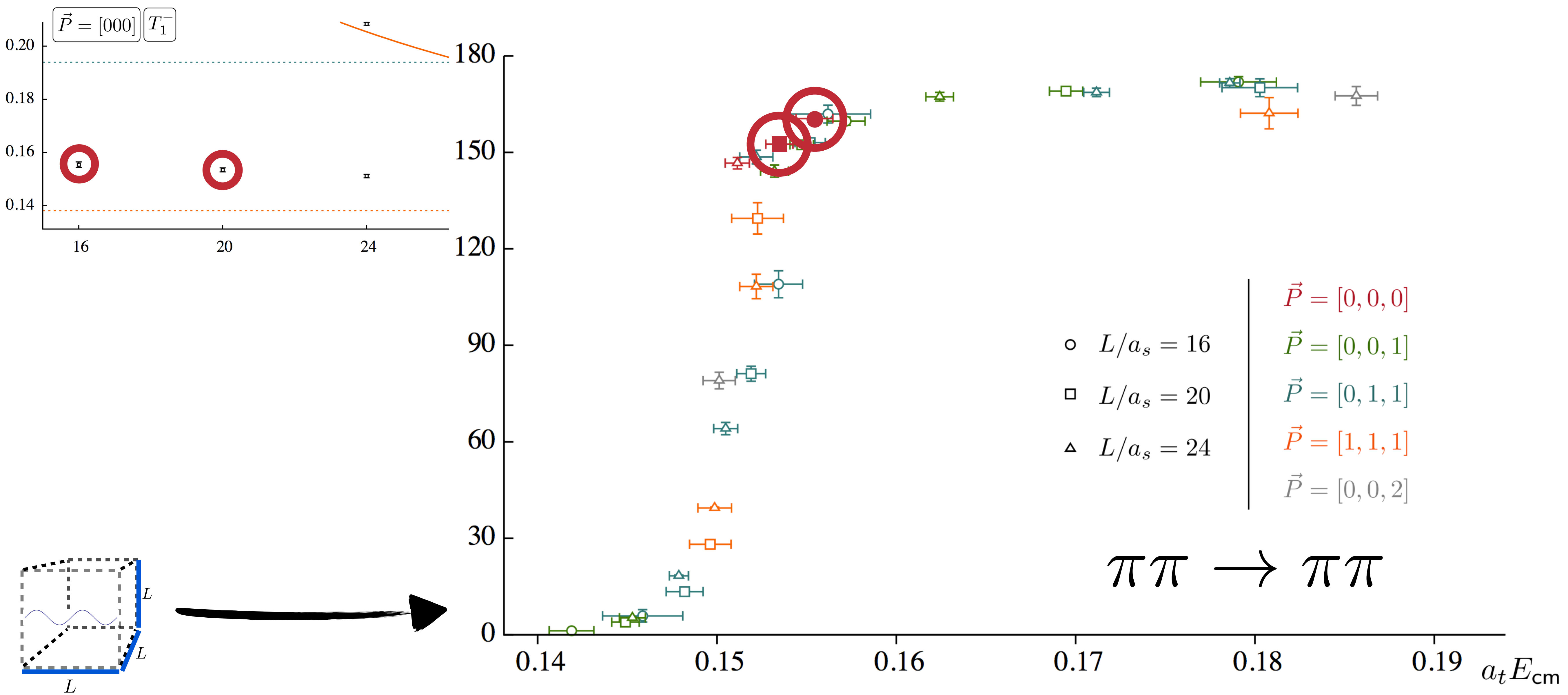

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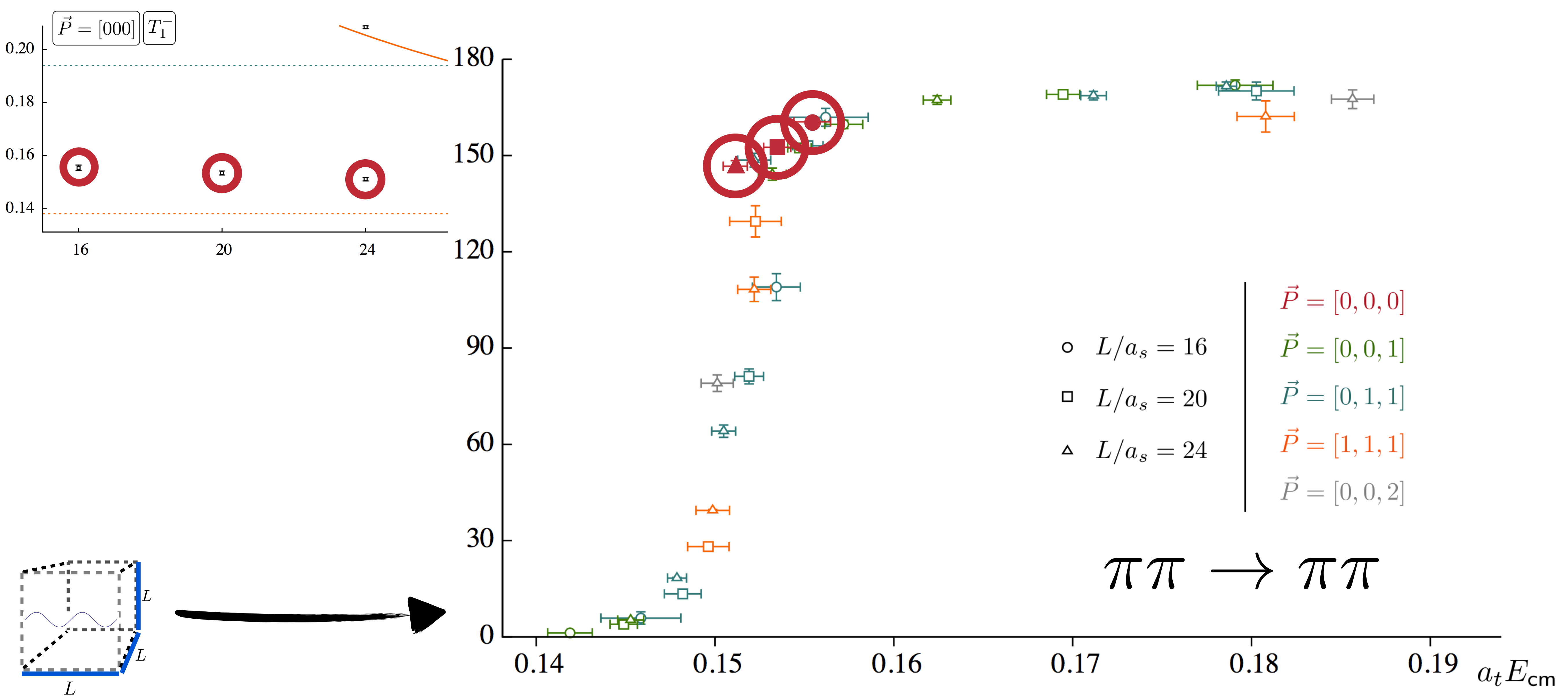

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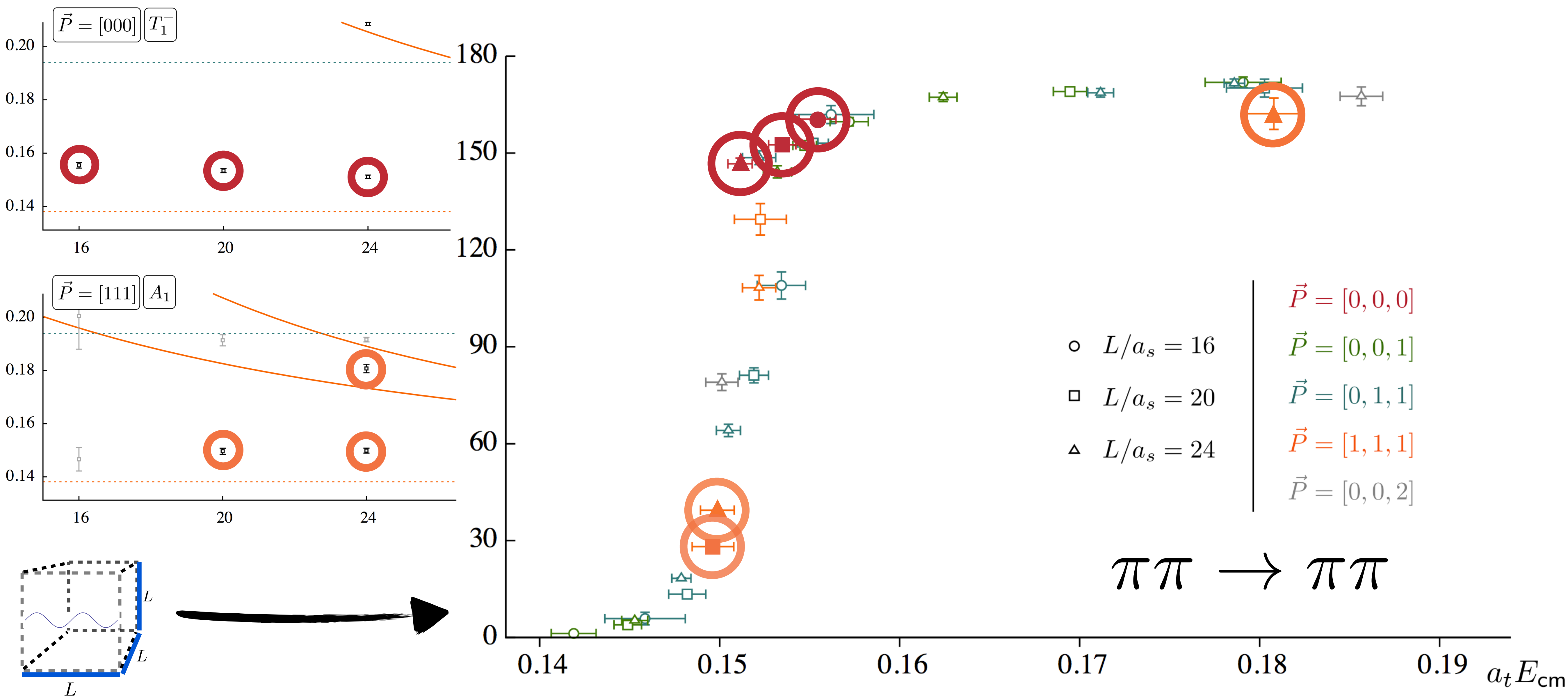

Lüscher's result has since been generalized to accommodate **moving frames, non-indentical particles, multiple two-particle channels, particles with spin**

Rummukainen and Gottlieb, *Nucl. Phys.* B450, 397 (1995)
Beane, Bedaque, Parreno, and Savage, *Nucl. Phys.* A747, 55 (2005)
Kim, Sachrajda, and Sharpe, *Nucl. Phys.* B727, 218 (2005)
Christ, Kim, Yamazaki, *Phys. Rev.* D72, 114506 (2005)
Bernard, Lage, Meißner, and Rusetsky, JHEP, 1101, 019 (2011)
MTH and Sharpe, *Phys.Rev. D86* (2012) 016007
Briceño and Davoudi, *Phys.Rev. D88* (2013) 094507
Li and Liu, Phys. Rev. D87, 014502 (2013)
Briceño, *Phys. Rev.* D 89, 074507 (2014)

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The **numerical implementation** of the formalism has reached an **impressive level**

Wilson, et.al. (2015) 1507.02599
Talks yesterday by Ben Hörz, John Bulava, Tadeusz Janowski, Gordon Donald, Dehua Guo

However, there is **no general method** for extracting scattering amplitudes involving **more than two hadrons**.

This limits LQCD investigation of...

resonances which decay into more than two hadrons

$$\omega(782) \to \pi\pi\pi \qquad N(1440) \to N\pi\pi$$

two-particle scattering above three-particle thresholds

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weak decays and transitions ala Lellouch-Lüscher

$$K \to \pi\pi\pi$$

Lellouch and Lüscher, *Com.Math.Phys.* 219, 31 (2001)
Meyer, (2012) 1202.6675
Agadjanov, Bernard, Meißner, Rusetsky, Nucl. Phys. B886 (1014) 1199
Briceño, MTH, Walker-Loud, *Phys. Rev.* D 91, 034501 (2015)
MTH and Briceño, (2015) 1502.04314

Outline

$1/L$ expansions

Nonperturbative studies in
non-relativistic quantum theories

Three-particle bound state

Relativistic QFT in finite volume

$1/L$ expansions

In 1957, Huang and Yang determined energy shift for n identical bosons in a box

K. Huang and C. Yang, *Phys. Rev.* 105 (1957) 767-775

$$\boxed{E_0(n,L) = \frac{4\pi\, a}{M\, L^3}\left\{\binom{n}{2} - \left(\frac{a}{\pi\, L}\right)\binom{n}{2}\mathcal{I} + \left(\frac{a}{\pi\, L}\right)^2\left\{\binom{n}{2}\mathcal{I}^2 - \left[\binom{n}{2}^2 - 12\binom{n}{3} - 6\binom{n}{4}\right]\mathcal{J}\right\}\right\} + \mathcal{O}\left(L^{-6}\right)}$$

where a is the two-particle scattering length and

$$\mathcal{I} = \lim_{\Lambda\to\infty}\sum_{\mathbf{i}\neq\mathbf{0}}^{|\mathbf{i}|\leq\Lambda}\frac{1}{|\mathbf{i}|^2} - 4\pi\Lambda = -8.91363291781 \qquad \mathcal{J} = \sum_{\mathbf{i}\neq\mathbf{0}}\frac{1}{|\mathbf{i}|^4} = 16.532315959$$

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In 2007 Beane, Detmold and Savage pushed the order to $1/L^6$ and the latter two calculated to $1/L^7$ the next year

Beane, S., Detmold, W. & Savage, M. *Phys. Rev.* D76 (2007) 074507
Detmold, W. & Savage, M. *Phys. Rev.* D77 (2008) 057502

At $1/L^6$ a three-particle contact term appears

$1/L$ expansions

Last year Detmold and Flynn performed a similar calculation for matrix elements

Detmold and Flynn, *Phys. Rev.* D91, 074509 (2015)

$$\begin{aligned}\langle n|J|n\rangle \;=\;& n\alpha_1 + \frac{n\alpha_1 a^2}{\pi^2 L^2}\binom{n}{2}\mathcal{J} + \frac{\alpha_2}{L^3}\binom{n}{2} \\ &+ \frac{2n\alpha_1 a^3}{\pi^3 L^3}\binom{n}{2}\left\{\mathcal{K}\binom{n}{2} - \left[\mathcal{I}\mathcal{J} + 4\mathcal{K}\binom{n-2}{1} + \mathcal{K}\binom{n-2}{2}\right]\right\} - \frac{2\alpha_2 a}{\pi L^4}\binom{n}{2}\mathcal{I} \\ &+ \frac{n\alpha_1 a^4}{\pi^4 L^4}\left[3\mathcal{I}^2\mathcal{J} + \mathcal{L}\left(186 - \frac{241n}{2} + \frac{29}{2}n^2\right) + \mathcal{J}^2\left(\frac{n^2}{4} + \frac{3n}{4} - \frac{7}{2}\right)\right. \\ &\qquad \left. + \mathcal{I}\mathcal{K}(4n-14) + \mathcal{U}(32n-64) + \mathcal{V}(16n-32)\right] + \mathcal{O}(1/L^5)\,.\end{aligned}$$

Here $\mathcal{I}, \mathcal{J}, \cdots$ are known geometric constants and α_1, α_2 are one- and two-boson current couplings

Nonperturbative and non-relativistic

Non-relativistic Faddeev analysis

In 2012, Polejaeva and Rusetsky derived a Lüscher-like result using non-relativistic Faddeev equations

Polejaeva and Rusetsky, *Eur. Phys. J.* A48, 67 (2012)

Demonstrates that on-shell S-matrix determines spectrum

Difficult to extract scattering from the formalism

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Dimer formalism

In 2013, Briceño and Davoudi studied three-particles in finite-volume using the Dimer formalism

Briceño and Davoudi, *Phys. Rev.* D87, 094507 (2013)

Recovered Lüscher result when two of the three become bound

$$k \cot \delta = -k \cot \phi + \eta \frac{e^{-\gamma L}}{L}$$

Final result involves an integral equation that one needs to solve numerically

Three-particle bound state

This year Meißner, Rios and Rusetsky determined the finite-volume energy shift to a three-body bound state

$$\boxed{\Delta E = c\frac{\kappa^2}{m}\frac{|A|^2}{(kL)^{3/2}}\ \exp(-2\kappa L/\sqrt{3}) + \cdots}$$

Meißner, Rios and Rusektsky. *Phys. Rev. Lett.* 114, 091602 (2015)

Assumes the unitary limit for two-particle scattering

Result derived using non-relativistic quantum mechanics

Relativistic QFT in finite volume

based on

MTH and Sharpe, *Phys. Rev.* D90, 116003 (2014)
MTH and Sharpe, (2015) 1504.04248
MTH and Sharpe, *to appear*

guided by

Kim, Sachrajda, and Sharpe, *Nucl. Phys.* B727, 218 (2005)

Deriving Lüscher's Result

Finite volume	**Infinite volume**

Deriving Lüscher's Result

Finite volume | **Infinite volume**

single scalar, mass m

Deriving Lüscher's Result

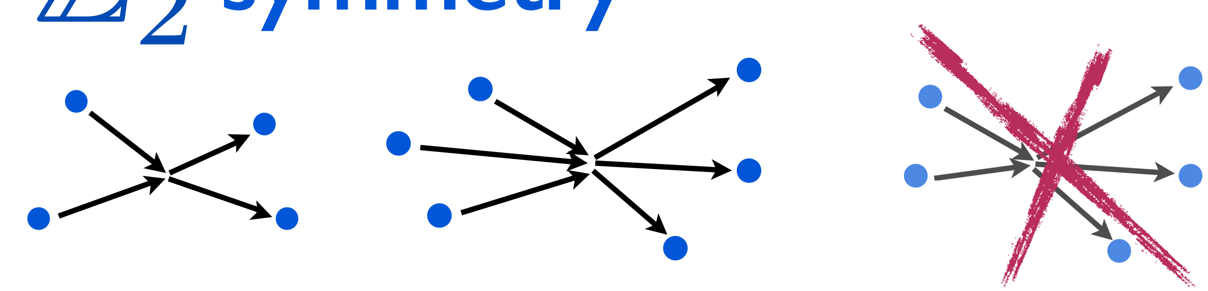
Finite volume

Infinite volume

single scalar, mass m

relativistic field theory

$\mathbb{Z}_2$ **symmetry**



(For pions in QCD this is G-parity)

Deriving Lüscher's Result

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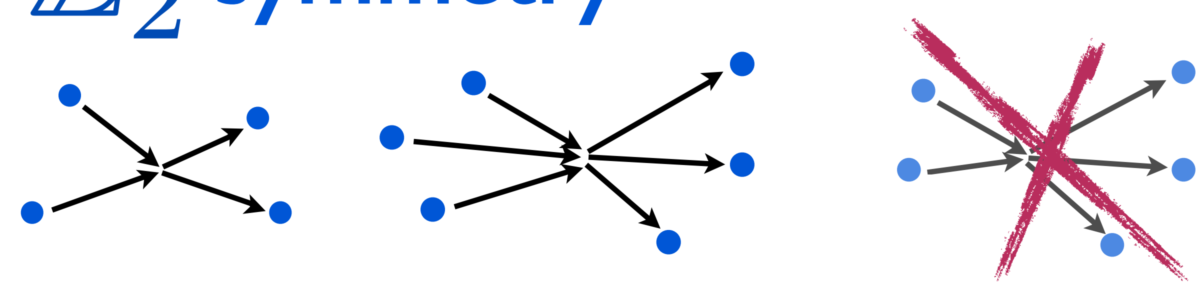
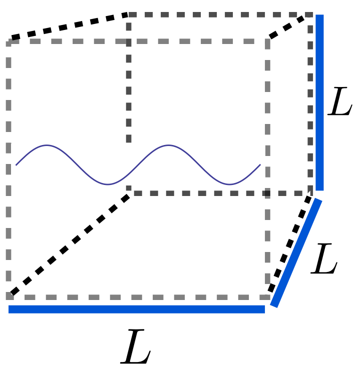


(For pions in QCD this is G-parity)

Include all vertices
with even number of legs

Deriving Lüscher's Result

Finite volume



cubic, spatial volume (extent L)

periodic boundary conditions

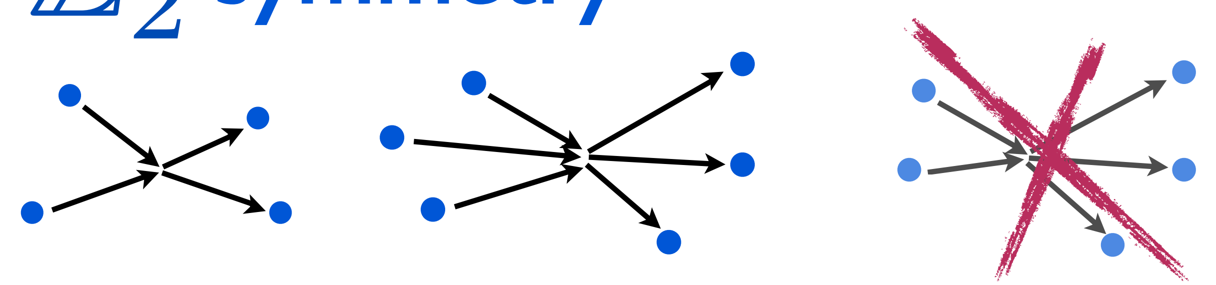
$\vec{p} \in (2\pi/L)\mathbb{Z}^3$

Infinite volume

single scalar, mass m

relativistic field theory

$\mathbb{Z}_2$ **symmetry**



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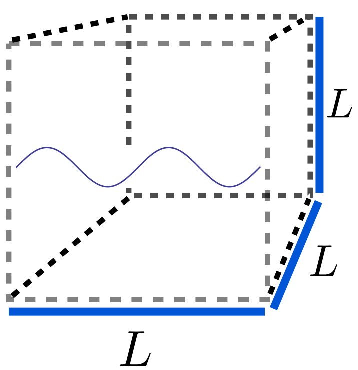
Finite volume



cubic, spatial volume (extent L)

periodic boundary conditions

$\vec{p} \in (2\pi/L)\mathbb{Z}^3$

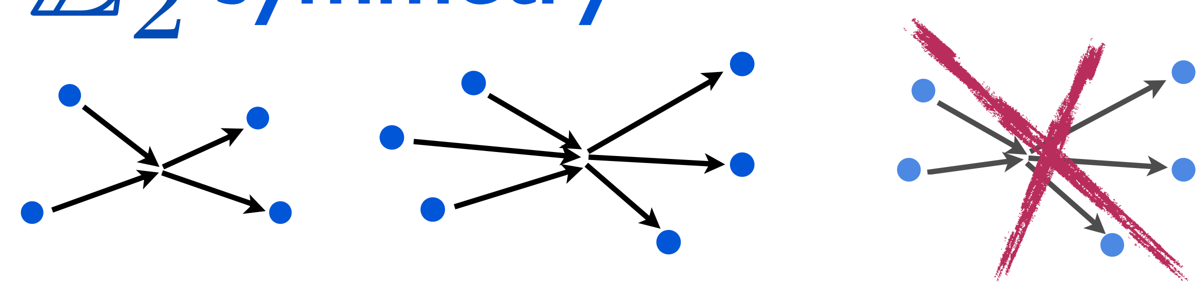
time direction **infinite** and **Minkowski**

Infinite volume

single scalar, mass m

relativistic field theory

$\mathbb{Z}_2$ **symmetry**

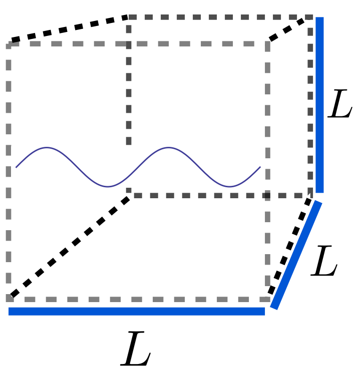


(For pions in QCD this is G-parity)

Include all vertices
with even number of legs

Deriving Lüscher's Result

Finite volume



cubic, spatial volume (extent L)

periodic boundary conditions

$$\vec{p} \in (2\pi/L)\mathbb{Z}^3$$

time direction **infinite** and **Minkowski**

Take L large enough to ignore

dropped throughout!

