(Dimensional) twisted reduction in large N gauge theories

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Overview

REDUCTION IN LARGE N GAUGE THEORIES

Is $SU(\infty)$ gauge theory independent of the spatial volume?

- Equations for Wilson loops in $U(N)$ are volume independent for $N \to \infty$ **[Eguchi, Kawai]**.
- Proof relies on factorization and $[U(1)]^d$ symmetry.
- Soon **[Bhanot, Heller, Neuberger]** it was realized that $[U(1)]^d$ broken at weak coupling.
- Partial reduction **[Narayanan, Neuberger]**: if $L > L_c$ then $SU(\infty)$ is volume independent ($L_c \sim 1/T_c$).

Twisted reduction [A. Gonzalez-Arroyo, M. Okawa]

- EK proof of volume independence valid for all boundary conditions
- Weak coupling behavior of $SU(N)$ depends on choice of b.c.

TWISTED BOUNDARY CONDITIONS AND WEAK COUPLING

Key idea

In a periodic world only gauge invariant quantities need to be periodic.

$$A_\mu(x + L\hat{\nu}) = \Omega_\nu(x) A_\mu(x) \Omega_\nu^+(x) + \imath \Omega_\nu(x) \partial_\mu \Omega_\nu^+(x).$$

- Periodic boundary conditions: Path integral dominated by torons

$$\langle A_\mu A_\nu \rangle \sim \int \mathcal{D}A \; A_\mu A_\nu e^{-(p_\mu A_\mu)^2 - \tilde{A}_\mu^4(0)}$$

- But twisted boundary conditions can be chosen so that zero mode vanish. ej: Symmetric twist ($\sqrt{N} \in \mathbb{Z}$, and $z_{\mu\nu} = e^{\imath k/\sqrt{N}}$ with k and $\sqrt{N}$) coprime.

$$\Omega_\mu \Omega_\nu = z_{\mu\nu} \Omega_\nu \Omega_\mu$$

- Using a particular Lie algebra basis **[M. Garcia Perez et. al.]**

$$A_\mu(x) = \frac{1}{L^4} \sum_p e^{\imath p x} \tilde{A}_\mu(p) \Gamma(p)$$

Twisted Boundary conditions and weak coupling

The N^2 glouns "build" an $\sqrt{N}^4$ space

(In the continuum) The gauge field can be expanded as

$$A_\mu(x) = \frac{1}{L^4} \sum_p e^{\imath px} \tilde{A}_\mu(p) \Gamma(p) \,,$$

- $\tilde{A}_\mu(p)$ is a number.
- $\Gamma(p)$ is a matrix.
- $p = \frac{2\pi n}{\sqrt{N}L} \rightarrow$ Momenta quantized as if the volume were $L_{\text{eff}} = \sqrt{N}L$.
- Feynmann rules in momentum space depend on

$$\sin\left(\frac{\theta_{\mu\nu}}{2} p_\mu q_\nu\right) \,, \tag{1}$$

with $\theta_{\mu\nu} = \frac{L^2 N}{2\pi} \bar{k}/\sqrt{N}$ and $k\bar{k} = 1 \mod \sqrt{N}$.

Choice of twist (k) very important

- Keeping $k = 1$ does not work **[Teper, Vairinhos; Ishikawa, Okawa]**.

TEK MODEL

Single site model (d=0)

$$\mathcal{Z} = \int \mathcal{D}U_\mu \;\exp\left\{ bN \sum_{\mu\neq\nu} \mathrm{Tr}\left[1 - z_{\mu\nu} U_\mu U_\nu U_\mu^\dagger U_\nu^\dagger\right]\right\}$$

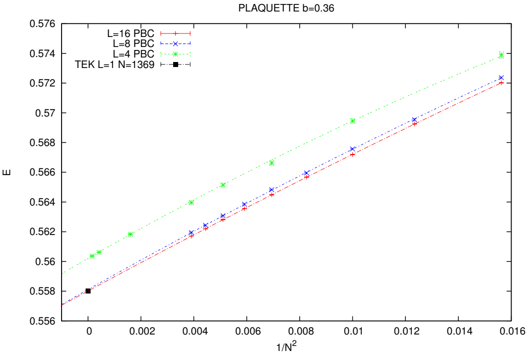


Figure: Source: arXiv:1410.6405 **[A. Gonzalez-arroyo, M. Okawa]**

OVERVIEW

WHAT ABOUT A NON-SYMMETRIC TWIST?

Can we play with other choices of twist?

