Canonical simulations of Supersymmetric Yang-Mills Quantum Mechanics

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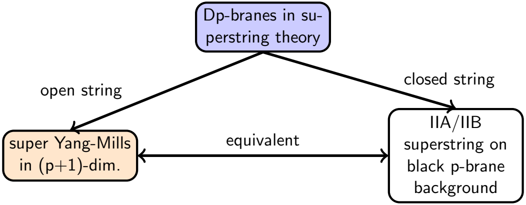
Dualities, black holes and all that

Gauge/gravity duality conjecture:

- $U(N)$ gauge theories as a low energy effective theory of N D-branes
- Dimensionally reduced large-N super Yang-Mills might provide a nonperturbative formulation of the string/M-theory
- Connection to black p-branes allows studying black hole thermodynamics through strongly coupled gauge theory:



Motivation

Super Yang-Mills quantum mechanics:

- Interesting physics:
  - testing gauge/gravity duality,
  - thermodynamics of black holes
- Interesting expectations:
  - discrete vs. continuous spectrum (depending on the fermion sector),
  - flat directions
- Interesting 'bosonisation':
  - fermion contribution decomposes into fermion sectors,
  - allows for a local fermion algorithm,
  - structure is the same as for QCD!

Continuum Model

- Start from $\mathcal{N} = 1$ SYM in $d = 4$ (or 10) dimensions
- Dimensionally reduce to 1-dim. $\mathcal{N} = 4$ (or 16) SYM QM:

$$S = \frac{1}{g^2} \int_0^\beta dt \, \mathrm{Tr} \left\{ (D_t X_i)^2 - \frac{1}{2} [X_i, X_j]^2 + \overline{\psi} D_t \psi - \overline{\psi} \sigma_i [X_i, \psi] \right\}$$

  - covariant derivative $D_t = \partial_t - i[A(t), \cdot]$,
  - time component of the gauge field $A(t)$,
  - spatial components become bosonic fields $X_i(t)$ with $i = 1, \ldots, d-1$,
  - anticommuting fermion fields $\overline{\psi}(t)$, $\psi(t)$,
  - σ_i are the γ-matrices in d dimensions

- all fields in the adjoint representation of SU(N)

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  - covariant derivative $D_t = \partial_t - i[A(t), \cdot]$,
  - time component of the gauge field $A(t)$,
  - spatial components become bosonic fields $X_i(t)$ with $i = 1, 2, 3$ (for $\mathcal{N} = 4$),
  - anticommuting fermion fields $\overline{\psi}(t)$, $\psi(t)$, (complex 2-component spinors for $\mathcal{N} = 4$)
  - σ_i are the γ-matrices in d dimensions (Pauli matrices for $\mathcal{N} = 4$)

- all fields in the adjoint representation of SU(N)

Lattice regularisation

- Discretise the bosonic part:

$$S_B = \frac{1}{g^2} \sum_{t=0}^{L_t - 1} \mathrm{Tr} \left\{ D_t X_i(t) D_t X_i(t) - \frac{1}{2} \left[X_i(t), X_j(t) \right]^2 \right\}$$

with $D_t X_i(t) = U(t) X_i(t+1) U^\dagger(t) - X_i(t)$

- Use Wilson term for the fermionic part,

$$S_F = \frac{1}{g^2} \sum_{t=0}^{L_t - 1} \mathrm{Tr} \left\{ \overline{\psi}(t) D_t \psi(t) - \overline{\psi}(t) \sigma_i \left[X_i(t), \psi(t) \right] \right\} \ ,$$

since

$$\partial^{\mathcal{W}} = \frac{1}{2}(\nabla^+ + \nabla^-) \pm \frac{1}{2} \nabla^+ \nabla^- \quad \overset{d=1}{\Longrightarrow} \quad \nabla^\pm$$

Lattice regularisation and reduced determinant

- Specifically, we have

$$S_F = \frac{1}{2g^2} \sum_{t=0}^{L_t-1} \left[-\overline{\psi}^a_\alpha(t) W^{ab}_{\alpha\beta}(t) e^{+\mu L_t} \psi^b_\beta(t+1) + \overline{\psi}^a_\alpha(t) \Phi^{ac}_{\alpha\beta}(t) \psi^c_\beta(t) \right]$$

where $W^{ab}_{\alpha\beta}(t) = 2\delta_{\alpha\beta} \otimes \mathrm{Tr}\{T^a U(t) T^b U(t)^\dagger\}$.