$$e^{-mL}$$

Take space to be continuous

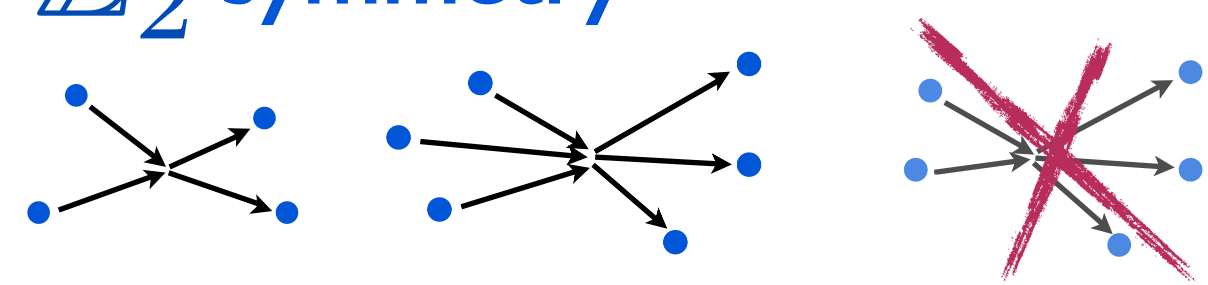
lattice spacing set to zero

Infinite volume

single scalar, mass m

relativistic field theory

$\mathbb{Z}_2$ **symmetry**



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Include all vertices
with even number of legs

$$C_L(E,\vec{P}) \equiv \int_L d^4x\, e^{i(Ex^0-\vec{P}\cdot\vec{x})} \langle 0|\mathrm{T}\sigma(x)\sigma^\dagger(0)|0\rangle$$

Kim, Sachrajda, and Sharpe, *Nucl. Phys.* B727, 218 (2005)

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energy E, momentum $\vec{P} = (2\pi/L)\vec{n}_P$

CM energy $E^{*2} \equiv E^2 - \vec{P}^2$

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interpolating field
even particle quantum numbers

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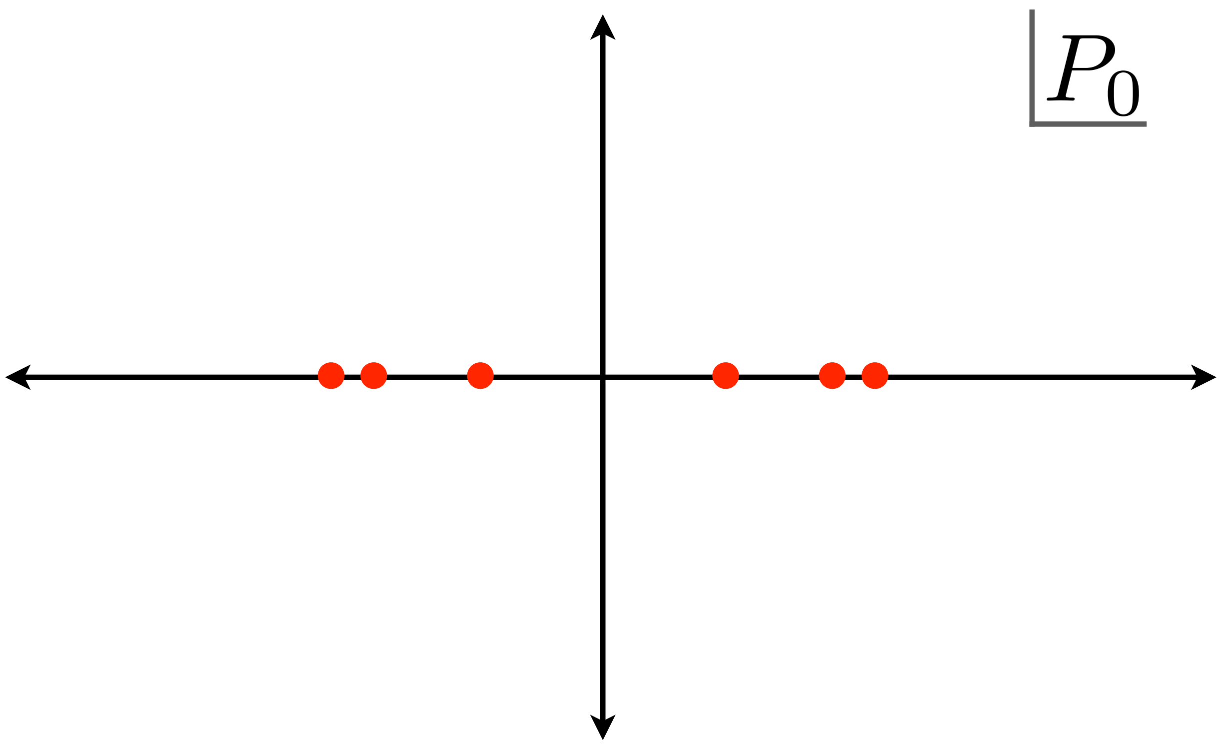
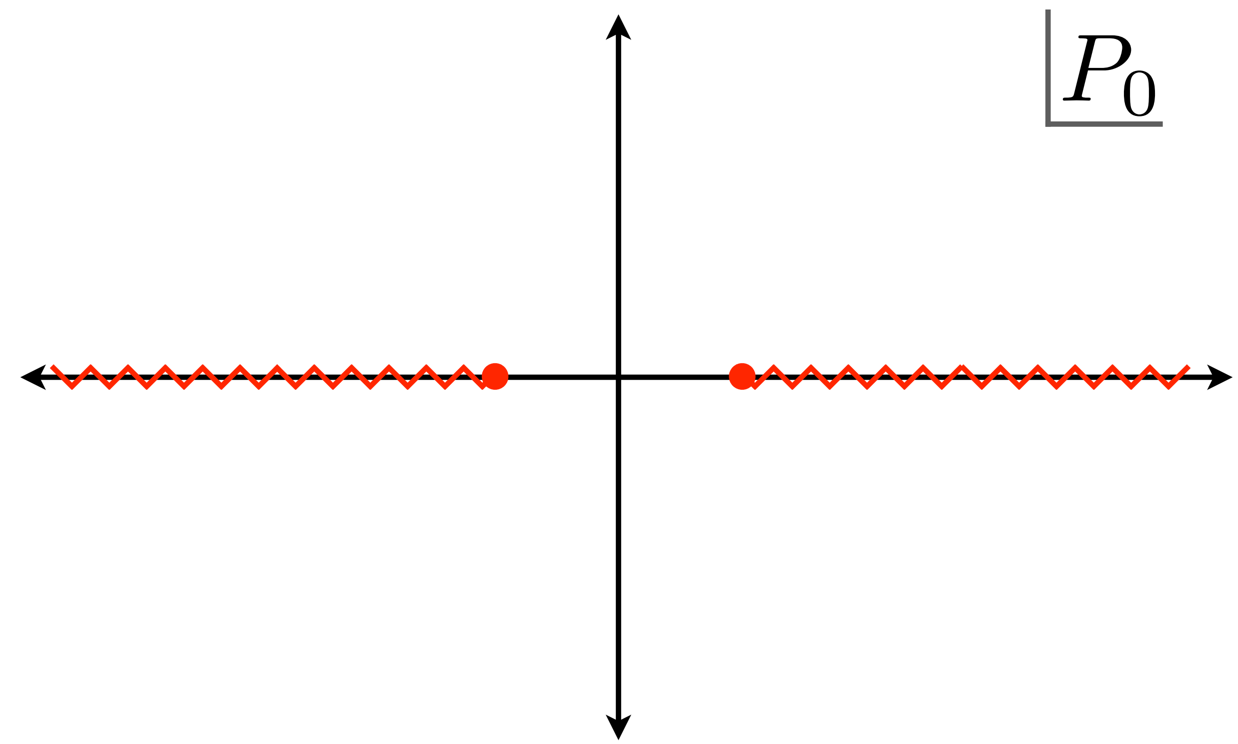
even particle quantum numbers

At fixed L, $\vec{P}$, poles in C_L give finite-volume spectrum



C_L **analytic structure**



C_∞ **analytic structure**

Kim, Sachrajda, and Sharpe, *Nucl. Phys.* B727, 218 (2005)

$$C_L(E,\vec{P}) \equiv \int_L d^4x\, e^{i(Ex^0-\vec{P}\cdot\vec{x})} \langle 0|\mathrm{T}\sigma(x)\sigma^\dagger(0)|0\rangle$$

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CM energy $E^{*2} \equiv E^2 - \vec{P}^2$

interpolating field
even particle quantum numbers

At fixed $L, \vec{P}$, poles in C_L give finite-volume spectrum

Calculate $C_L(E,\vec{P})$ to all orders in perturbation theory and determine condition of divergence.

Kim, Sachrajda, and Sharpe, *Nucl. Phys.* B727, 218 (2005)

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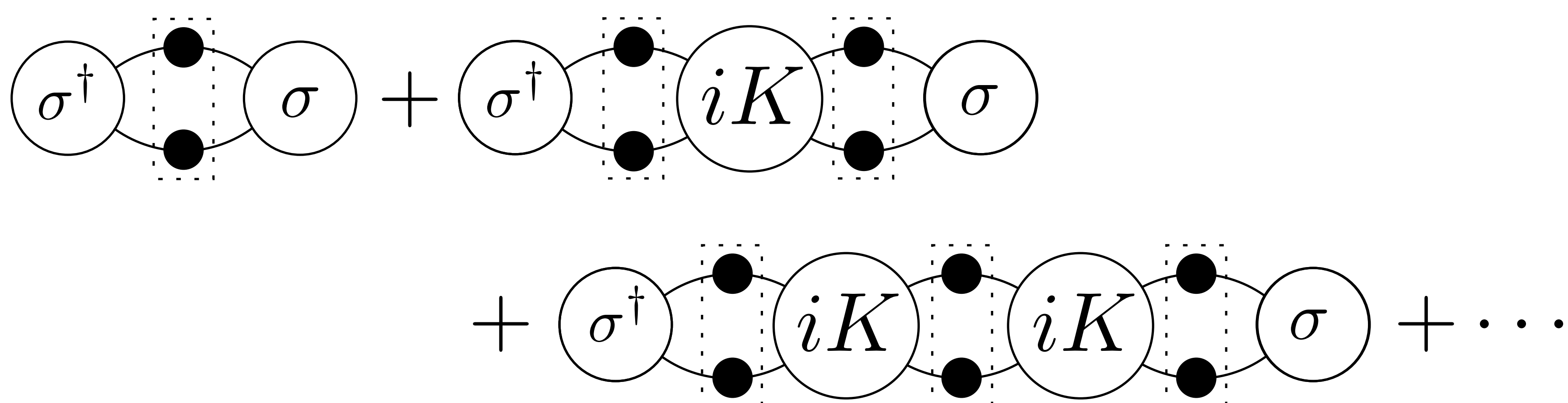
$C_L(E,\vec{P}) =$



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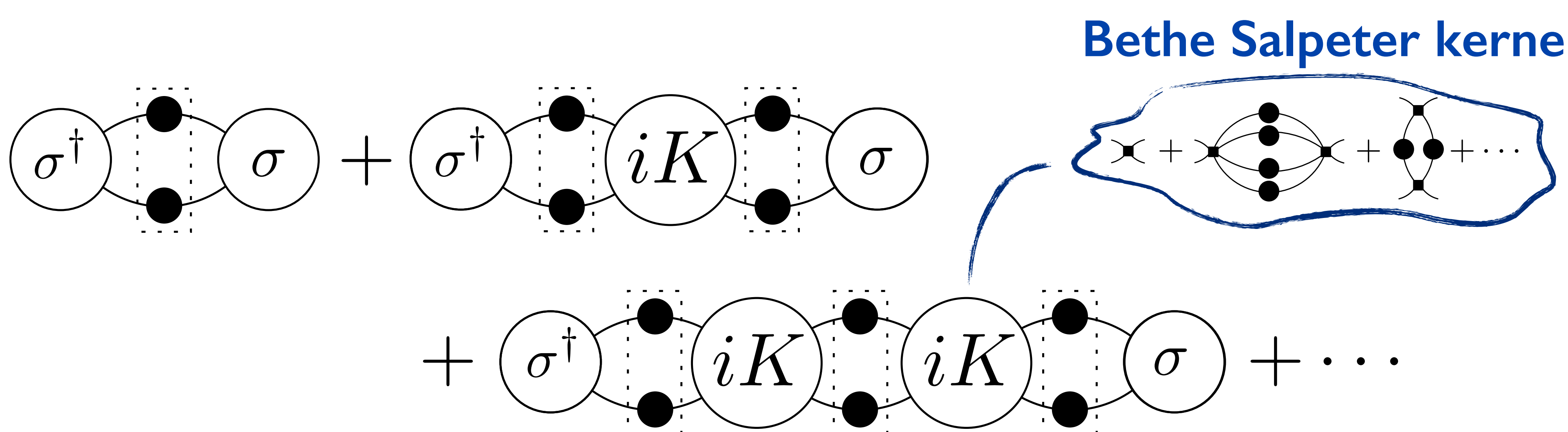

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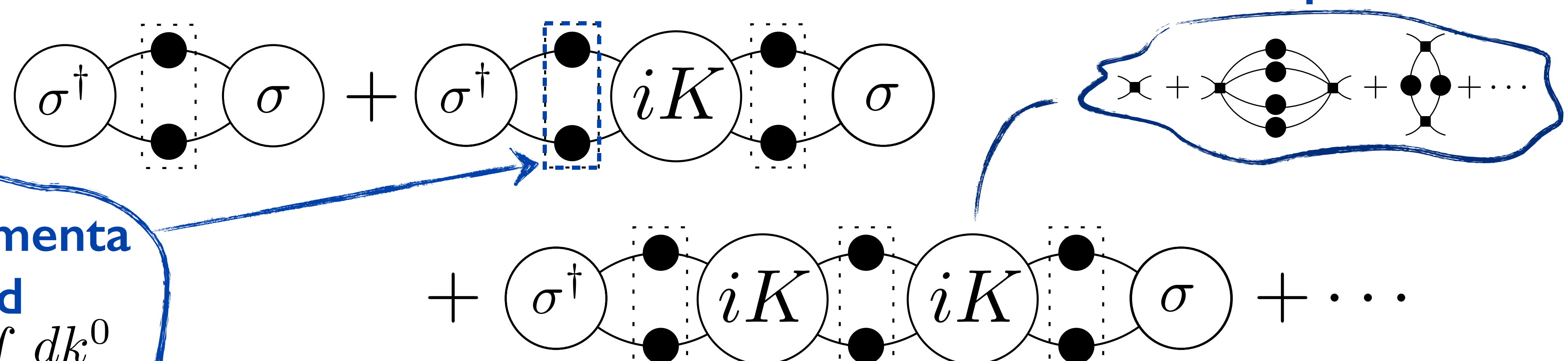
interpolating field
even particle quantum numbers

At fixed $L, \vec{P}$, poles in C_L give finite-volume spectrum

Calculate $C_L(E,\vec{P})$ to all orders in perturbation theory and determine condition of divergence.

Bethe Salpeter kernel

$C_L(E,\vec{P}) =$



spatial loop momenta are summed

$$\frac{1}{L^3}\sum_{\vec{k}\in(2\pi/L)\mathbb{Z}^3}\int\frac{dk^0}{2\pi}$$

$$C_L(E,\vec{P}) \equiv \int_L d^4x\, e^{i(Ex^0 - \vec{P}\cdot\vec{x})} \langle 0|\mathrm{T}\sigma(x)\sigma^\dagger(0)|0\rangle$$

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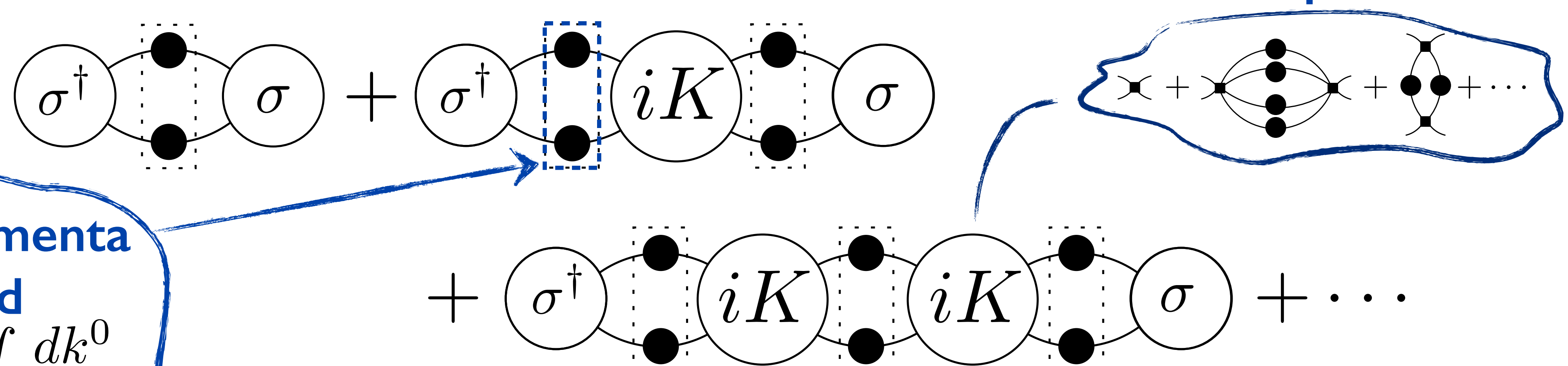
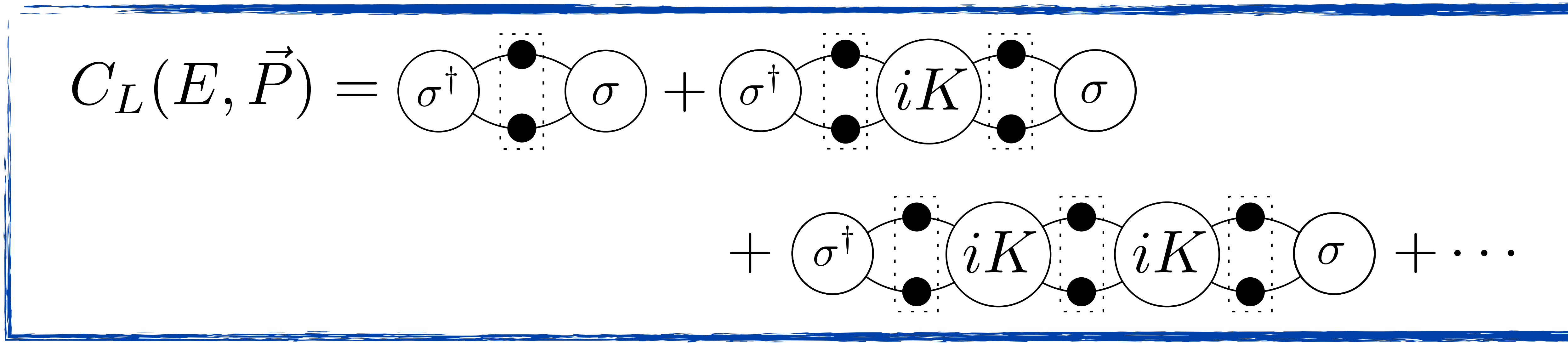


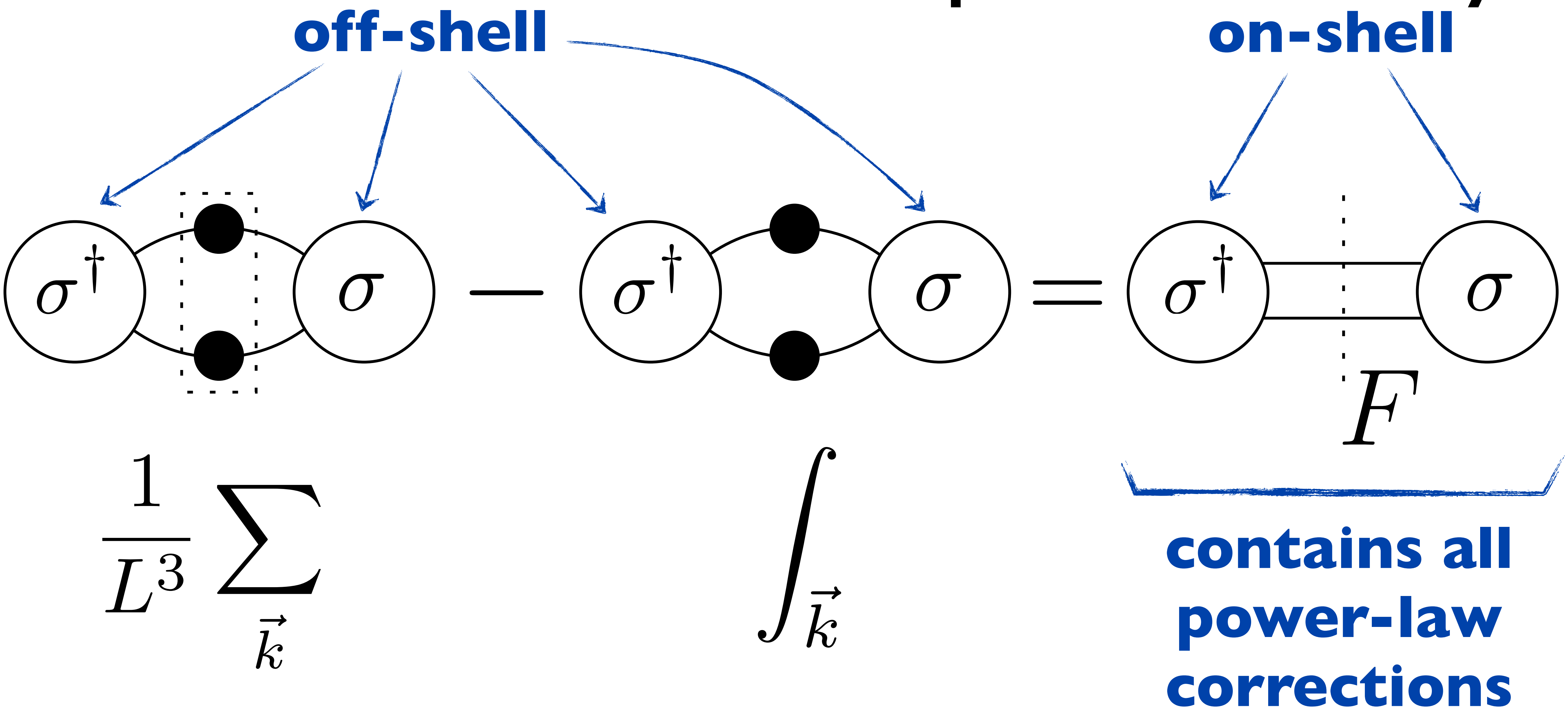
spatial loop momenta are summed

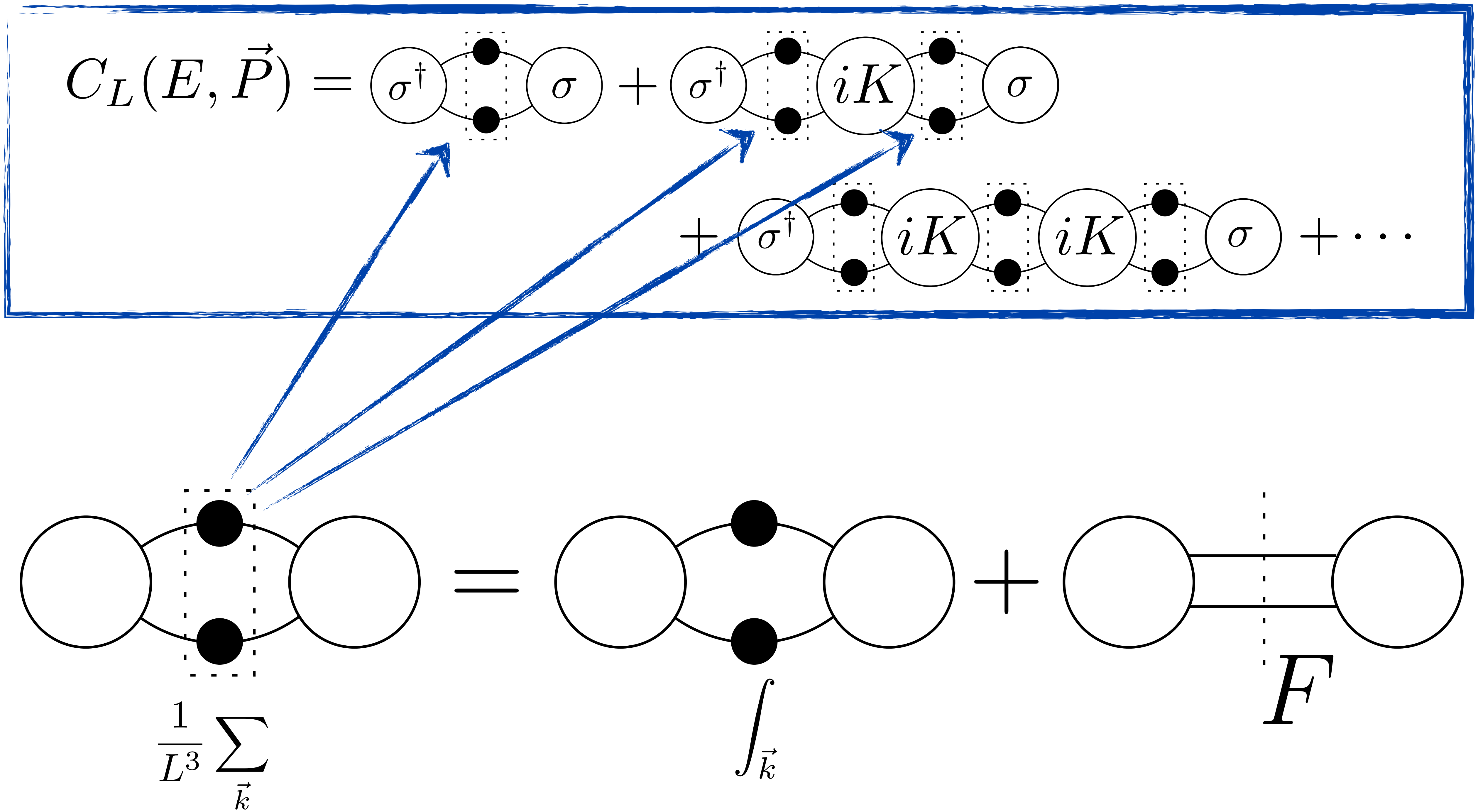
$$\frac{1}{L^3}\sum_{\vec{k}\in(2\pi/L)\mathbb{Z}^3}\int\frac{dk^0}{2\pi}$$