- Choose to twist only the plane $(1,2)$

$$n_{\mu\nu} = -n_{\nu\mu} = \begin{cases} k & \mu = 1, \nu = 2 \\ 0 & \text{otherwise} \end{cases}$$

- (In the continuum) The gauge field can be expanded as

$$A_\mu(x) = \frac{1}{L^4} \sum_p e^{\imath px} \tilde{A}_\mu(p) \Gamma(p) \,,$$

- $\tilde{A}_\mu(p)$ is a number.
- $\Gamma(p)$ is a matrix.
- $p_{1,2} = \frac{2\pi n}{NL_{1,2}}; p_{0,3} = \frac{2\pi n}{L_{0,3}} \to$ Momenta quantized as if $L_{1,2}^{\text{eff}} = NL_{1,2}$.
- Feynmann rules in momentum space depend on

$$\sin\left(\frac{\theta_{\mu\nu}}{2} p_\mu q_\nu\right) \,,$$

with $\theta_{\mu\nu} = \frac{L^2 N}{2\pi} \bar{k}/\sqrt{N}$ and $k\bar{k} = 1 \mod \sqrt{N}$.

What about a non-symmetric twist?

$SU(\infty)$ in $\mathbb{R}^4$

- Lattice simulation on a $L_0 \times 1 \times 1 \times L_3$ lattice
- Choose twist on the plane $(1,2)$: $L_{1,2}^{\text{eff}} = N$
- Recover $SU(\infty)$ in $\mathbb{R}^4$ by taking the limits $N, L \longrightarrow \infty$.
- Choose k with the same recipes as in symmetric twist **[A. Gonzalez-Arroyo, M. Okawa]**

$$k \text{ coprime with } N \qquad k/N > 1/9$$
$$k\bar{k} = 1 \mod N \qquad \text{large } \theta = \bar{k}/N$$

Why is such a choice interesting?

- Thermodynamics: $L_4 = \beta$ and $N, L_3 \to \infty$
- Extract masses from correlators:

$$\sum_{\mathbf{x}} \langle O(\mathbf{x}, x_0) O(0) \rangle \longrightarrow e^{-mx_0}$$

- Numerically efficient

$SU(L)$ on $1 \times 1 \times L \times L$

$L^3 \times 1^2 \times L^2 \to \mathcal{O}(L^5)$

$SU(L^2)$ on $1 \times 1 \times 1 \times 1$

$(L^2)^3 \times 1^4 \to \mathcal{O}(L^6)$

IS CENTER SYMMETRY RESPECTED?

Open paths

- In a normal lattice gauge theory gauge symmetry implies

$$U_1^{[\Lambda]}(x) = \Lambda^\dagger(x)U_1(x)\Lambda(x+\hat{1}) \Longrightarrow \langle \mathrm{Tr} U_1(x)\rangle = 0$$

- Not anymore for reduced directions

$$U_1^{[\Lambda]}(x) = \Lambda^\dagger(x)U_1(x)\Lambda(x) \not\Longrightarrow \langle \mathrm{Tr} U_1(x)\rangle = 0$$

- If reduction works, open paths must have zero expectation value. Note that this means that $[U(1)]^d$ symmetry is *not* broken.
- Measure order parameters

$$\frac{1}{N}\langle |U_{1,2}(x)|\rangle$$

- Simulations:

N	Lattice size	k	$\bar{k}$	$\bar{k}/N$
24	$1^2 \times 24^{\ 2}$	7	7	0.291666..
36	$1^2 \times 36^2$	13	25	0.305555..
40	$1^2 \times 40^2$	11	11	0.275
56	$1^2 \times 56^2$	23	39	0.303571..

IS CENTER SYMMETRY RESPECTED?