- Φ is a $2(N^2-1) \times 2(N^2-1)$ Yukawa interaction matrix:

$$\Phi^{ac}_{\alpha\beta}(t) = (\sigma_0)_{\alpha\beta} \otimes \delta^{ac} - 2\,(\sigma_i)_{\alpha\beta} \otimes \mathrm{Tr}\{T^a [X_i(t), T^c]\}$$

- Dimensional reduction of determinant at finite density $\mu \neq 0$:

$$\det \mathcal{D}_{p,a}[U, X_i; \mu] = \det \left[\prod_{t=0}^{L_t-1} \Phi(t) W(t) \mp e^{+\mu L_t} \right]$$

Fugacity expansion

- Dimensional reduction of determinant gives: (for finite density $\mu \neq 0$)

$$\det \mathcal{D}_{p,a}[U, X; \mu] = \det \left[\prod_{t=0}^{L_t - 1} \Phi(t) W(t) \mp e^{+\mu L_t} \right]$$

- Fugacity expansion is easy:

$$\det \mathcal{D}_{p,a}[U, X_i; \mu] = \sum_{n_f=0}^{2(N^2-1)} (\mp e^{\mu L_t})^{n_f} \det \mathcal{D}_{n_f}[U, X_i]$$

  - diagonalise $\mathcal{T} \equiv \prod_{t=0}^{L_t-1} \Phi(t) W(t) \quad \rightarrow$ eigenvalues $\{\tau_i\}$
  - calculate coefficients of the characteristic polynomial:

$$\det \mathcal{D}_{p,a}[U, X_i; \mu] = \prod_{j=1}^{2(N^2-1)} \left(\tau_j \mp e^{\mu L_t} \right)$$

Fugacity expansion and transfer matrices

- Canonical determinants are expressed in terms of elementary symmetric functions S_k of order k of $\{\tau_i\}$:

$$\det \mathcal{D}_{n_f}[U, X_i] = S_{n_f^{\max} - n_f}(\mathcal{T})$$

where

$$S_k(\mathcal{T}) \equiv S_k(\{\tau_i\}) = \sum_{1 \leq i_1 < \cdots < i_k \leq n_f^{\max}} \prod_{j=1}^{k} \tau_{i_j} \, .$$

Crucial object:

$$\mathcal{T} \equiv \prod_{t=0}^{L_t - 1} \Phi(t) W(t) \quad \Leftrightarrow \quad \text{product of transfer matrices}$$

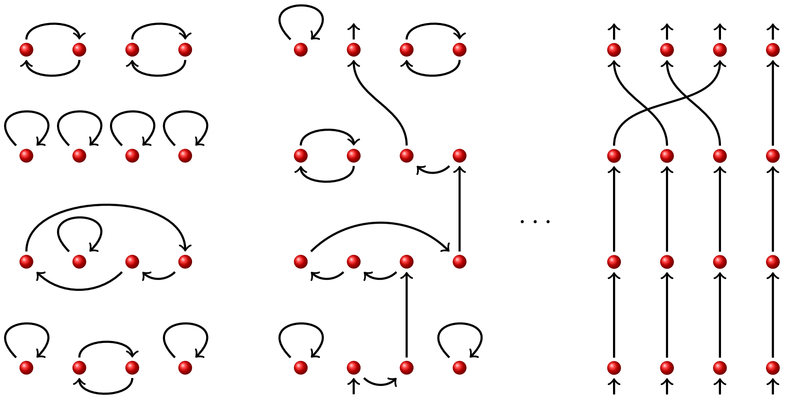
- Proof via fermion loop formulation:

$\Rightarrow$ explicit construction in each fermion sector

Fermion loop formulation ⇔ hopping expansion to all orders

- Configurations can be classified according to the number of propagating fermions n_f:

$n_f = 0$ $n_f = 1$ $\cdots$ $n_f = 2(N^2 - 1)$



Transfer matrices

- Propagation of fermions described by transfer matrices:

$T^{\Phi}_{n_f}(t) \quad \Rightarrow \quad$ sums up local vacuum contibutions,

$T^{W}_{n_f}(t) \quad \Rightarrow \quad$ projects onto gauge invariant states

Explicitly:

$$\left(T^{\Phi}_{n_f}\right)_{AB} = (-1)^{p(A,B)} \det \Phi^{\not{B}\not{A}} \qquad \text{cofactor } C_{\not{B}\not{A}}(\Phi)$$

$$\left(T^{W}_{n_f}\right)_{AB} = \det W^{AB} \qquad \text{minor } M_{AB}(W)$$

- Size of $T^{\Phi,W}_{n_f}$ is given by $N_{\text{states}} = n_f^{\max}!/(n_f^{\max} - n_f)! \cdot n_f!$
- Fermion contribution to the partition function is simply

$$\det \mathcal{D}_{n_f}[U, X_i] = \text{Tr}\left[\prod_{t=0}^{L_t-1} T^{\Phi}_{n_f}(t) \cdot T^{W}_{n_f}(t)\right]$$