If $E^* < 4m$ then $K_L = K_\infty + \mathcal{O}(e^{-mL})$

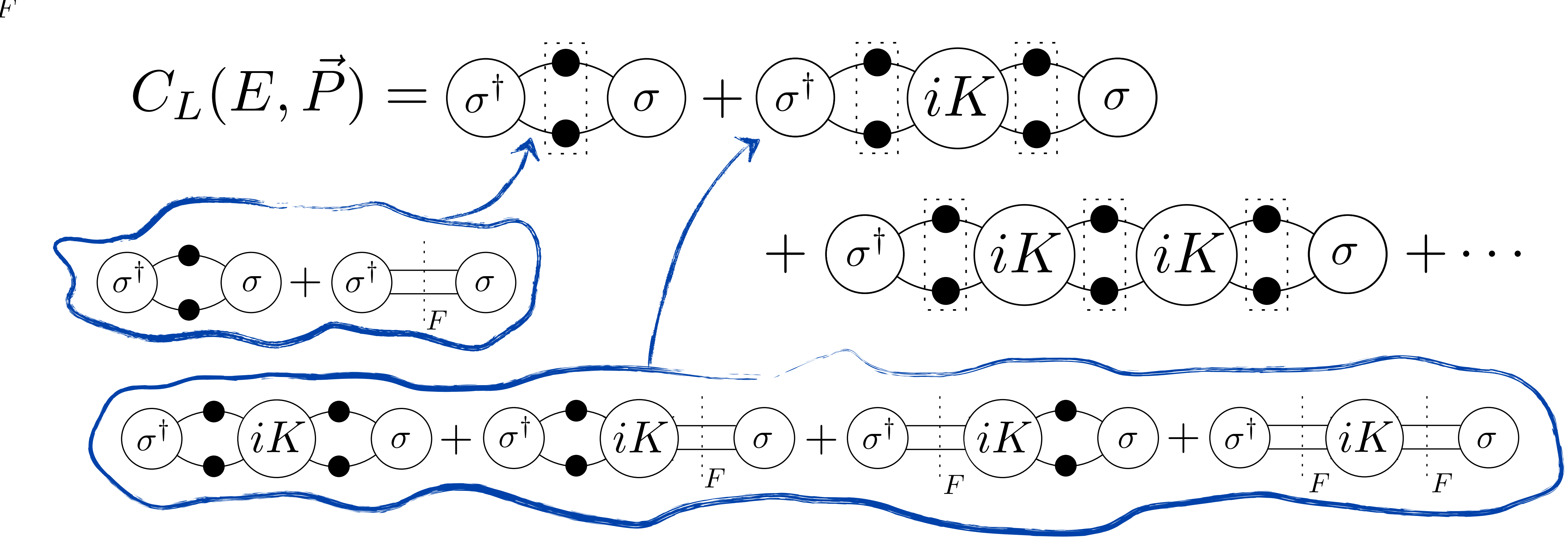


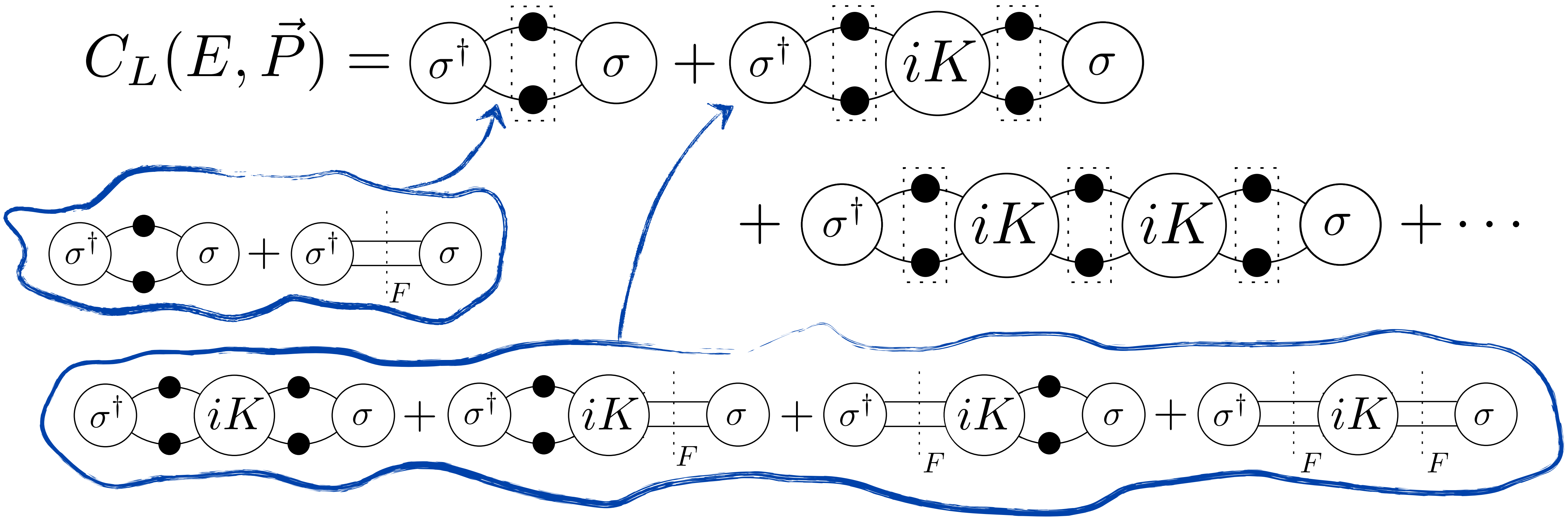
Next we introduce an important identity



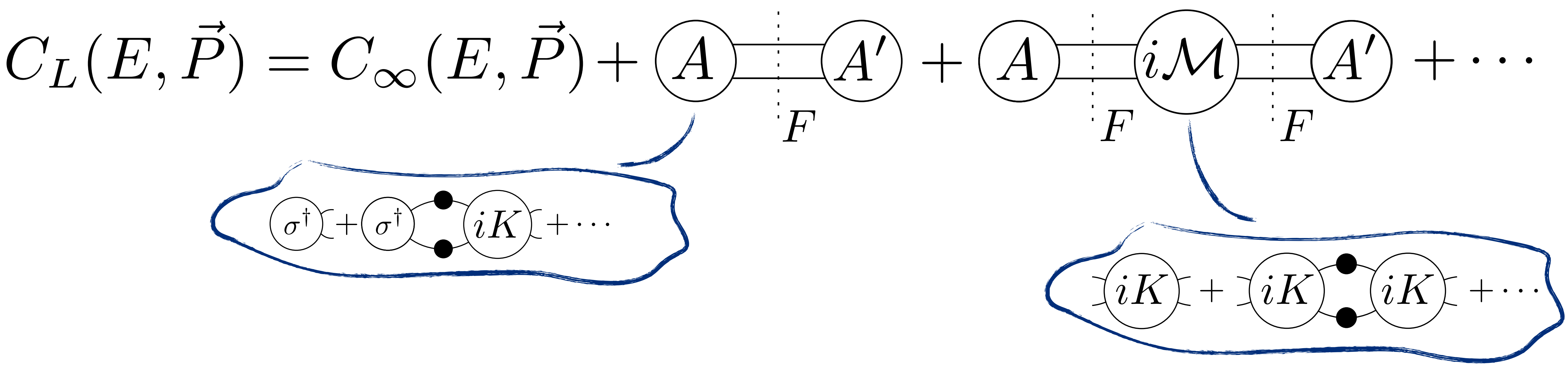


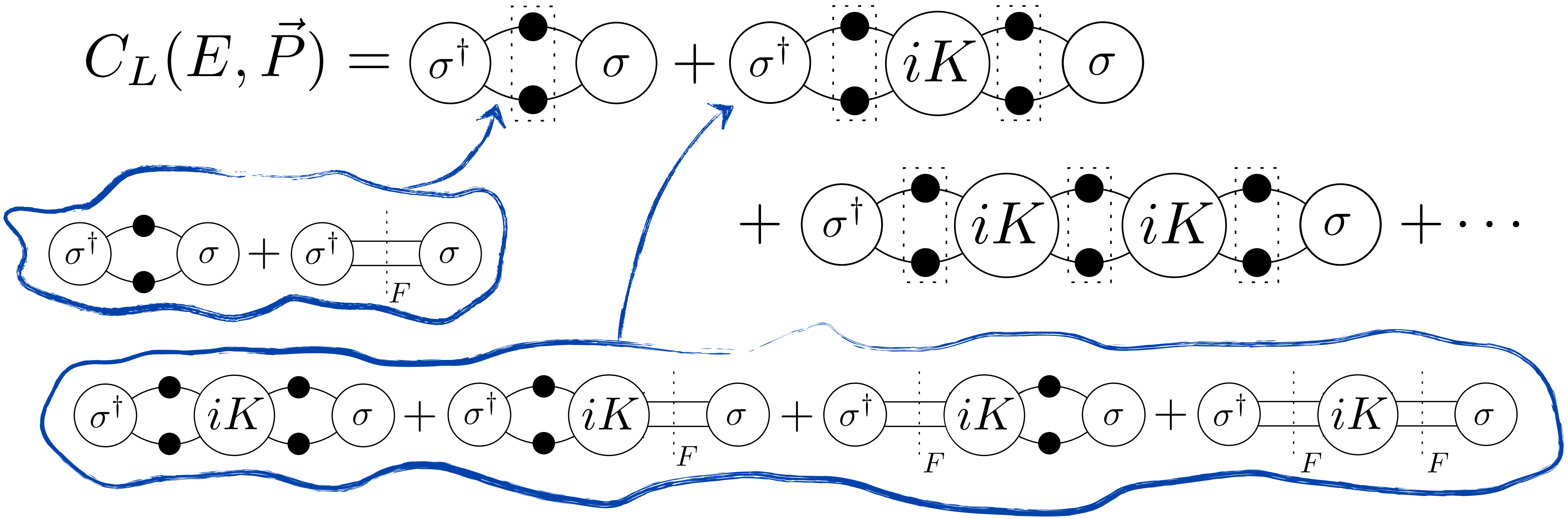
Can be applied in all two-particle loops


$C_L(E,\vec{P}) =$
$\sigma^\dagger$
σ
$+$
iK
$+\cdots$
F



Now regroup by number of F cuts





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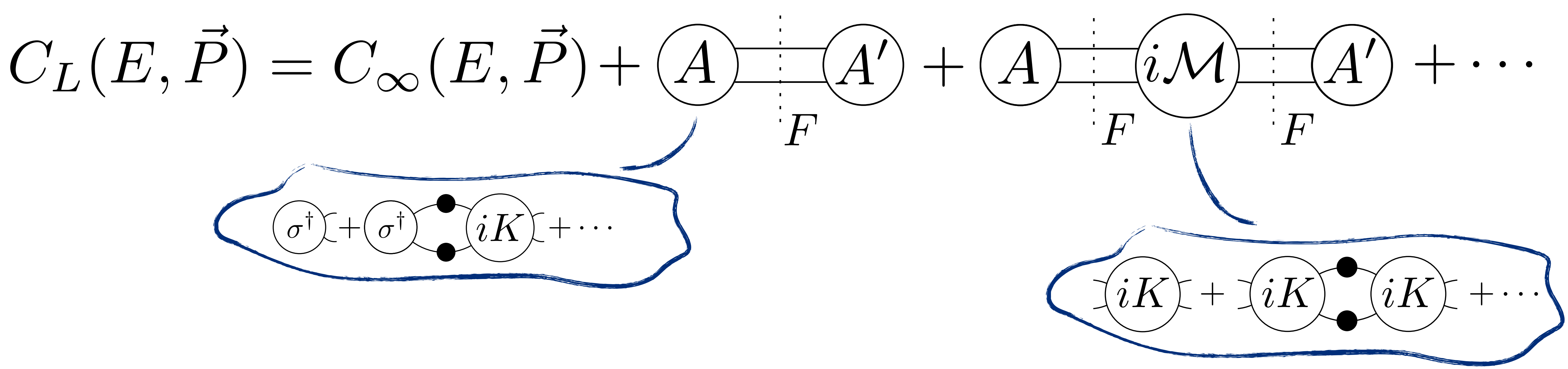
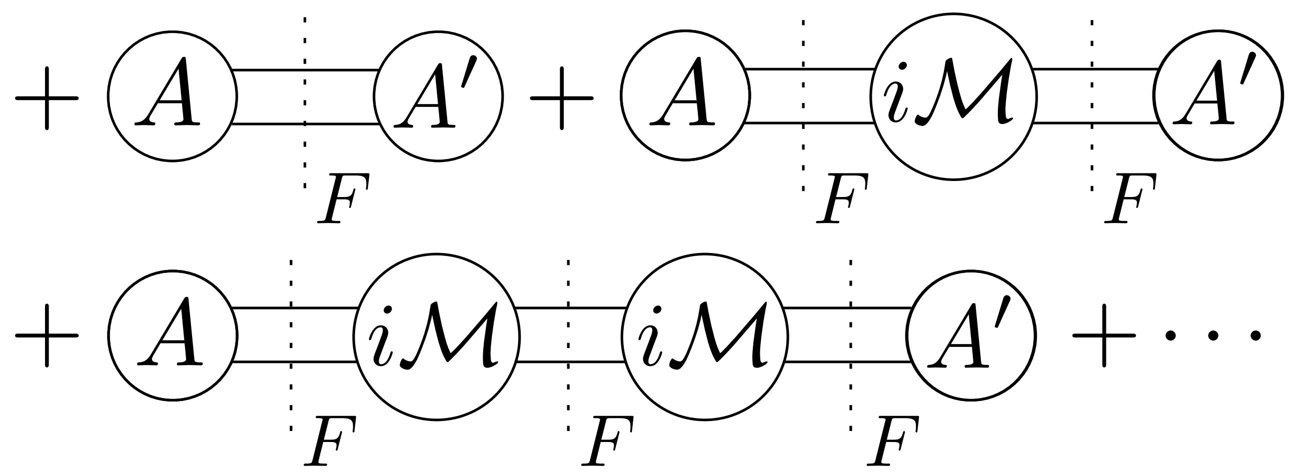


As Promised!

Infinite-volume on-shell two-to-two scattering amplitude

$$C_L(E,\vec{P}) = C_\infty(E,\vec{P})$$



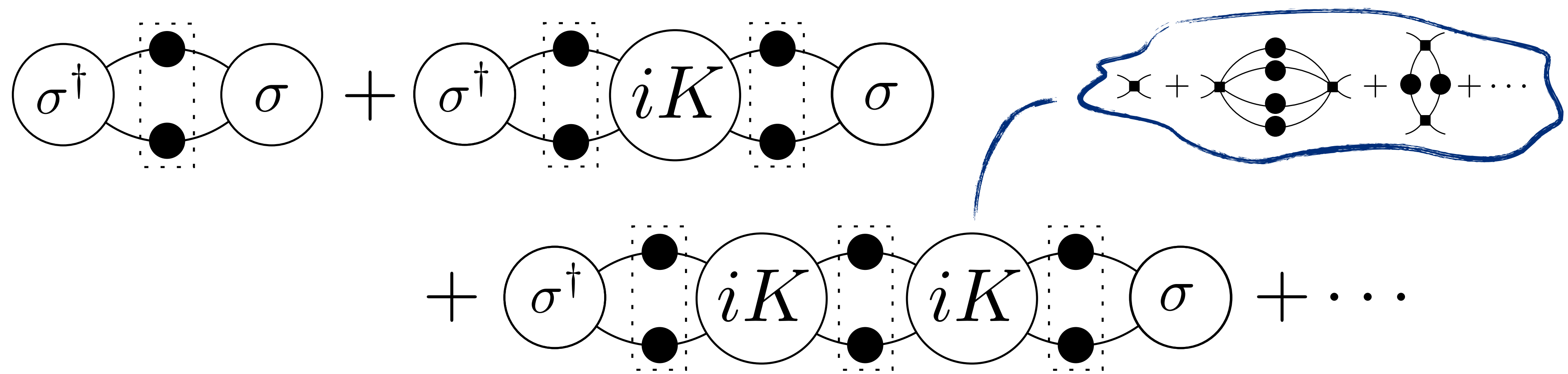
$$C_L(E,\vec{P}) = C_\infty(E,\vec{P}) + \sum_{n=0}^{\infty} A' iF [i\mathcal{M}_{2\to 2} iF]^n A$$

$$\boxed{C_L(E,\vec{P}) = C_\infty(E,\vec{P}) + A' iF \frac{1}{1 - i\mathcal{M}_{2\to 2} iF} A}$$