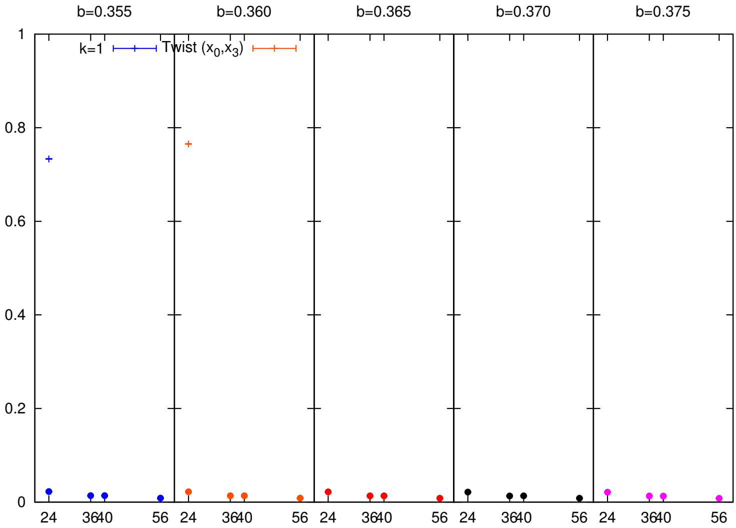
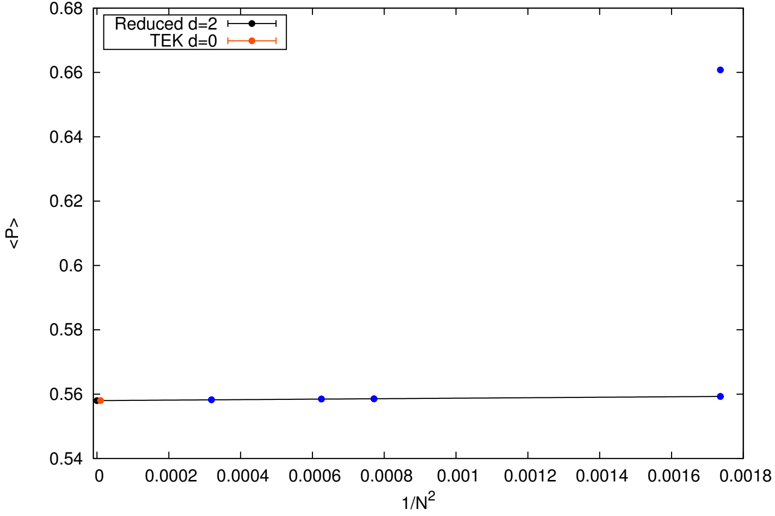


Figure: $\frac{1}{N}\langle|U_{1,2}(x)|\rangle$ For different b and N.

Overview

Plaquette values $b = 0.360$



PLAQUETTE VALUES $b = 0.360$

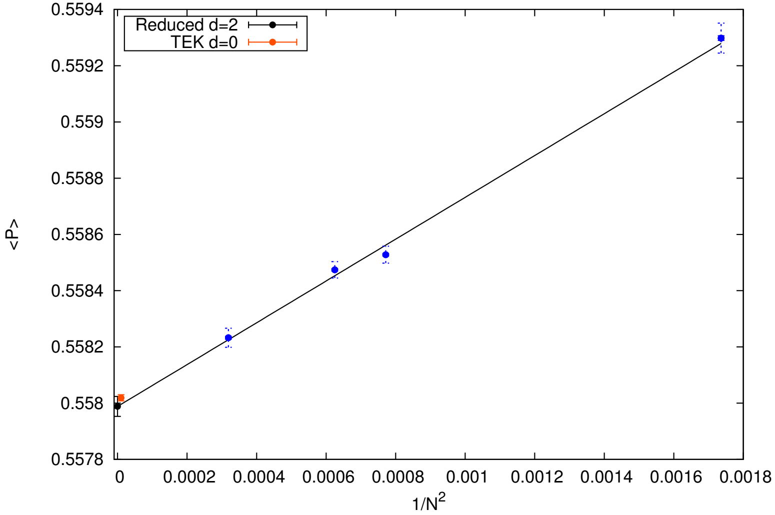


Figure: Comparison with Literature **[arXiv:1410.6405]** ($SU(1369)$).

Plaquette values $b = 0.370$

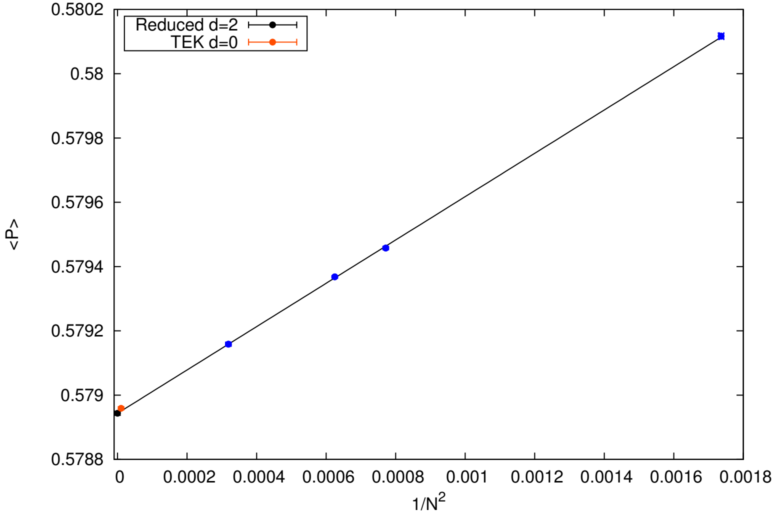


Figure: Comparison with Literature **[arXiv:1410.6405]** ($SU(1369)$).