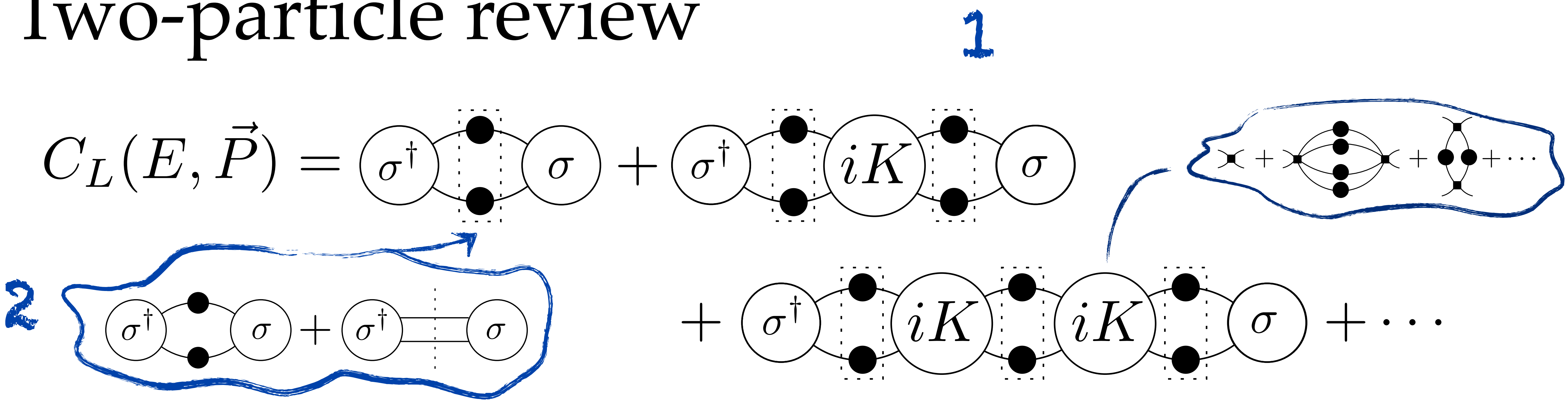
Two-particle review

1

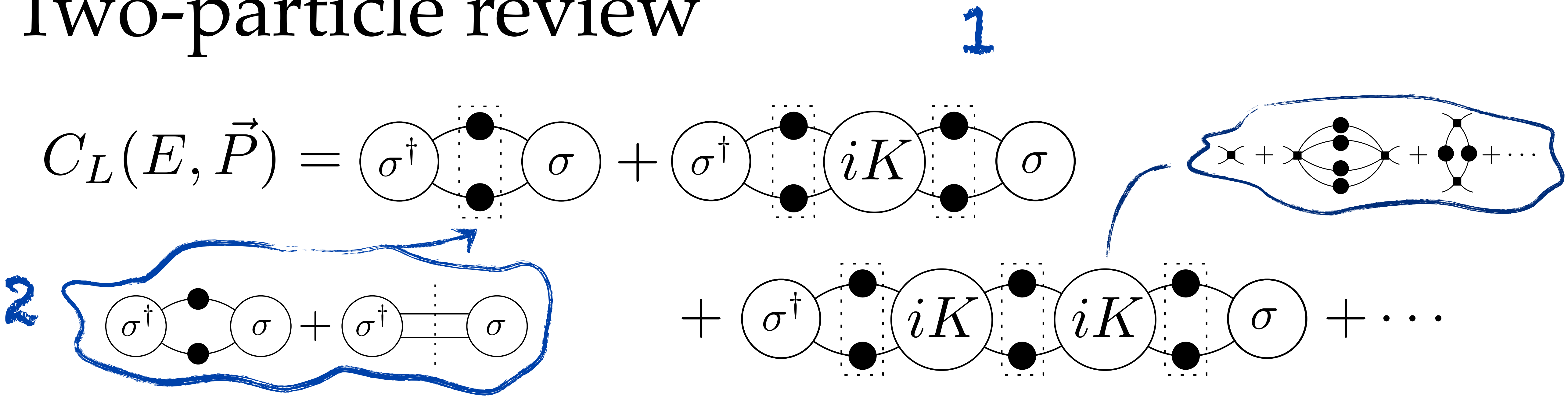
$C_L(E, \vec{P}) =$

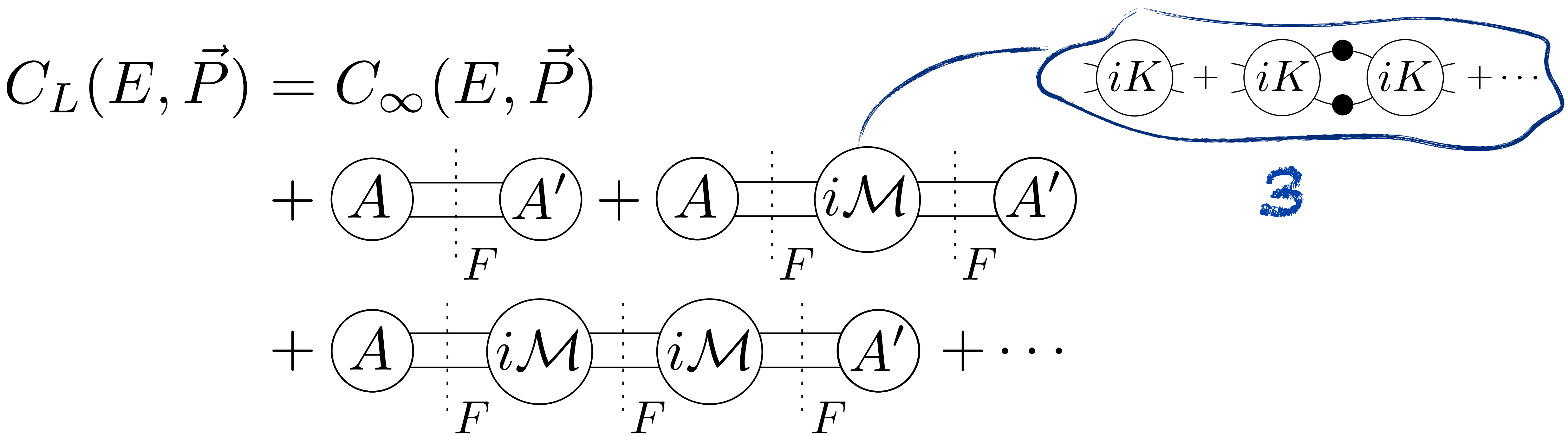


Two-particle review

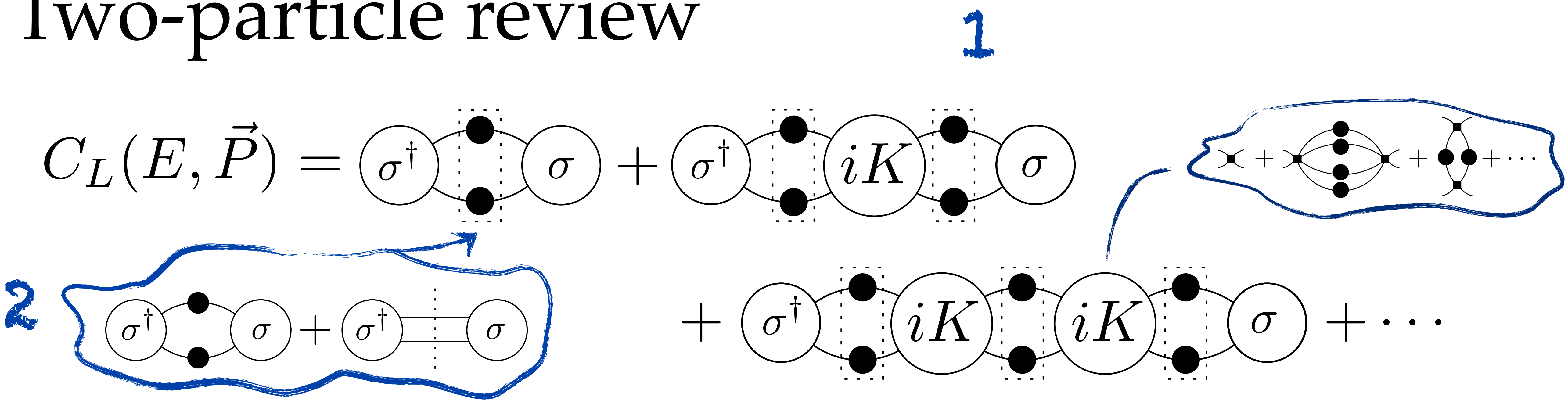


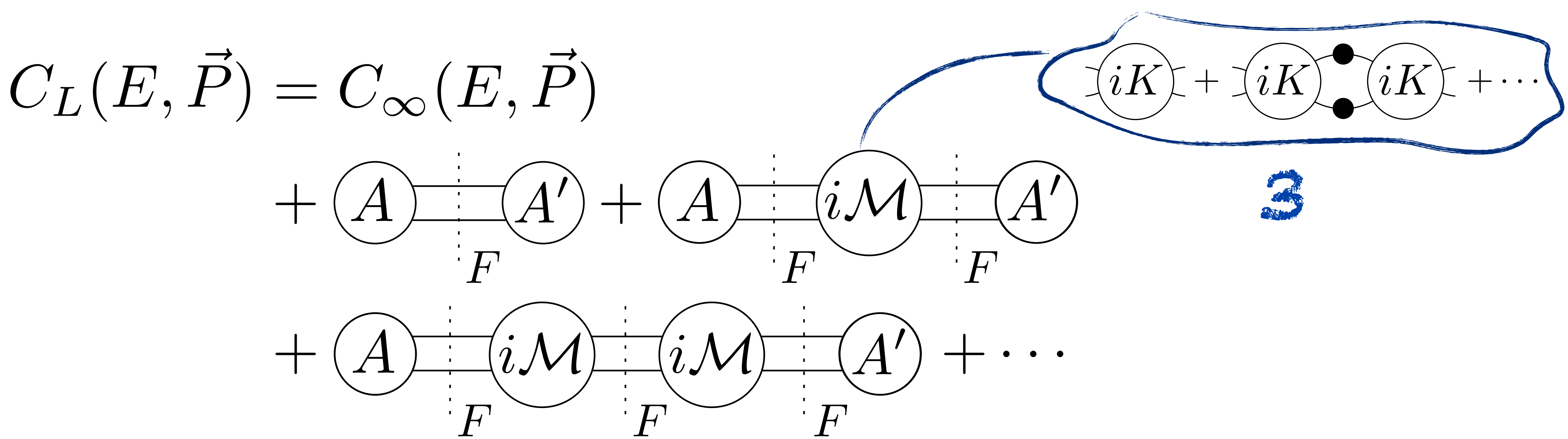
Two-particle review





Two-particle review





$$C_L(E, \vec{P}) = C_\infty(E, \vec{P}) + A' iF \frac{1}{1 - i\mathcal{M}_{2\to 2} iF} A$$

4

no poles

no poles

no poles

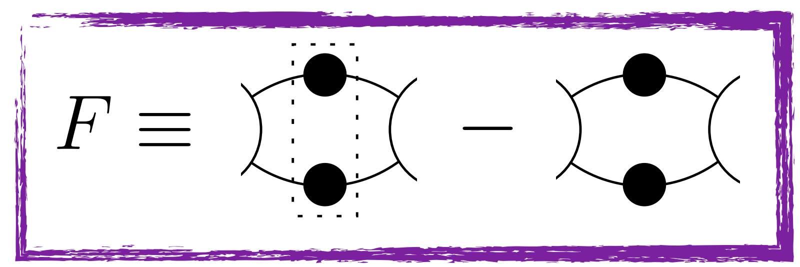
Two-particle result

At fixed $(L, \vec{P})$, finite-volume spectrum is all solutions to

$$\det[1 - \underline{i\mathcal{M}_{2\to2}}\,\underline{iF}] = 0$$

diagonal matrix in angular momentum space

kinematic, not diagonal (related to Lüscher Zeta function)



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diagonal matrix in angular momentum space

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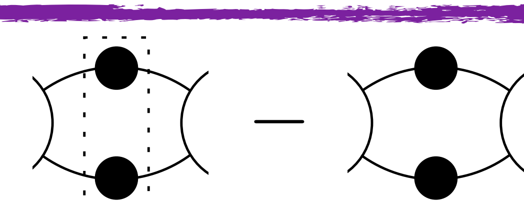
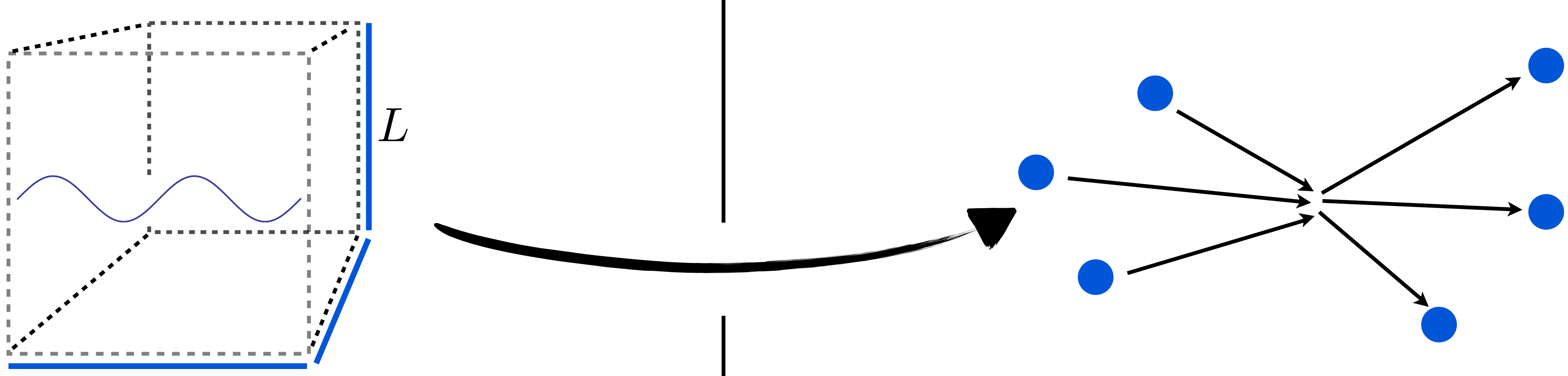
$F \equiv$ 

Can be rewritten as

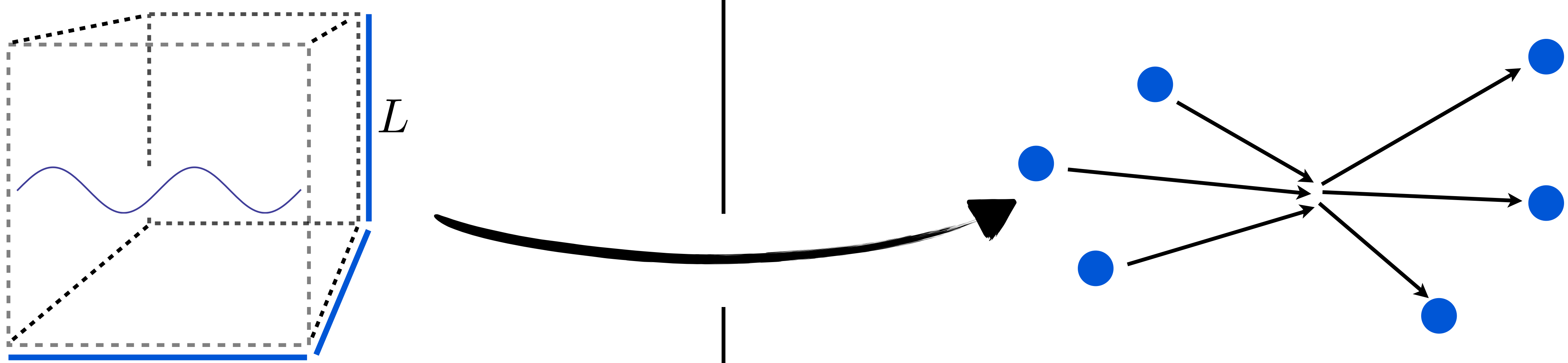
$$\boxed{\det\left[\cot\delta(E_n^*) + \cot\phi(E_n, \vec{P}, L)\right] = 0}$$

Now, three particles in a box



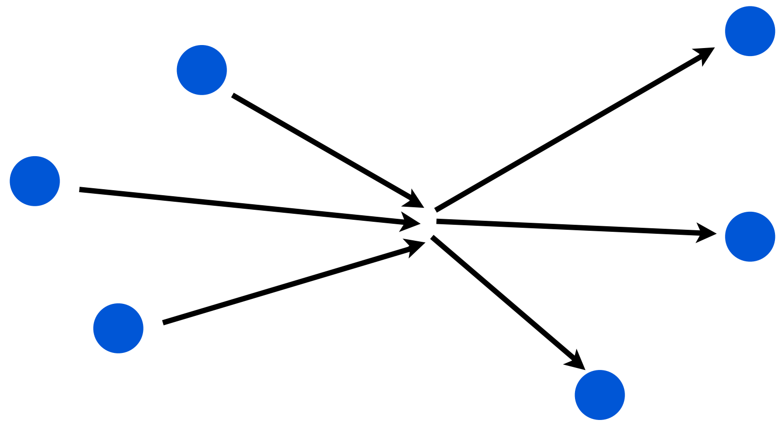
Finite volume **Infinite volume**

Now, three particles in a box

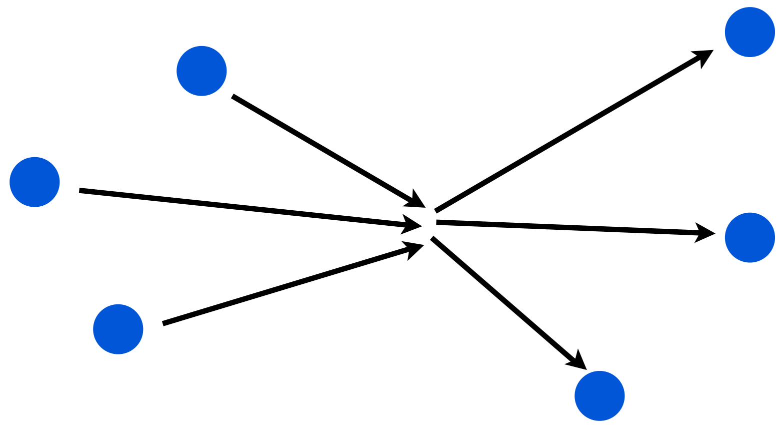


Finite volume | **Infinite volume**

Infinite volume

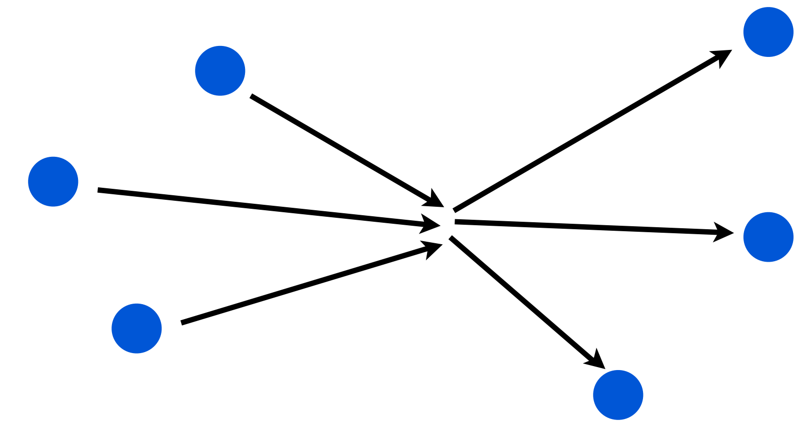


Infinite volume



Degrees of freedom for three on-shell particles with $(E, \vec{P})$

Infinite volume



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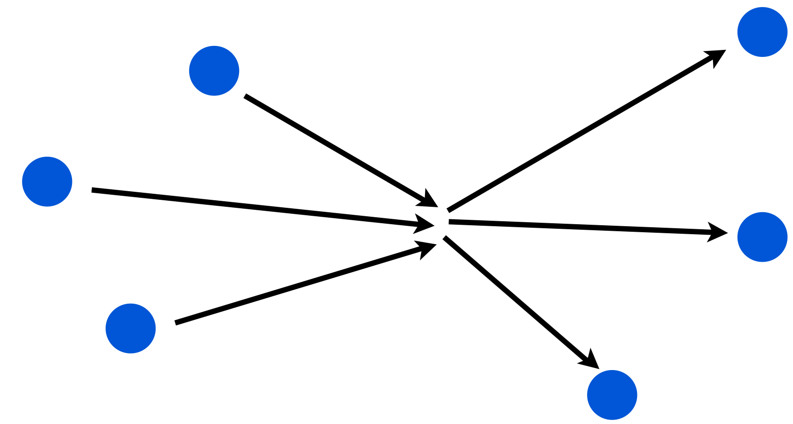
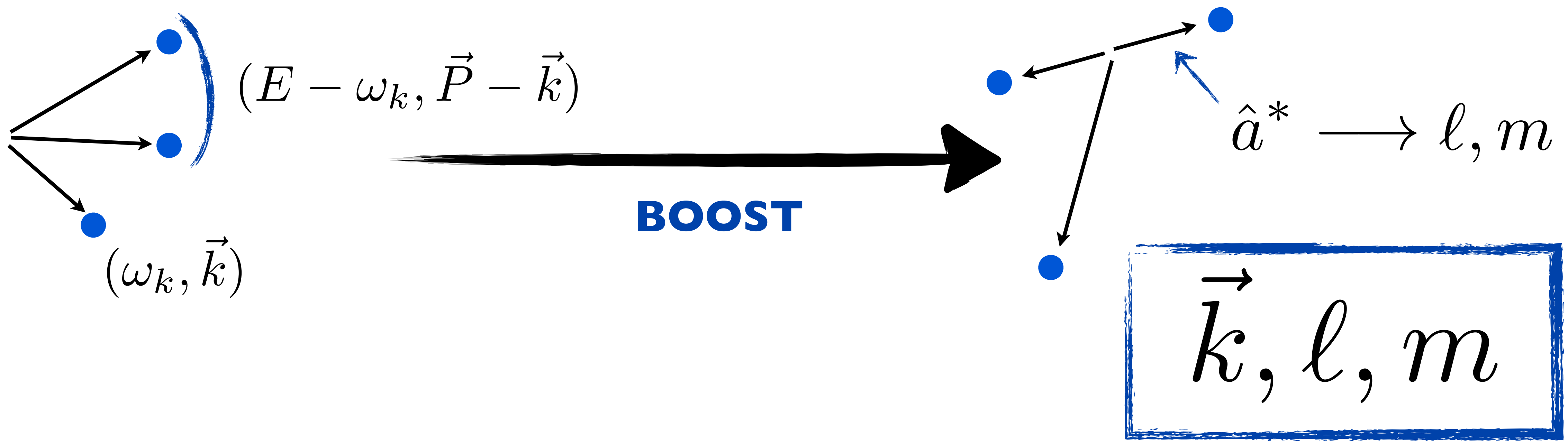
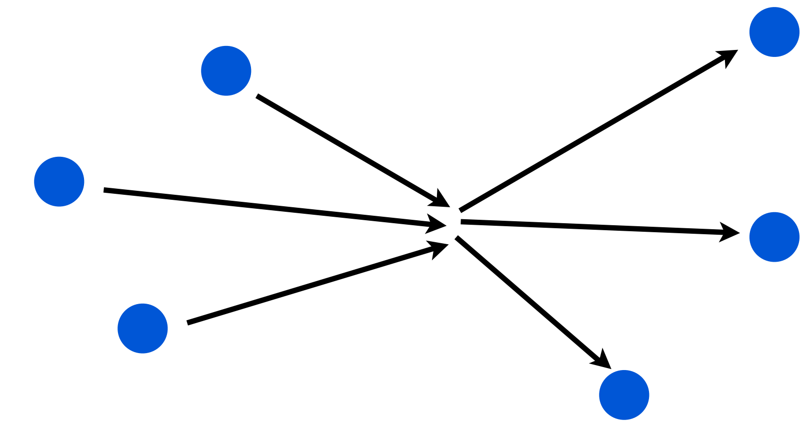


Degrees of freedom for three on-shell particles with $(E, \vec{P})$



Infinite volume

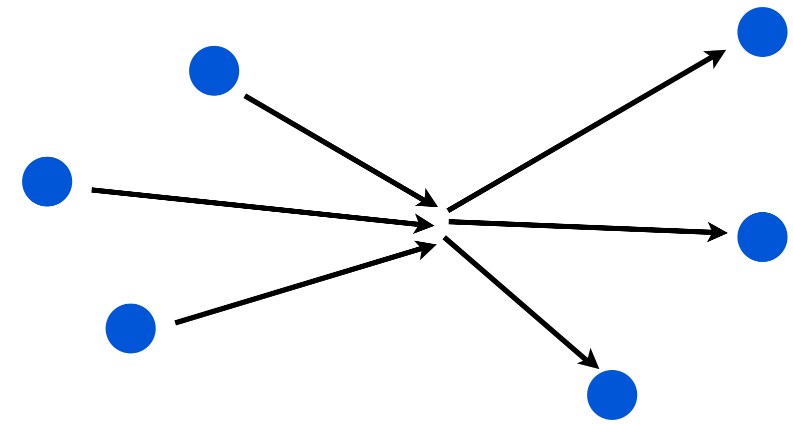


Degrees of freedom for three on-shell particles with $(E, \vec{P})$

$$\boxed{\vec{k}, \ell, m}$$

Three particle divergences

Infinite volume

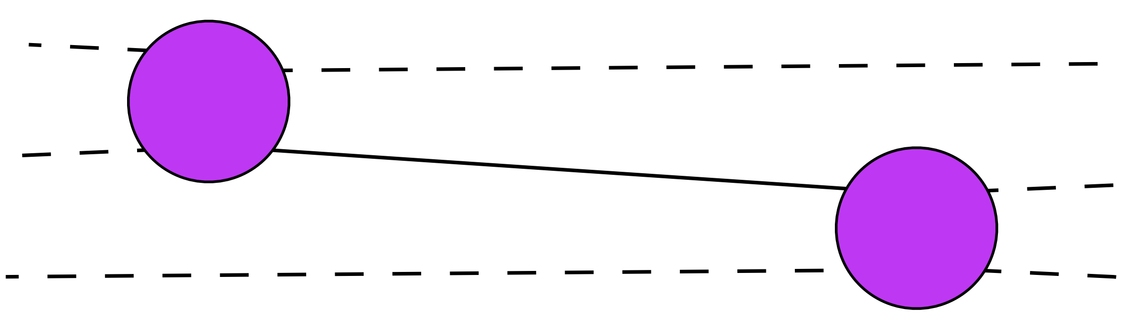


Degrees of freedom for three on-shell particles with $(E, \vec{P})$

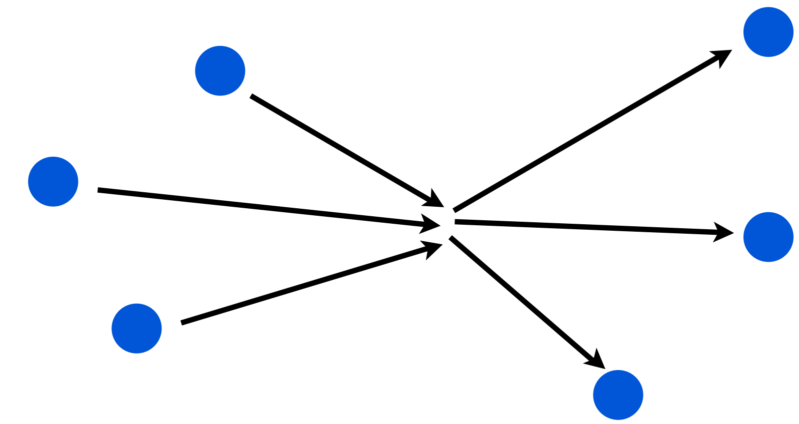
$$\boxed{\vec{k}, \ell, m}$$

Three particle divergences

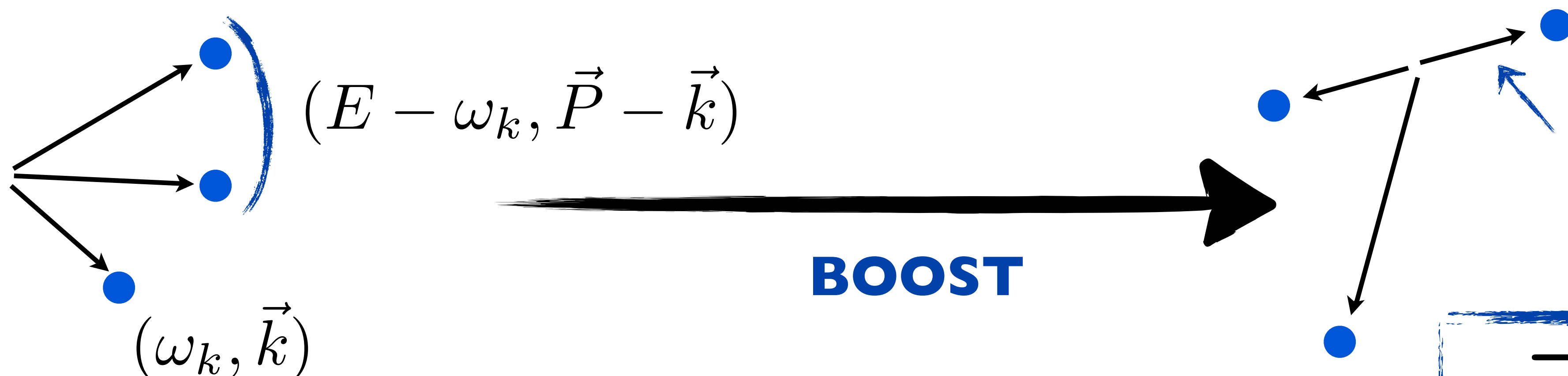
$\mathcal{M}_{3\to 3}$ contains the diagram



Infinite volume



Degrees of freedom for three on-shell particles with $(E, \vec{P})$

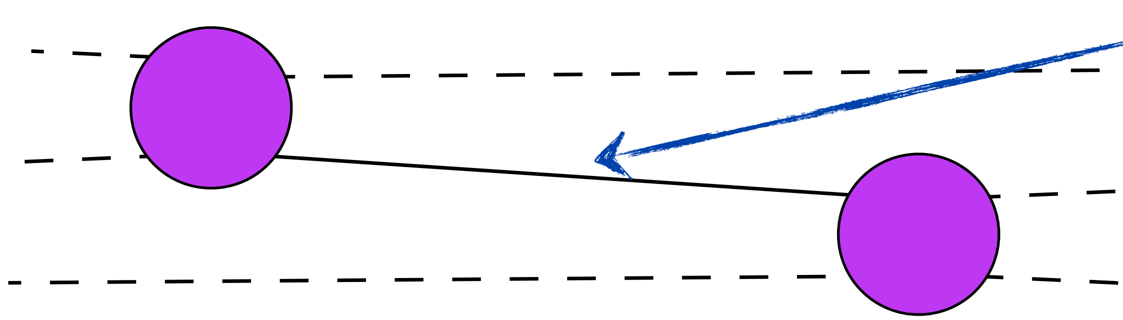


$\hat{a}^* \longrightarrow \ell, m$

$$\boxed{\vec{k}, \ell, m}$$

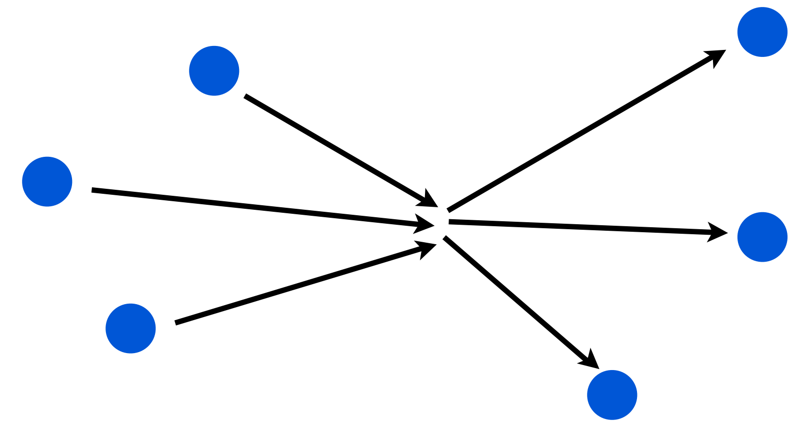
Three particle divergences

$\mathcal{M}_{3\to3}$ contains the diagram

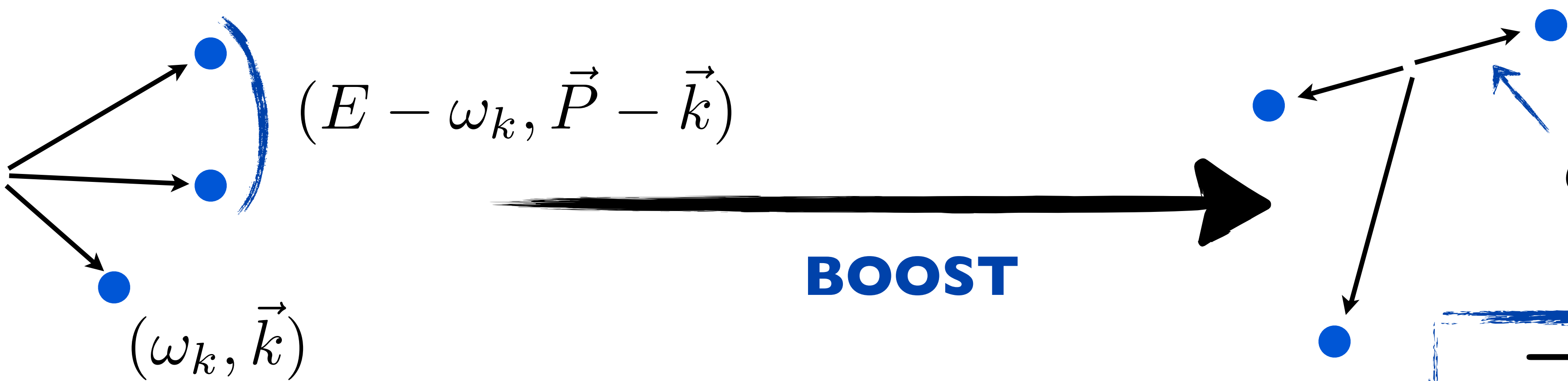


Certain external momenta put this on-shell! → $\mathcal{M}_{3\to3}$ has singularities

Infinite volume



Degrees of freedom for three on-shell particles with $(E, \vec{P})$



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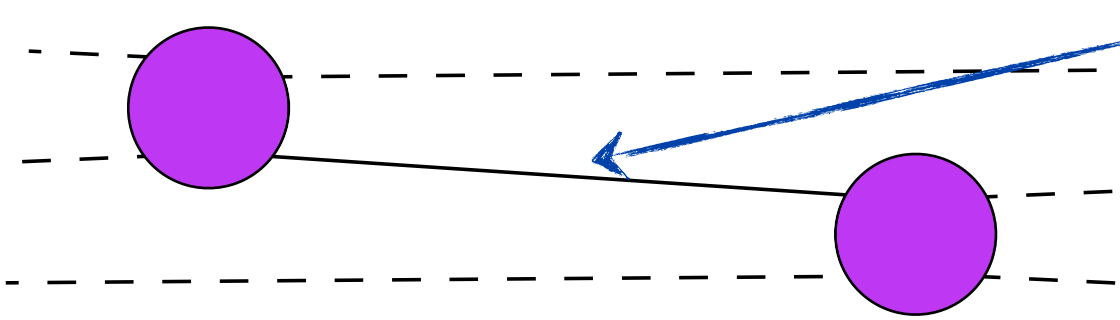
Three particle divergences

$\mathcal{M}_{3\to 3}$ contains the diagram



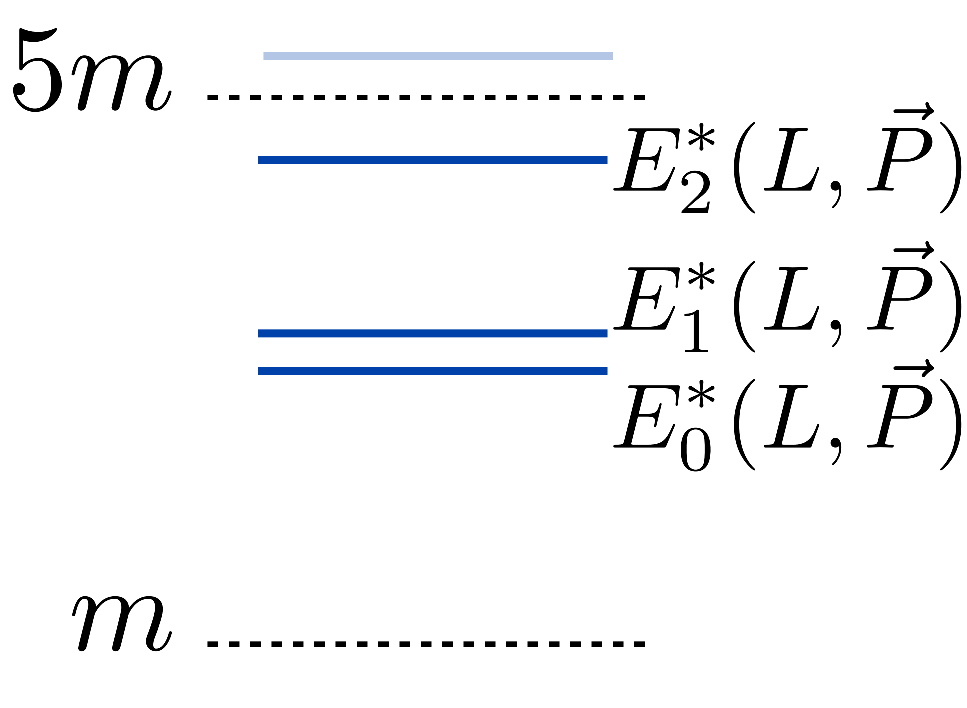
Certain external momenta put this on-shell! $\longrightarrow$ $\mathcal{M}_{3\to 3}$ has singularities

No dominance of lowest partial waves

Three particles in a box

$$C_L(E,\vec{P}) \equiv \int_L d^4x\, e^{i(Ex^0 - \vec{P}\cdot\vec{x})} \langle 0|\mathrm{T}\sigma(x)\sigma^\dagger(0)|0\rangle$$

Require $m < E^* < 5m$ **odd-particle quantum numbers**

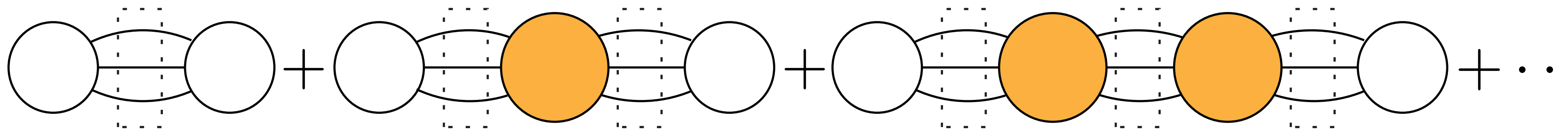


$$i\mathcal{M}_{2\to 2}$$

$$i\mathcal{M}_{3\to 3}$$

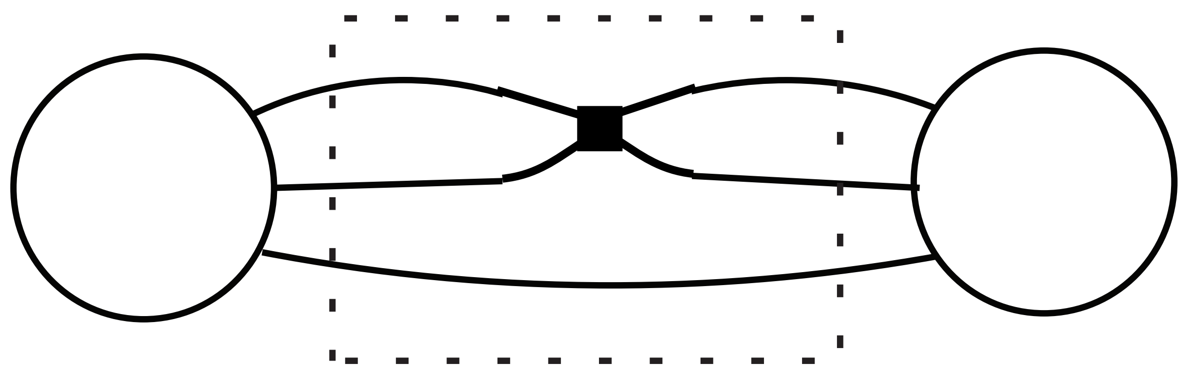
Assume no two-particle bound state or resonance

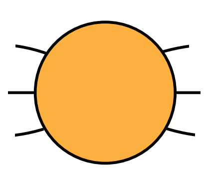
New skeleton expansion

$C_L(E, \vec{P}) \stackrel{?}{=}$ 

(propagators still fully dressed)

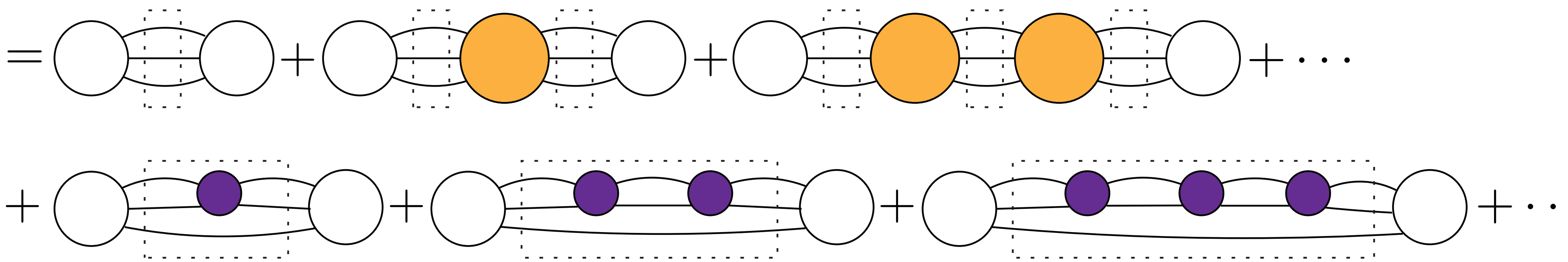
No! We also need diagrams like



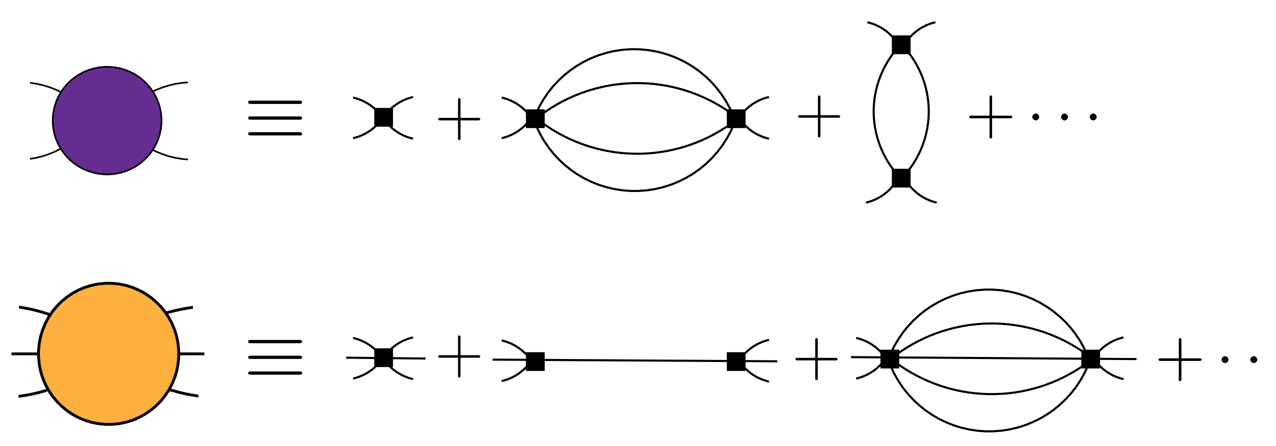
( **should only contain connected diagrams)**

New skeleton expansion

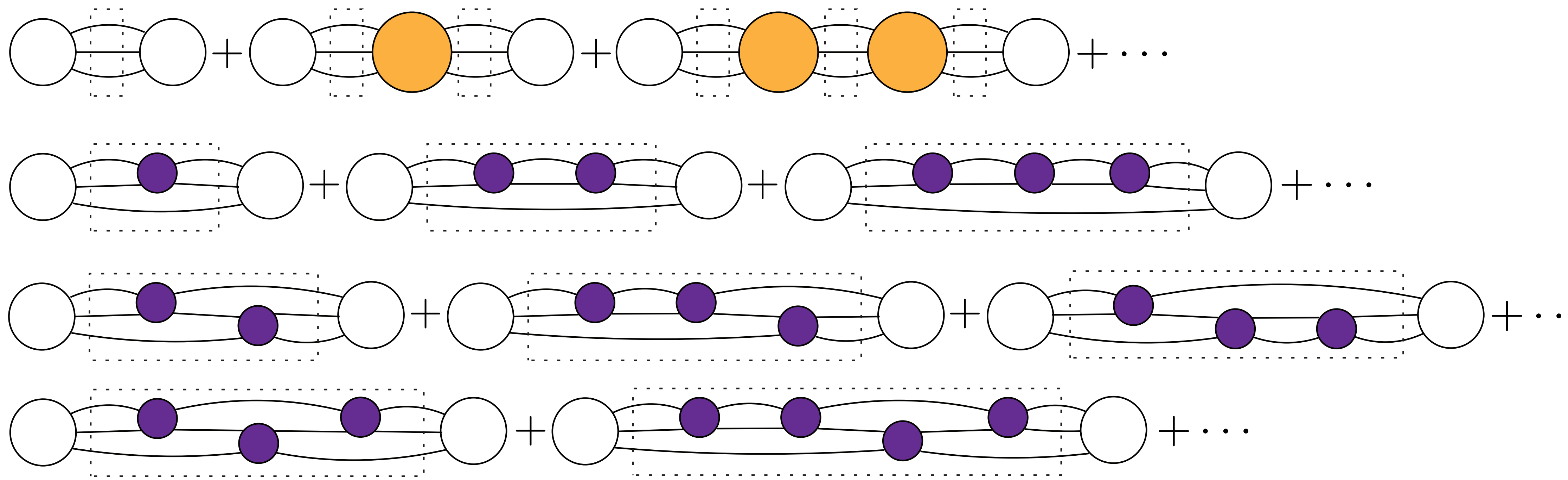
$C_L(E, \vec{P}) =$



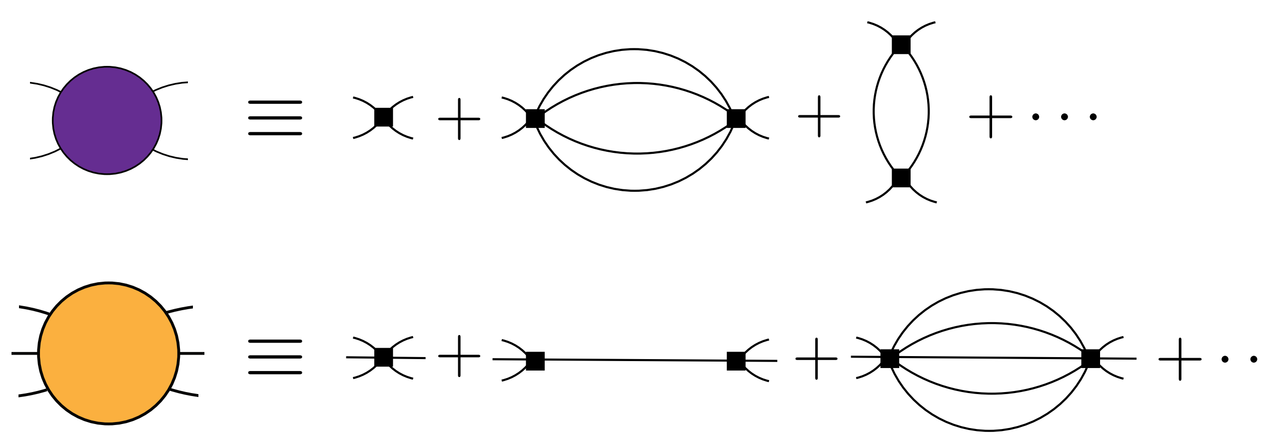
Kernel definitions:



New skeleton expansion

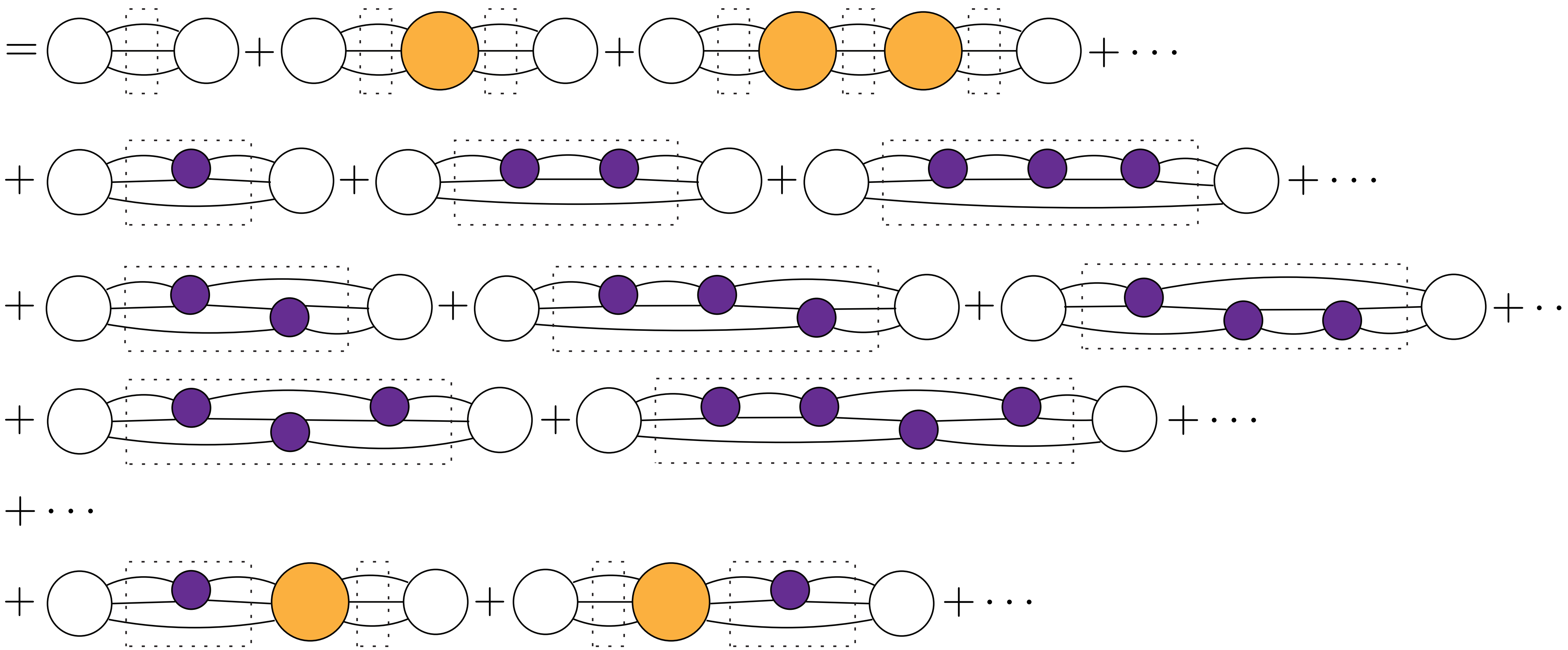
$C_L(E,\vec{P}) =$ 

Kernel definitions:

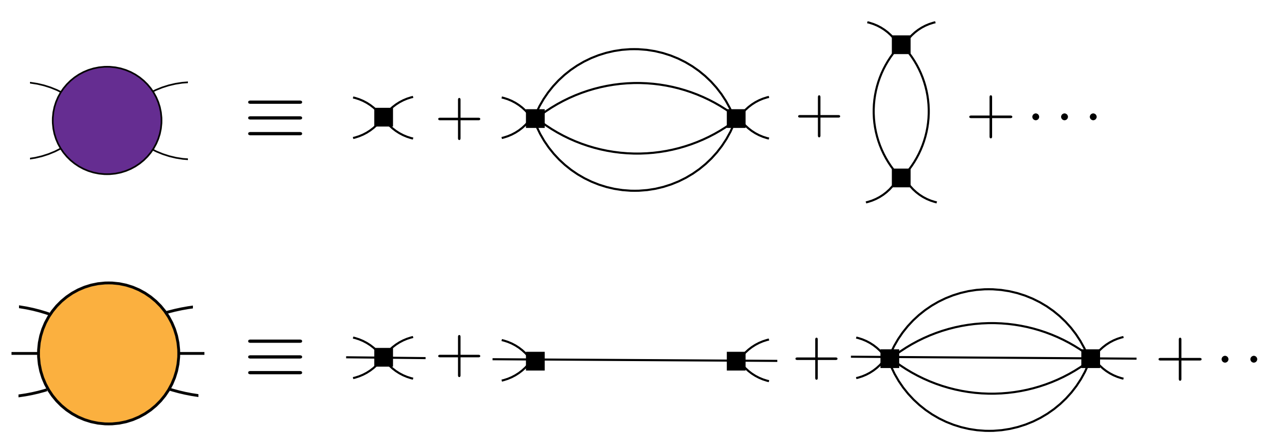


New skeleton expansion

$C_L(E,\vec{P}) =$

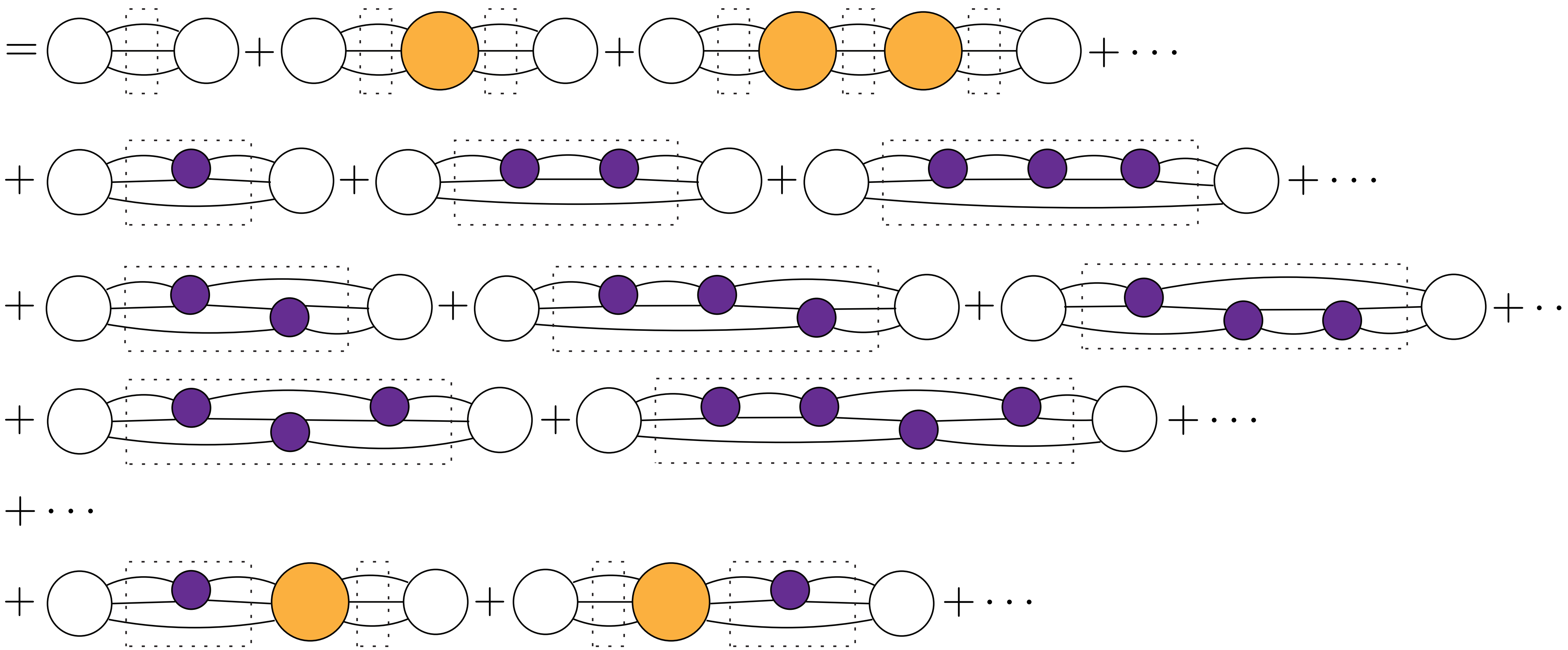


Kernel definitions:



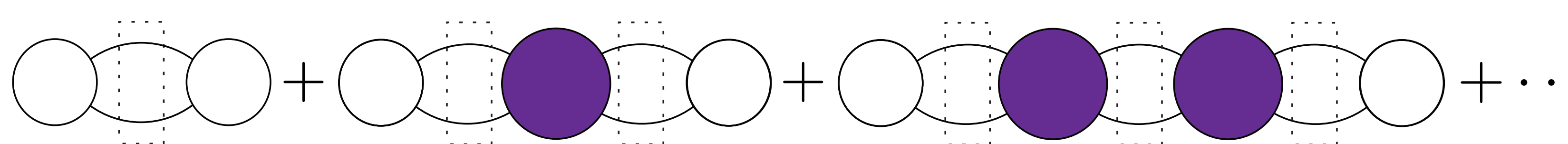
New skeleton expansion

$C_L(E, \vec{P}) =$



Compare to two-particle skeleton expansion

$C_L(E, \vec{P}) =$



What is new here?

1. Degrees of freedom are different

two particles

two-particle angular momentum

three particles

$\vec{k}$ **+ two-particle angular momentum**

Our result only depends on finite-volume momentum $\vec{k} = \frac{2\pi}{L}\vec{n}$

What is new here?

1. Degrees of freedom are different

two particles

two-particle angular momentum

three particles

$\vec{k}$ **+ two-particle angular momentum**

Our result only depends on finite-volume momentum $\vec{k} = \dfrac{2\pi}{L}\vec{n}$

Quantization condition expressed using matrices with indices

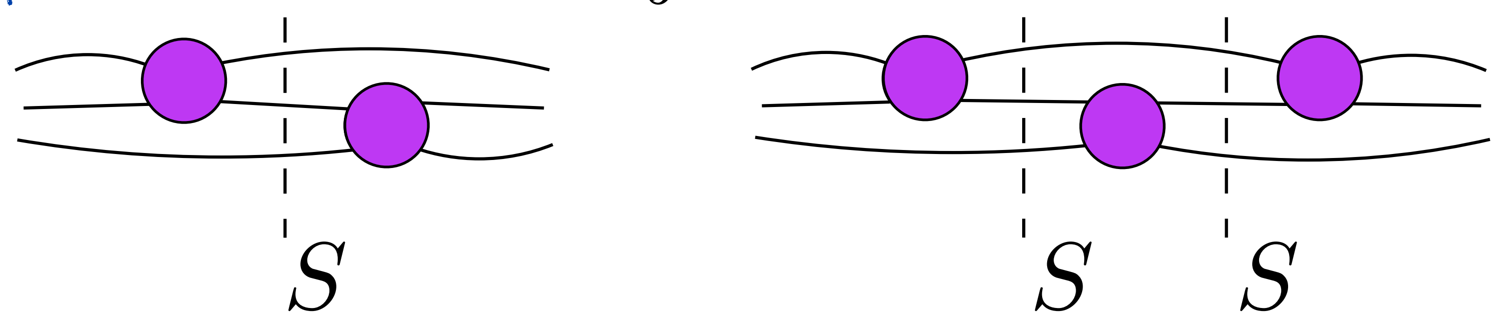
$$\vec{k}, \ell, m$$

What is new here?

2. Three particle divergences

Define $i\mathcal{M}_{\mathrm{df},3\to3}$

$$\equiv i\mathcal{M}_{3\to3} - \left[i\mathcal{M}_{2\to2} S i\mathcal{M}_{2\to2} + \int i\mathcal{M}_{2\to2} S i\mathcal{M}_{2\to2} S i\mathcal{M}_{2\to2} + \cdots \right]$$



only on-shell amplitudes here

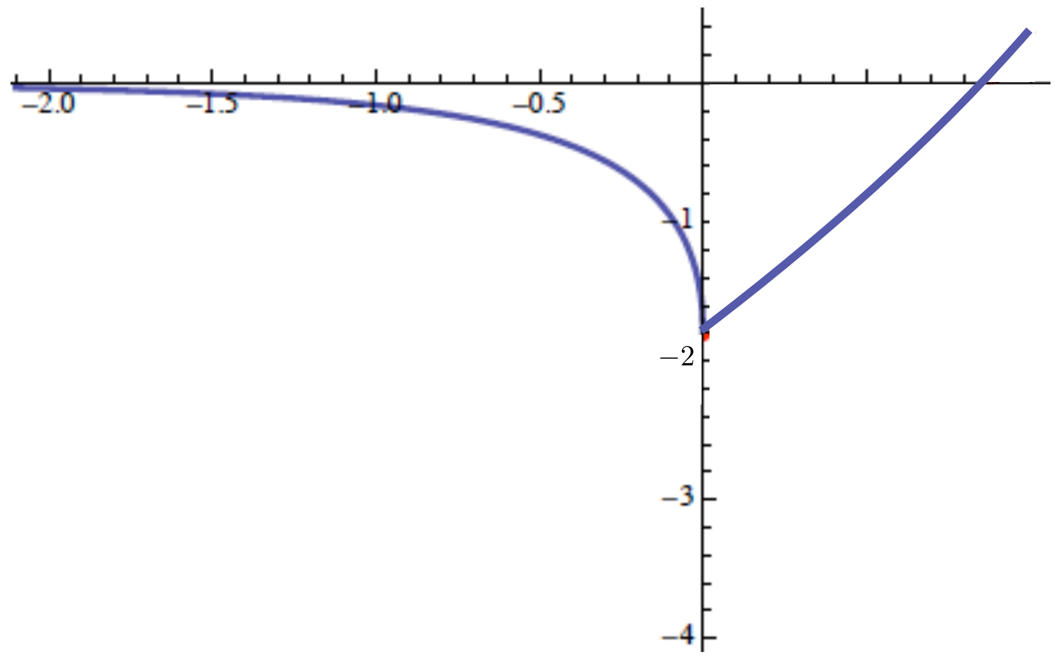
infinite series built with factors of $S i\mathcal{M}_{2\to2}$

This subtraction emerges naturally in our finite-volume analysis

What is new here?

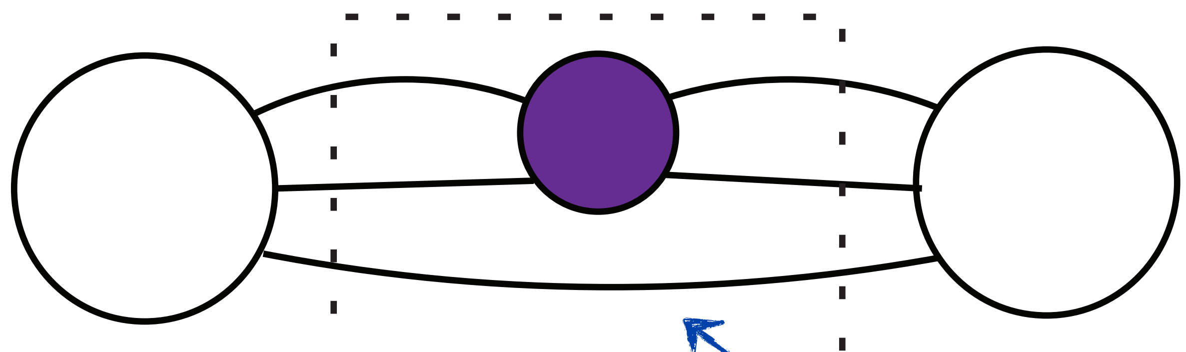
3. Must now worry about sum crossing two-particle unitary cusp

two-particle scattering (real part)



depends on k

two particle energy

$$\frac{1}{L^3}\sum_{\vec{k}}$$ 

k

What is new here?

3. Must now worry about sum crossing two-particle unitary cusp

To remove cusp

$i\epsilon$ prescription → principal value $\widetilde{\mathrm{PV}}$

Analytically continue principal value below threshold then interpolate to prescription-free subthreshold form

Polejaeva, K. and Rusetsky, A. *Eur. Phys. J.* A48 (2012) 67

What is new here?

3. Must now worry about sum crossing two-particle unitary cusp

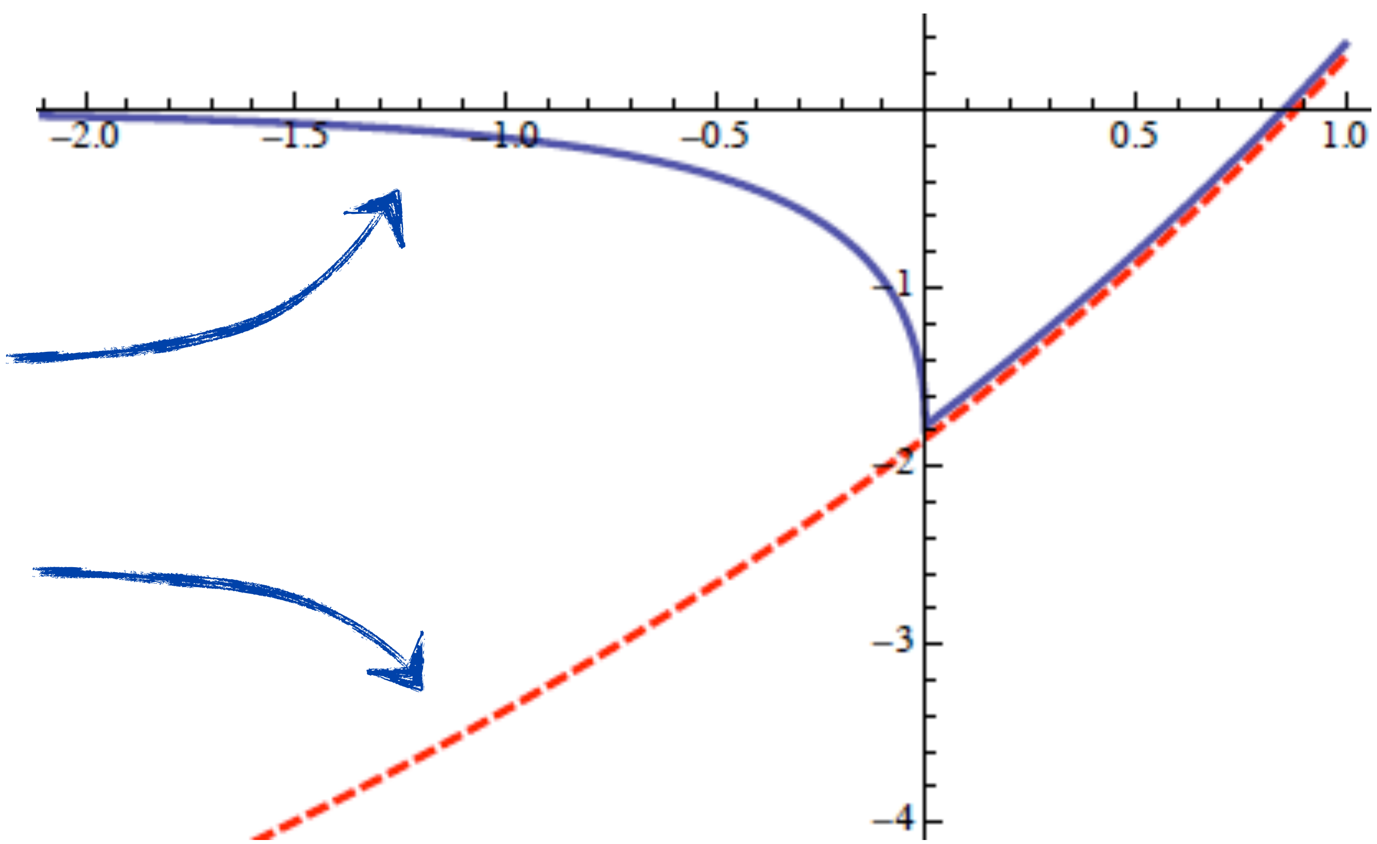
To remove cusp

$i\epsilon$ prescription

principal value $\widetilde{\mathrm{PV}}$

standard definition

modification $\widetilde{\mathrm{PV}}$

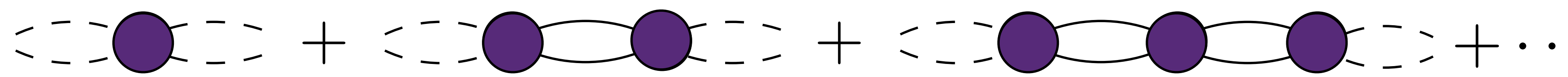


What is new here?

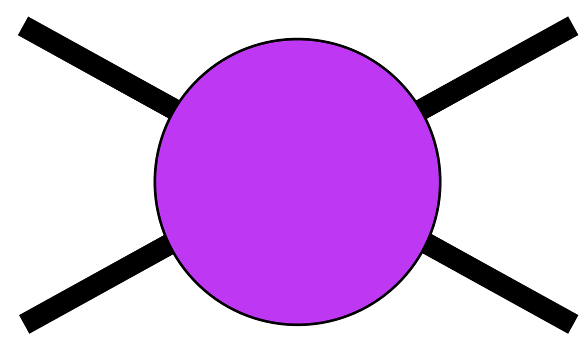
3. Must now worry about sum crossing two-particle unitary cusp

has a cusp

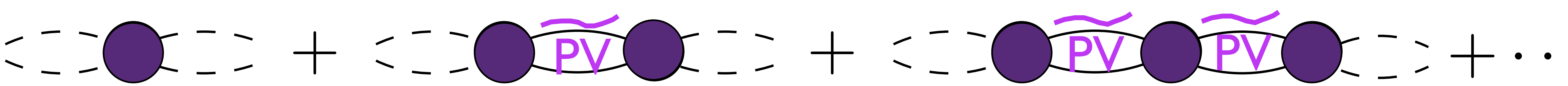
$$i\mathcal{M}_{2\to 2} =$$



$$i\widetilde{\mathcal{K}}_{2\to 2} =$$



$$=$$



What is new here?

3. Must now worry about sum crossing two-particle unitary cusp

$$i\mathcal{M}_{2\to2} \qquad i\mathcal{K}_{2\to2}$$

$$i\mathcal{M}_{\mathrm{df},3\to3} \qquad i\mathcal{K}_{\mathrm{df},3\to3}$$

We relate these infinite-volume quantities to the finite-volume spectrum

Three-particle result

$$\boxed{C_L(E,\vec{P}) = C_\infty(E,\vec{P}) + A_3' iF_3 \frac{1}{1 - i\mathcal{K}_{\mathrm{df},3\to 3}\; iF_3} A_3}$$

$$iF_3 \equiv \frac{iF}{2\omega L^3}\left[\frac{1}{3} + \frac{1}{1 - i\mathcal{M}_{L,2\to 2}\; iG}\; i\mathcal{M}_{L,2\to 2}\; iF\right]$$

$$i\mathcal{M}_{L,2\to 2} \equiv i\mathcal{K}_{2\to 2}\frac{1}{1 - iFi\mathcal{K}_{2\to 2}}$$

All factors are matrices with indices $\vec{k}, \ell, m$

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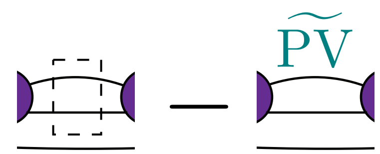
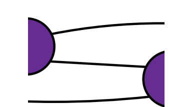
sum-integral difference



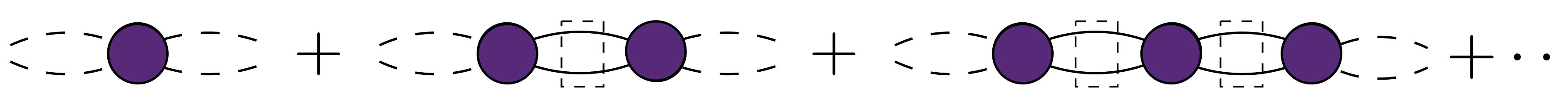
$$iF_3 \equiv \frac{iF}{2\omega L^3}\left[\frac{1}{3} + \frac{1}{1 - i\mathcal{M}_{L,2\to2}\, iG}\, i\mathcal{M}_{L,2\to2}\, iF\right]$$

encodes switches



$$i\mathcal{M}_{L,2\to2} \equiv i\mathcal{K}_{2\to2}\frac{1}{1 - iF i\mathcal{K}_{2\to2}}$$

sum of all two-particle loops (with summed momenta)



All factors are matrices with indices $\vec{k}, \ell, m$

Three-particle result

At fixed $(L, \vec{P})$, finite-volume spectrum is all solutions to

$$\det\left[1 - i\mathcal{K}_{\mathrm{df},3\to 3}\, iF_3\right] = 0$$

$$iF_3 \equiv \frac{iF}{2\omega L^3}\left[\frac{1}{3} + \frac{1}{1 - i\mathcal{M}_{L,2\to 2}\, iG}\, i\mathcal{M}_{L,2\to 2}\, iF\right] \qquad i\mathcal{M}_{L,2\to 2} \equiv i\mathcal{K}_{2\to 2}\frac{1}{1 - iF\, i\mathcal{K}_{2\to 2}}$$

MTH and Sharpe, *Phys. Rev.* D90, 116003 (2014)

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Model independent general result of relativistic scalar field theory

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Assumes no two-body bound states or resonances

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Infinite matrices truncate if we truncate in angular momentum

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MTH and Sharpe, *Phys. Rev.* D90, 116003 (2014)

Model independent general result of relativistic scalar field theory

Assumes no two-body bound states or resonances

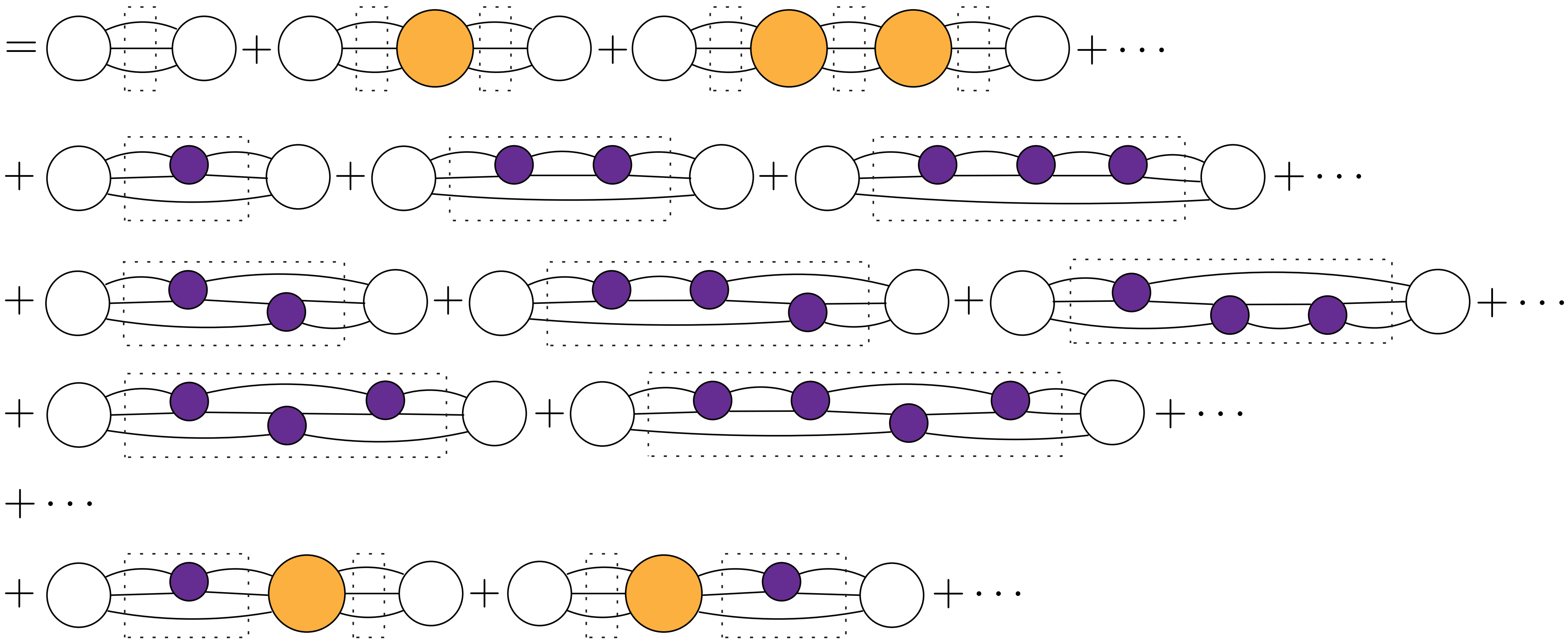
Infinite matrices truncate if we truncate in angular momentum

Strongest truncation is the isotropic limit, gives simple result

$$\mathcal{K}_{\mathrm{df},3\to 3}(E_n^*) = -\left[F_{3,\mathrm{iso}}(E_n, \vec{P}, L)\right]^{-1}$$