PLAQUETTE VALUES

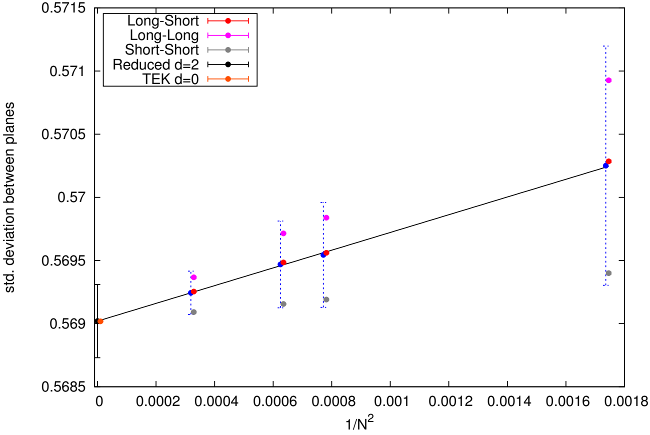
Reduction	$b = 0.355$	$b = 0.360$	$b = 0.365$	$b = 0.370$
$d = 4^*$	0.545417(63)	0.558012(12)	0.569021(41)	0.578978(17)
$d = 2$	0.545319(51)	0.557988(35)	0.569018(17)	0.5789434(64)
$d = 0^*$	0.545336(11)	0.558019(11)	0.569018(4)	0.578959(5)

* **[A. Gonzalez-Arroyo, M. Okawa. arXiv:1410.6405]**

RESTORATION OF $O(4)$ SYMMETRY

$O(4)$ Symmetry is restored

Taking $L, N \to \infty$ we should recover $O(4)$ symmetry



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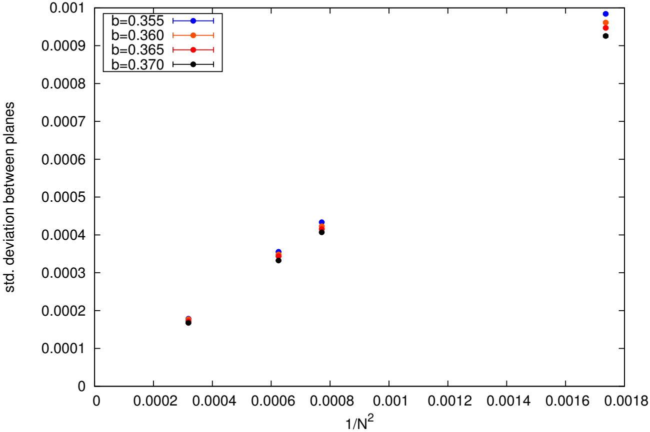


Overview

Yang-Mills flow quantities

Yang-Mills gradient flow

$$
\begin{aligned}
G_{\nu\mu}(x,t) &= \partial_\nu B_\mu(x,t) - \partial_\nu B_\mu(x,t) + [B_\nu(x,t), B_\mu(x,t)] \\
\frac{dB_\mu(x,t)}{dt} &= D_\nu G_{\nu\mu}(x,t) \quad \left(\sim -\frac{\delta S_{\mathrm{YM}}[B]}{\delta B_\mu}\right)
\end{aligned}
$$

with initial condition $B_\mu(x, t = 0) = A_\mu(x)$. Energy density has a continuum limit

$$\langle E(t,x)\rangle = -\frac{1}{2}\langle \mathrm{Tr} G_{\mu\nu}(t,x) G_{\mu\nu}(t,x)\rangle$$