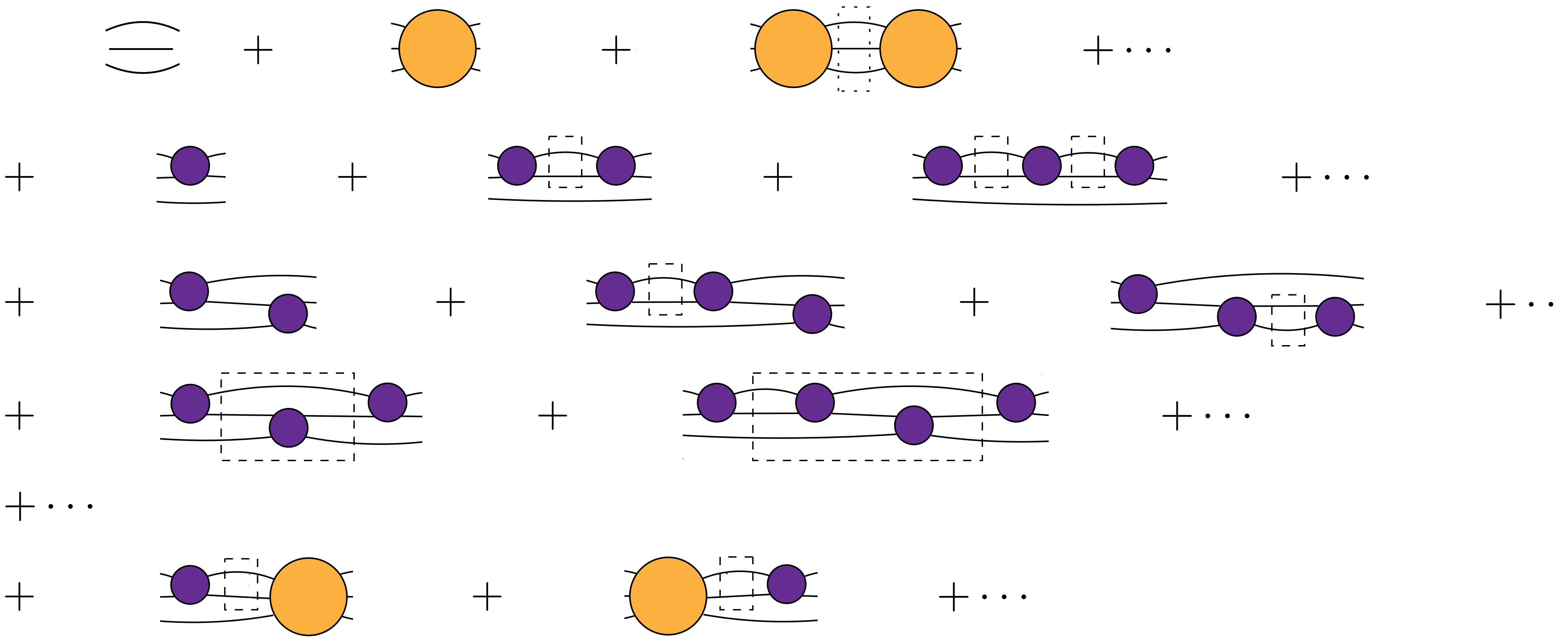
Relating $i\mathcal{K}_{\mathrm{df},3\to3}$ to $i\mathcal{M}_{3\to3}$

$C_L(E,\vec{P}) =$



First we modify $C_L(E,\vec{P})$ to define $i\mathcal{M}_{L,3\to3}$

Relating $i\mathcal{K}_{\mathrm{df},3\to3}$ to $i\mathcal{M}_{3\to3}$



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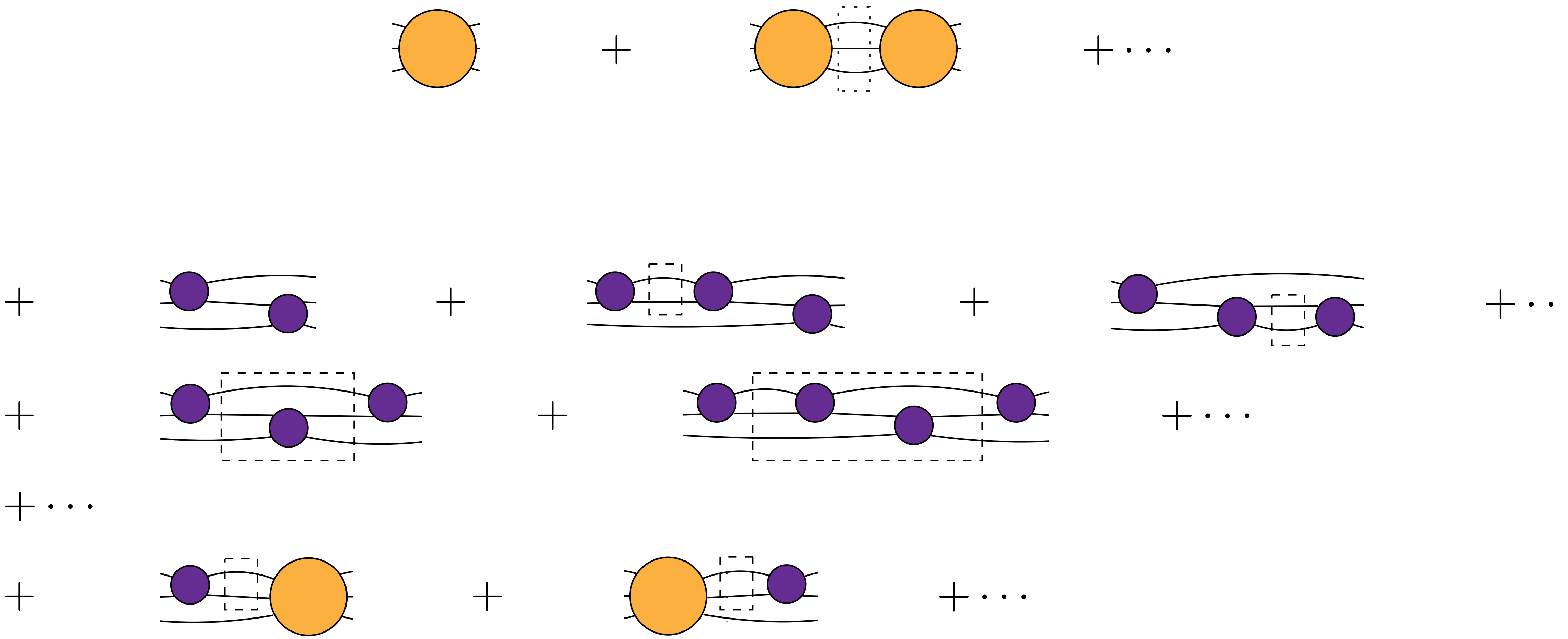
1. Amputate interpolating fields

Relating $i\mathcal{K}_{\mathrm{df},3\to3}$ to $i\mathcal{M}_{3\to3}$

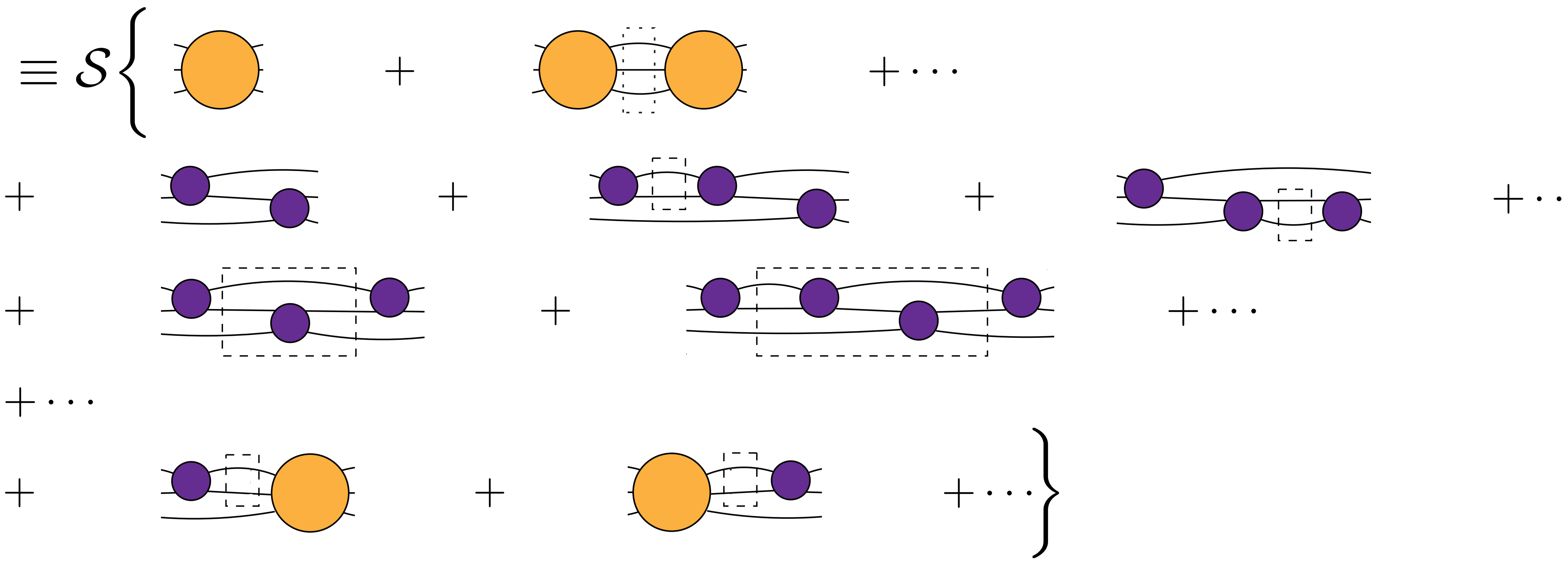


First we modify $C_L(E,\vec{P})$ to define $i\mathcal{M}_{L,3\to3}$

2. Drop disconnected diagrams

Relating $i\mathcal{K}_{\mathrm{df},3\to3}$ to $i\mathcal{M}_{3\to3}$

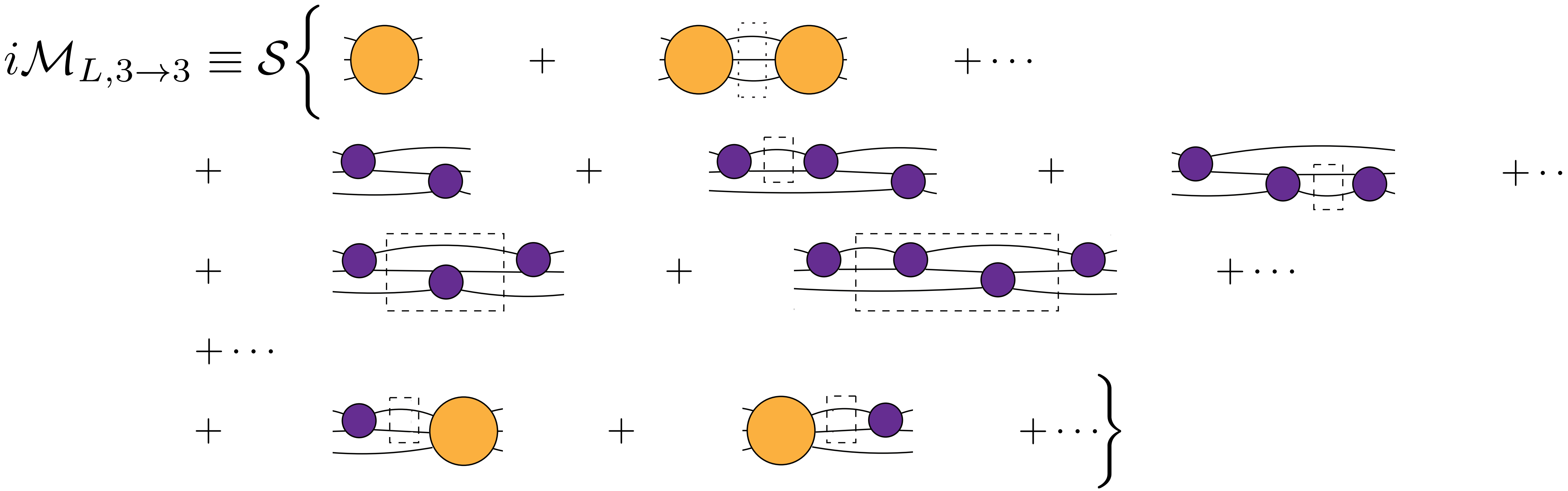
$$i\mathcal{M}_{L,3\to3} \equiv \mathcal{S}\Big\{$$



First we modify $C_L(E,\vec{P})$ to define $i\mathcal{M}_{L,3\to3}$

3. Symmetrize

Relating $i\mathcal{K}_{\mathrm{df},3\to3}$ to $i\mathcal{M}_{3\to3}$



Replacing all loop momentum sums with **i-epsilon prescription integrals** gives **physical three-to-three scattering amplitude**

$$i\mathcal{M}_{3\to3} = \lim_{L\to\infty}\Big|_{i\epsilon} i\mathcal{M}_{L,3\to3}$$

Relating $i\mathcal{K}_{\mathrm{df},3\to3}$ to $i\mathcal{M}_{3\to3}$

$$i\mathcal{M}_{L,3\to3} = i\mathcal{D}_L + \mathcal{S}\left[\mathcal{L}_L \; i\mathcal{K}_{\mathrm{df},3\to3} \frac{1}{1 - iF_3 \; i\mathcal{K}_{\mathrm{df},3\to3}} \; \mathcal{R}_L\right]$$

$$i\mathcal{M}_{3\to3} = \lim_{L\to\infty}\Big|_{i\epsilon} \; i\mathcal{M}_{L,3\to3}$$

MTH and Sharpe, (2015) 1504.04248

Gives integral equation relating $i\mathcal{K}_{\mathrm{df},3\to3}$ to $i\mathcal{M}_{3\to3}$

Completes formal story (for the setup considered!)

Relation only depends on on-shell scattering quantities

Connecting to other work

Reproducing Beane, Detmold and Savage threshold expansion

$$E = 3m + \frac{12\pi a}{mL^3}\left[1 + A\frac{a}{L} + B\frac{a^2}{L^2}\right] + C_1\frac{1}{L^6} - \frac{\mathcal{M}_{\mathrm{df},3\to3,\mathrm{thr}}}{48m^3L^6} + C_2\frac{\log(mL)}{L^6}$$

We agree unambiguously A, B, C_2 **and relate** $C_1 - \dfrac{\mathcal{M}_{\mathrm{df},3\to3,\mathrm{thr}}}{48m^3}$

to a non-relativistic contact interaction

Beane, S., Detmold, W. & Savage, M. *Phys. Rev.* D76 (2007) 074507
Tan, S. Phys. Rev. A78 (2008) 013636

Meißner, Rios and Rusetsky three-body bound state

$$\Delta E = c(\kappa^2/m)(\kappa L)^{-3/2}|A|^2\exp(-2\kappa L/\sqrt{3})$$

would be interesting to check agreement

Meißner, Rios and Rusektsky. *Phys. Rev. Lett.* 114, 091602 (2015)

Summary

Lüscher formalism for *simplest* three-to-three system is complete

- relates on-shell scattering to finite-volume spectrum
- derived in general relativistic quantum field theory
- passes non-trivial checks

- no two-particle bound state or resonance
- identical bosons
- no even-odd coupling

Future work

Include two-particle bound states and resonances

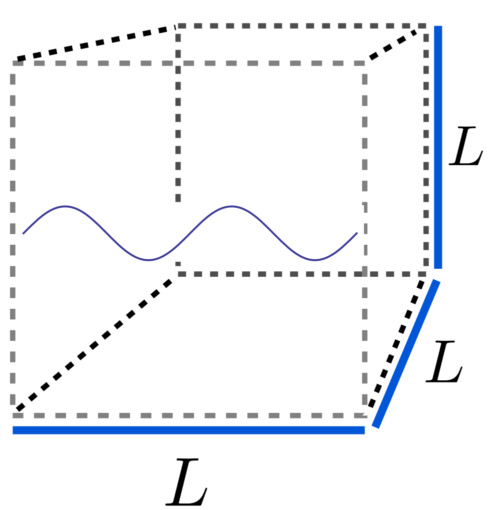
Include two-to-three coupling

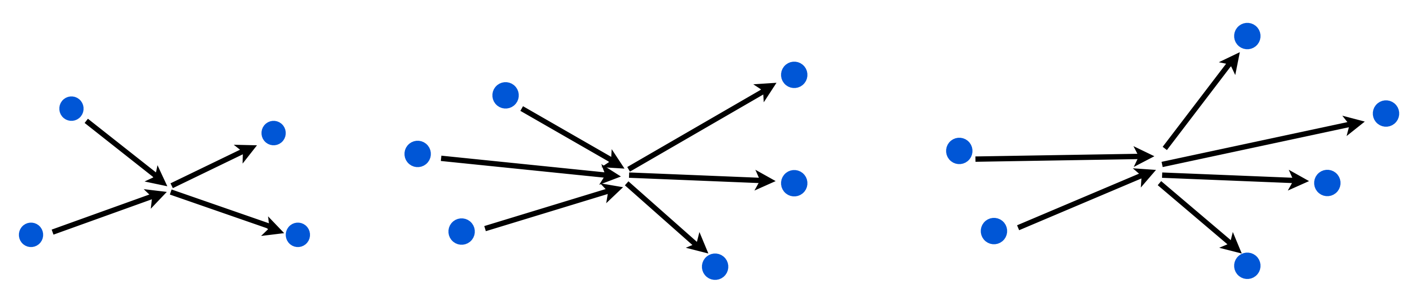
Generalize Lellouch-Lüscher method to extract three-particle weak decays

$$K \longrightarrow \pi\pi\pi$$

Include non-identical, non-degenerate and spin-half particles

Extend mapping to four-particle states



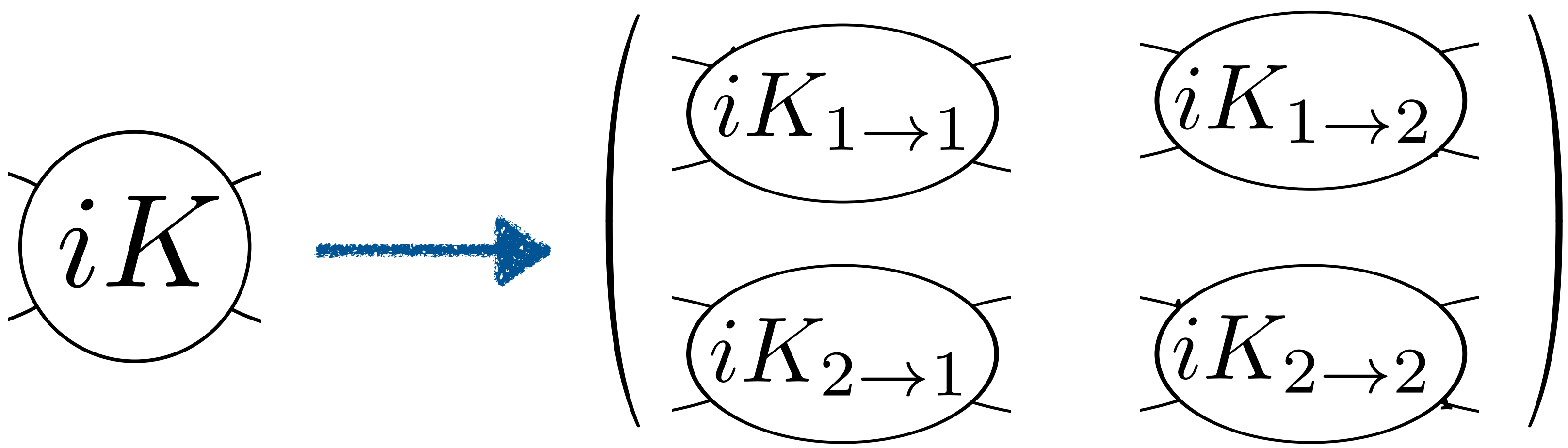


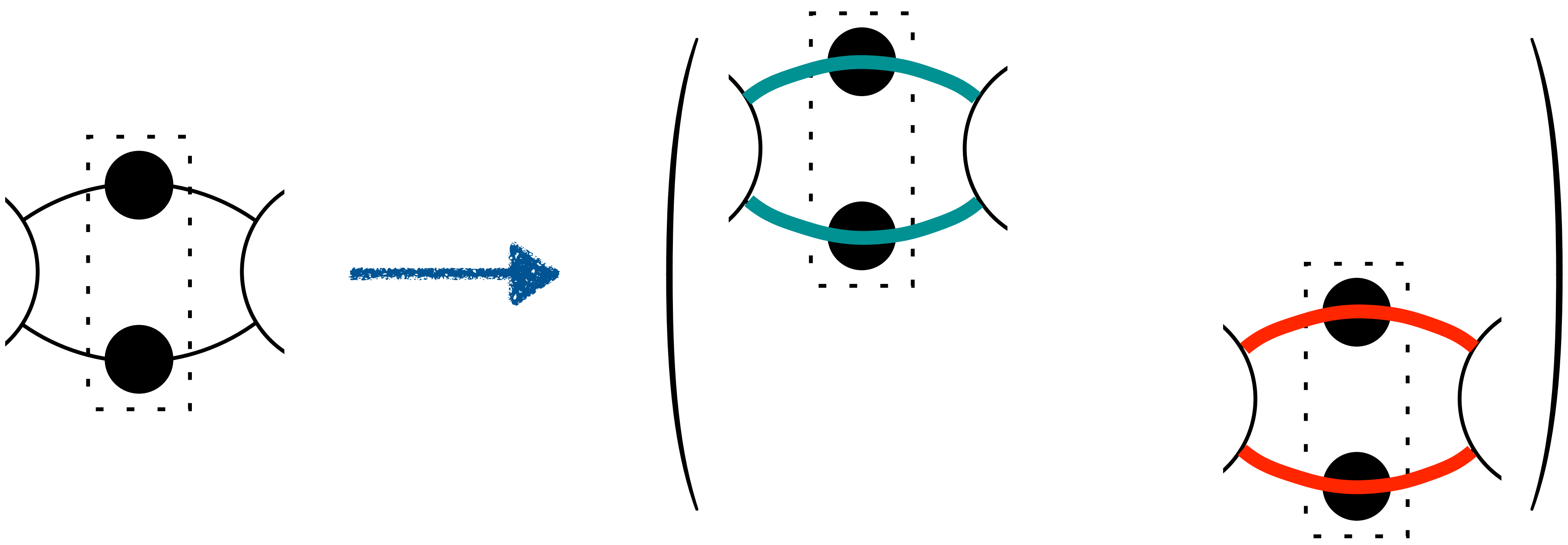
Backup Slides

Scattering of multiple two-particle channels

$$\pi\pi \to \overline{K}K \qquad \pi K \to \eta K$$

Make following replacements





Scattering of multiple two-particle channels

$$\pi\pi \to \overline{K}K \qquad \pi K \to \eta K$$

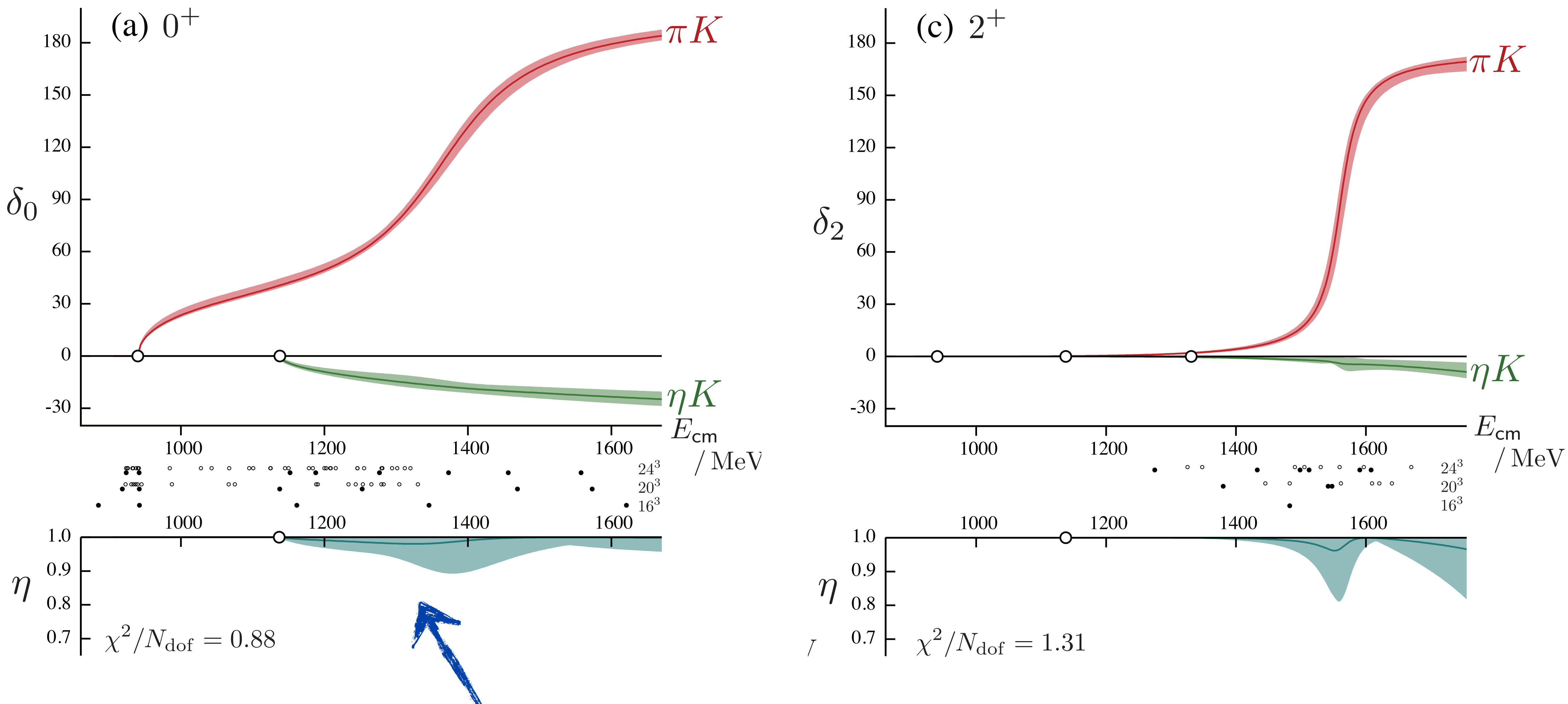
One finds

$$\det\left[1 - \begin{pmatrix} i\mathcal{M}_{1\to 1} & i\mathcal{M}_{1\to 2} \\ i\mathcal{M}_{2\to 1} & i\mathcal{M}_{2\to 2} \end{pmatrix} \begin{pmatrix} iF_1 & 0 \\ 0 & iF_2 \end{pmatrix}\right] = 0$$