Reference scales

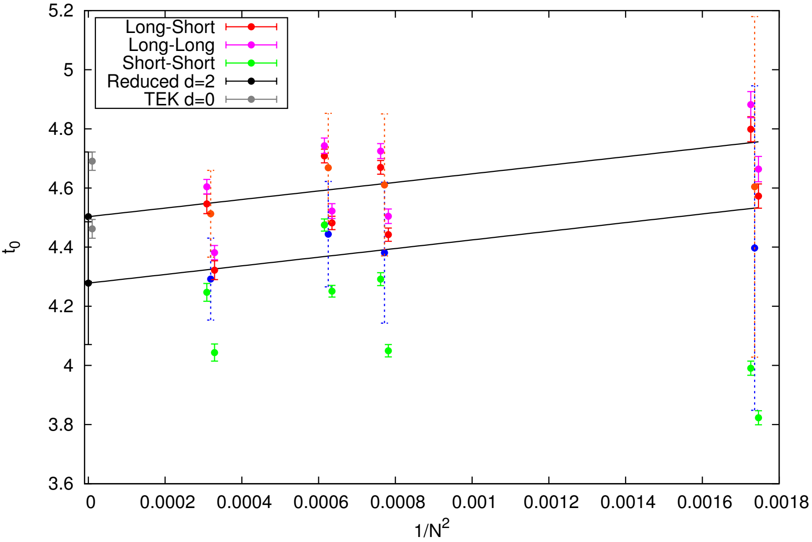
With the definition $\mathcal{E}(t) = \frac{1}{N} t^2 \langle E(t)\rangle$

$$
\begin{aligned}
t_0 &: \quad \mathcal{E}(t_0) = 0.1 \\
w_0 &: \quad w_0^2 \frac{d}{dt}\mathcal{E}(t)\Big|_{t=w_0^2} = 0.1
\end{aligned}
$$

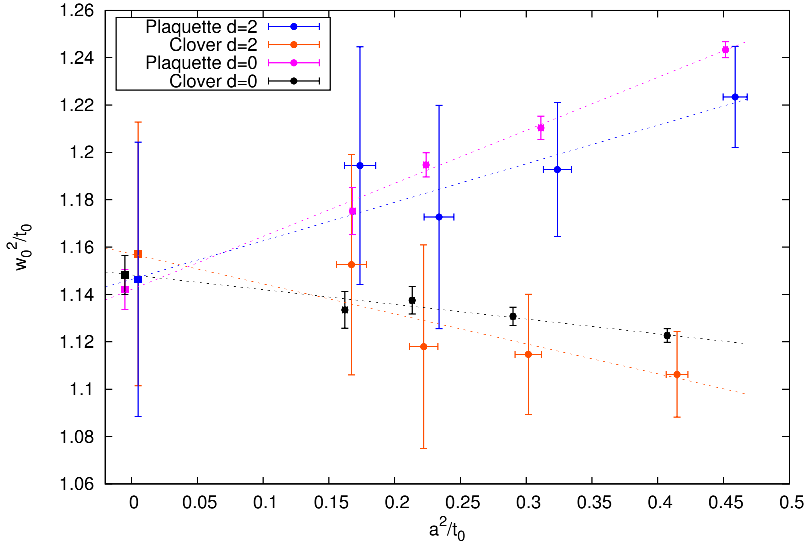
we have

$$\sqrt{t_0} \sim 0.15\text{fm} \qquad w_0 \sim 0.18\text{fm}$$

$b = 0.365$


Long-Short
Long-Long
Short-Short
Reduced d=2
TEK d=0
5.2
5
4.8
4.6
4.4
4.2
4
3.8
3.6
t0
0
0.0002
0.0004
0.0006
0.0008
0.001
0.0012
0.0014
0.0016
0.0018
1/N2

Ratio w_0^2/t_0


Plaquette d=2
Clover d=2
Plaquette d=0
Clover d=0
1.26
1.24
1.22
1.2
1.18
1.16
1.14
1.12
1.1
1.08
1.06
w0^2/t0
0
0.05
0.1
0.15
0.2
0.25
0.3
0.35
0.4
0.45
0.5
a^2/t0

Overview

CONCLUSIONS

Exploration of $d = 2$ twisted reduction

- Twisted boundary conditions allow to push the idea of reduction to the limit (TEK model).
- Choice of twist allows a variety of models with different dimensional reductions ($d = 4, 2, 0$).
- $d = 2$ reduced models have some advantages
  - Computing masses/correlators
  - Phase diagram and finite T of $SU(\infty)$
  - Numericallly efficient: paralellization/better scaling
- Reduction works for $d = 2$ if twist choosen properly: No center symmetry breaking, $O(4)$ symmetry restoration.

Drawbacks

- For precise quantities (plaquette, flow), $\mathcal{O}(1/N^2)$, $\mathcal{O}(1/L)$ corrections are not negligible
- Better understanding on these corrections is desirable and necessary for a precise determination of continuum quantities.