M. Lage, U.-G. Meißner, and A. Rusetsky, Phys.Lett., B681, 439 (2009)
V. Bernard, M. Lage, U.-G. Meißner, and A. Rusetsky, JHEP, 1101, 019 (2011)
M. Döring, U.-G. Meißner, E. Oset, and A. Rusetsky, Eur.Phys.J., A47, 139 (2011)
MTH, S. R. Sharpe, *Phys.Rev. D86* (2012) 016007
R. A. Briceño, Z. Davoudi, *Phys.Rev. D88* (2013) 094507

Already implemented in LQCD calculation

$$\pi K \to \eta K$$

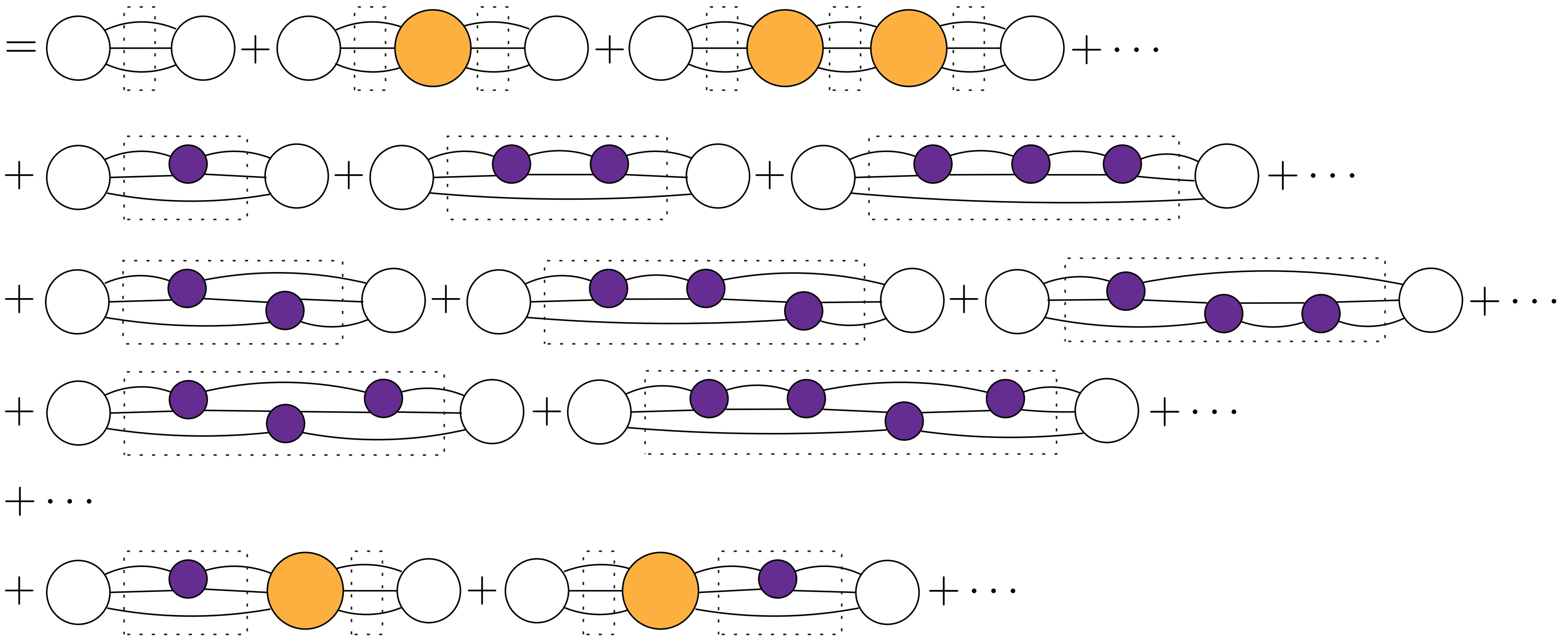


$$\mathcal{M}(\pi K \to \eta K) \sim \sqrt{1-\eta^2}$$

from Dudek, Edwards, Thomas, Wilson in arXiv:1406:4158

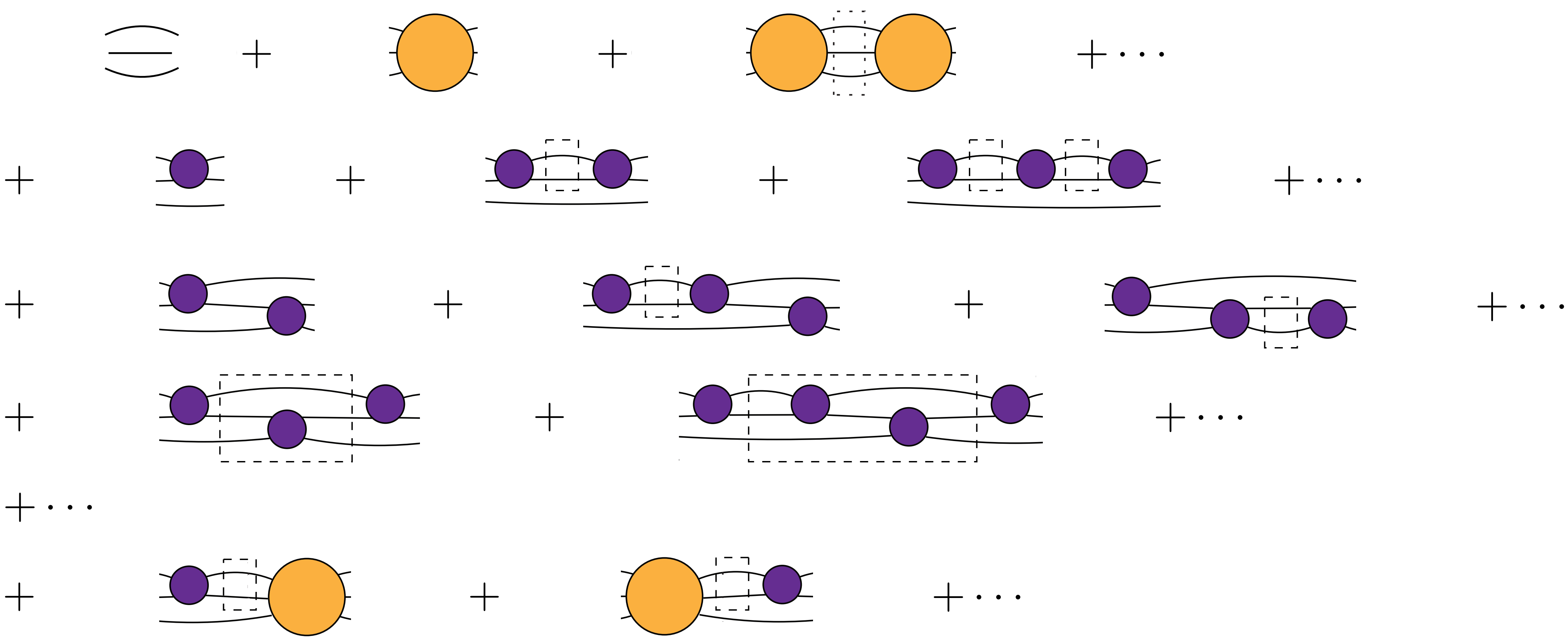
Relating $i\mathcal{K}_{\mathrm{df},3\to3}$ to $i\mathcal{M}_{3\to3}$

$C_L(E,\vec{P}) =$



First we modify $C_L(E,\vec{P})$ to define $i\mathcal{M}_{L,3\to3}$

Relating $i\mathcal{K}_{\mathrm{df},3\to3}$ to $i\mathcal{M}_{3\to3}$



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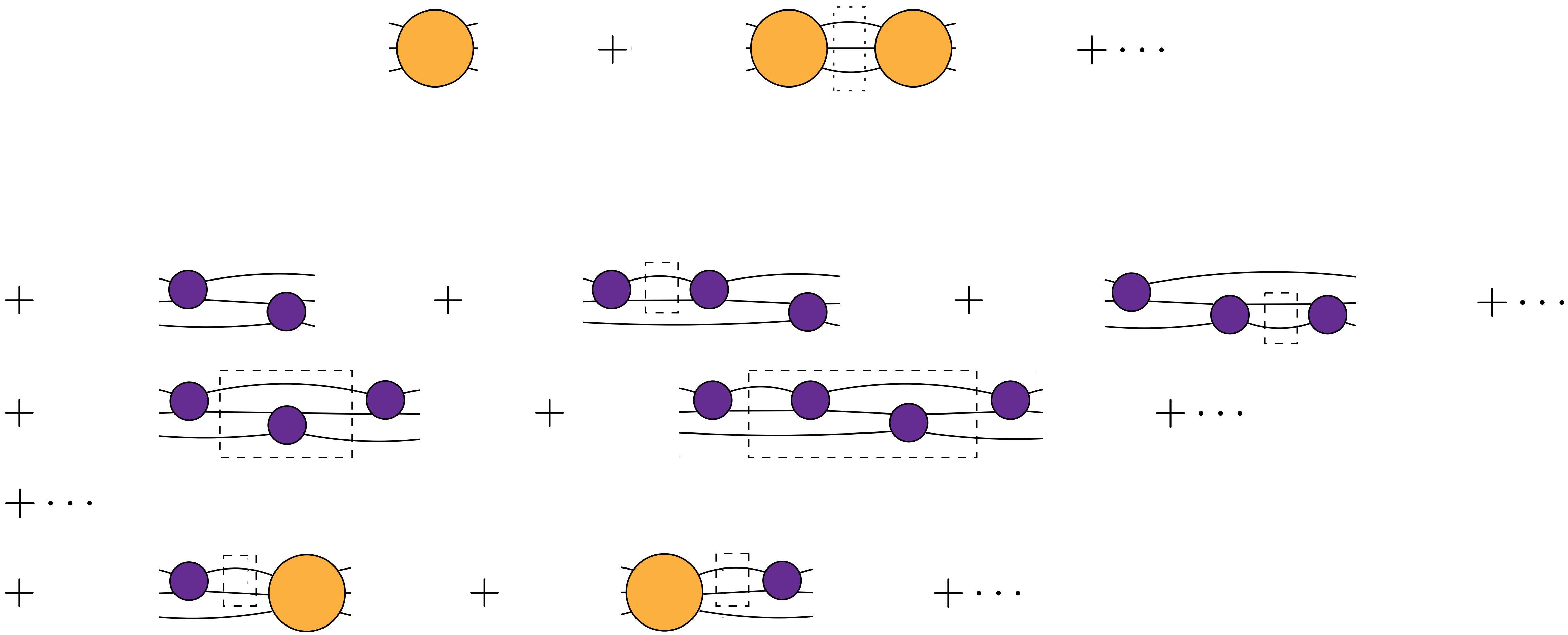
Relating $i\mathcal{K}_{\mathrm{df},3\to3}$ to $i\mathcal{M}_{3\to3}$



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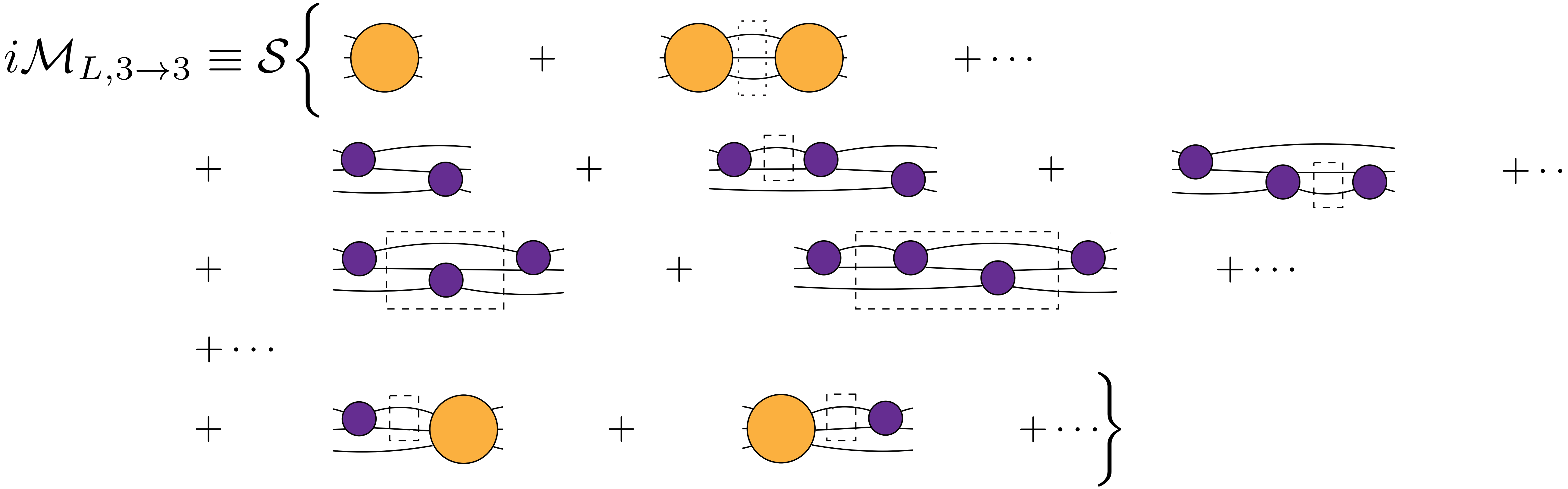
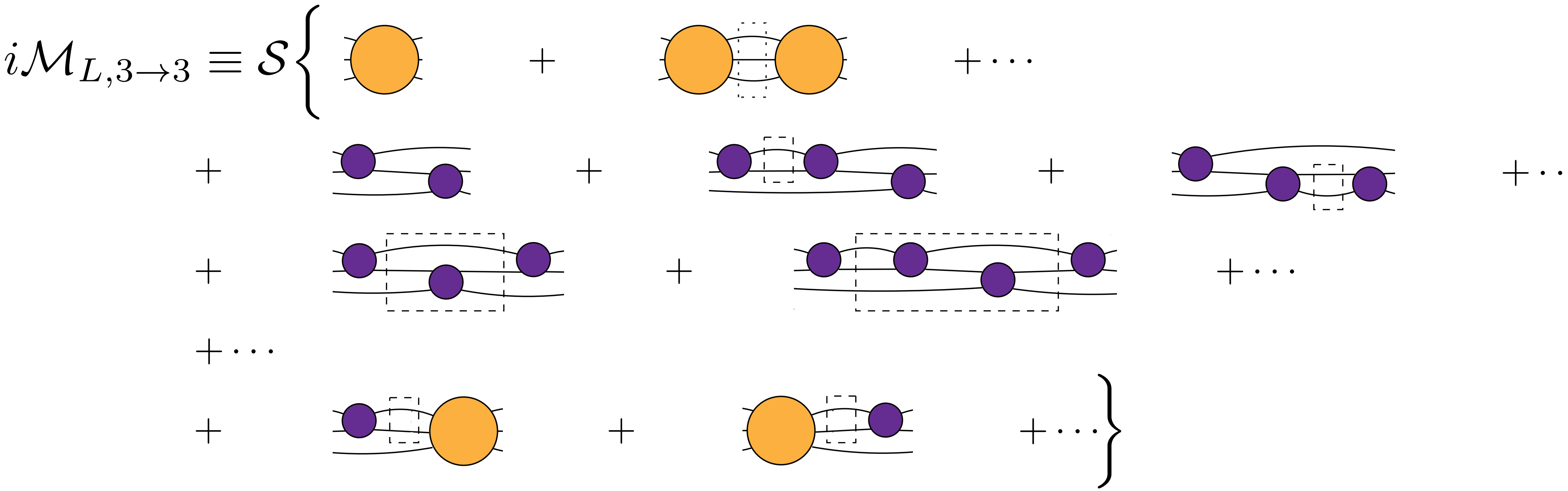
Relating $i\mathcal{K}_{\mathrm{df},3\to3}$ to $i\mathcal{M}_{3\to3}$



First we modify $C_L(E,\vec{P})$ to define $i\mathcal{M}_{L,3\to3}$

3. Symmetrize

Relating $i\mathcal{K}_{\text{df},3\to3}$ to $i\mathcal{M}_{3\to3}$



Replacing all loop momentum sums with i-epsilon prescription integrals would give physical three-to-three scattering amplitude

Relating $i\mathcal{K}_{\mathrm{df},3\to3}$ to $i\mathcal{M}_{3\to3}$

We find a simple form for $i\mathcal{M}_{L,3\to3}$

$$\boxed{i\mathcal{M}_{L,3\to3} = i\mathcal{D}_L + \mathcal{S}\left[\mathcal{L}_L \; i\mathcal{K}_{\mathrm{df},3\to3} \frac{1}{1 - iF_3 \; i\mathcal{K}_{\mathrm{df},3\to3}} \; \mathcal{R}_L\right]}$$

$$i\mathcal{D}_L \equiv \mathcal{S}\left[\frac{1}{1 - i\mathcal{M}_{L,2\to2} \; iG} \; i\mathcal{M}_{L,2\to2} \; iG \; i\mathcal{M}_{L,2\to2}[2\omega L^3]\right]$$

$$iF_3 \equiv \frac{iF}{2\omega L^3}\left[\frac{1}{3} + \frac{1}{1 - i\mathcal{M}_{L,2\to2} \; iG} \; i\mathcal{M}_{L,2\to2} \; iF\right]$$

$$\equiv \frac{iF}{2\omega L^3}\mathcal{L}_L \equiv \mathcal{R}_L \frac{iF}{2\omega L^3}$$

Relating $i\mathcal{K}_{\mathrm{df},3\to3}$ to $i\mathcal{M}_{3\to3}$

We find a simple form for $i\mathcal{M}_{L,3\to3}$

$$\boxed{i\mathcal{M}_{L,3\to3} = i\mathcal{D}_L + \mathcal{S}\left[\mathcal{L}_L \; i\mathcal{K}_{\mathrm{df},3\to3} \frac{1}{1 - iF_3 \; i\mathcal{K}_{\mathrm{df},3\to3}} \; \mathcal{R}_L\right]}$$

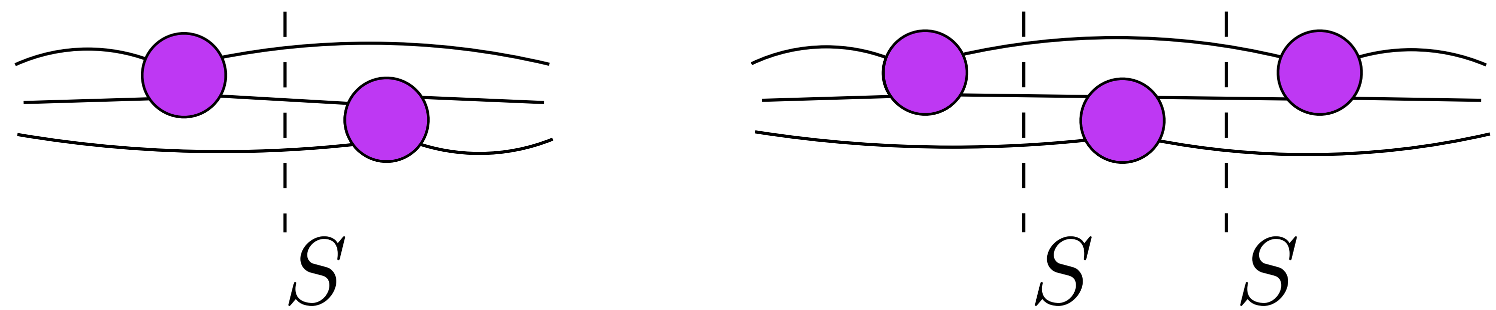
Complete analysis with infinite volume limit

$$\boxed{i\mathcal{M}_{3\to3} = \lim_{L\to\infty}\Big|_{i\epsilon} \; i\mathcal{M}_{L,3\to3}}$$

$$\boxed{i\mathcal{M}_{L,3\to3} = i\mathcal{D}_L + \mathcal{S}\left[\mathcal{L}_L \; i\mathcal{K}_{\mathrm{df},3\to3}\frac{1}{1 - iF_3 \; i\mathcal{K}_{\mathrm{df},3\to3}} \; \mathcal{R}_L\right]}$$

Recall $i\mathcal{M}_{\mathrm{df},3\to3}$

$$\equiv i\mathcal{M}_{3\to3} - \left[i\mathcal{M}_{2\to2}Si\mathcal{M}_{2\to2} + \int i\mathcal{M}_{2\to2}Si\mathcal{M}_{2\to2}Si\mathcal{M}_{2\to2} + \cdots\right]$$



It reappears here... $i\mathcal{M}_{df,3\to3} \equiv \lim_{L\to\infty}\Big|_{i\epsilon} \left[i\mathcal{M}_{L,3\to3} - i\mathcal{D}_L\right]$

$$i\mathcal{D}_L \equiv \mathcal{S}\left[\frac{1}{1 - i\mathcal{M}_{L,2\to2} \; iG} \; i\mathcal{M}_{L,2\to2} \; iG \; i\mathcal{M}_{L,2\to2}[2\omega L^3]\right]$$

encodes switches

Relating $i\mathcal{K}_{\mathrm{df},3\to3}$ to $i\mathcal{M}_{3\to3}$

$$i\mathcal{M}_{L,3\to3} = i\mathcal{D}_L + \mathcal{S}\left[\mathcal{L}_L \; i\mathcal{K}_{\mathrm{df},3\to3} \frac{1}{1 - iF_3 \; i\mathcal{K}_{\mathrm{df},3\to3}} \; \mathcal{R}_L\right]$$

$$i\mathcal{M}_{3\to3} = \lim_{L\to\infty}\Big|_{i\epsilon} \; i\mathcal{M}_{L,3\to3}$$

Gives integral equation relating $i\mathcal{K}_{\mathrm{df},3\to3}$ **to** $i\mathcal{M}_{3\to3}$

Completes formal story (for the setup considered!)

Relation only depends on on-shell scattering quantities