Transfer matrices and canonical determinants

- Fermion contribution to the partition function is simply

$$\det \mathcal{D}_{n_f}[U, X_i] = \mathrm{Tr}\left[\prod_{t=0}^{L_t-1} T^{\Phi}_{n_f}(t) \cdot T^{W}_{n_f}(t)\right]$$

- Use Cauchy-Binet formula (and some algebra):

$$\left(\prod_{t=0}^{L_t-1}\left[T^{\Phi}_{n_f}(t) \cdot T^{W}_{n_f}\right]\right)_{AB} = (-1)^{p(A,B)} \det \mathcal{T}^{\not{A}\not{B}} = C_{\not{A}\not{B}}(\mathcal{T})$$

Sum over principal minors:

$$\det \mathcal{D}_{n_f}[U, X_i] = \sum_{B} \det \mathcal{T}^{\not{B}\not{B}} \equiv E_{n_f}(\mathcal{T})\,.$$

- Finally one can proof by linear algebra

$$E_{n_f}(\mathcal{T}) = S_{n_f^{\max}-n_f}(\mathcal{T})\,.$$

Summary

- Canonical determinants are directly given by transfer matrices

$$\det \mathcal{D}_{n_f}[U, X_i] = \mathrm{Tr}\left[\prod_{t=0}^{L_t-1} T_{n_f}^{\Phi}(t) \cdot T_{n_f}^{W}(t)\right] = \sum_{B} \det \mathcal{T}^{\not{B}\not{B}}$$

  constructed from reduced matrix

$$\mathcal{T} \equiv \prod_{t=0}^{L_t-1} \Phi(t) W(t) \,.$$

- Proof is applicable to QCD, algebraic structure is the same!
- Remarks:
  - $\mathcal{T}$ describes the dimensionally reduced effective action for W,
  - our result allows for local fermion algorithm,
  - allows canonical simulations at fixed n_f.

Canonical simulations at fixed n_f

- Canonical determinants are real: $\det \mathcal{D}_{n_f}[U, X_i] = \det \mathcal{D}_{n_f}[U, X_i]^*$
- Furthermore, for $n_f = 0$ and $n_f = n_f^{\max}$ (quenched):

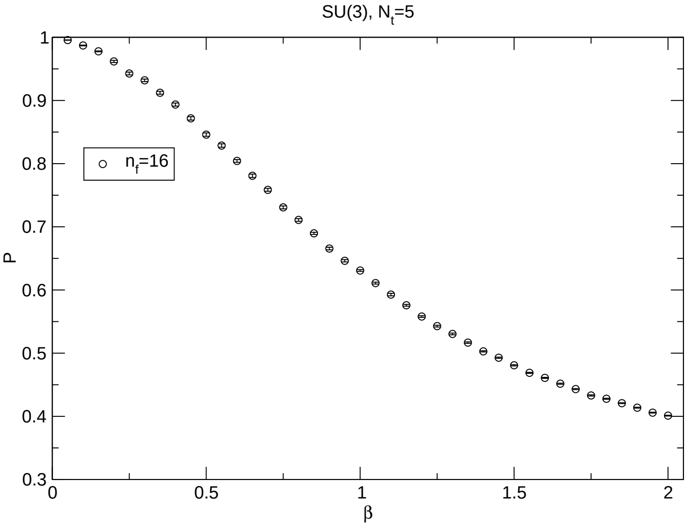
$$\det \mathcal{D}_{n_f}[U, X_i] \geq 0 \quad \text{positive}$$

- Charge conjugation ensures symmetry between sectors:
  - broken by the Wilson discretisation,
  - restored in the continuum.
- Simulations for $N = 3$ with $n_f^{\max} = 2(N^2 - 1) = 16$:
  - SU(3) adjoint $\Rightarrow$ sectors $n_f = \{0, 1, \ldots, 16\}$
- Measure moduli of Polyakov loop and scalar field:

$$P = \left| \mathrm{Tr}_F \left[\prod_t U(t) \right] \right|, \qquad R^2 \equiv |X|^2 = X_i^a X_i^a$$

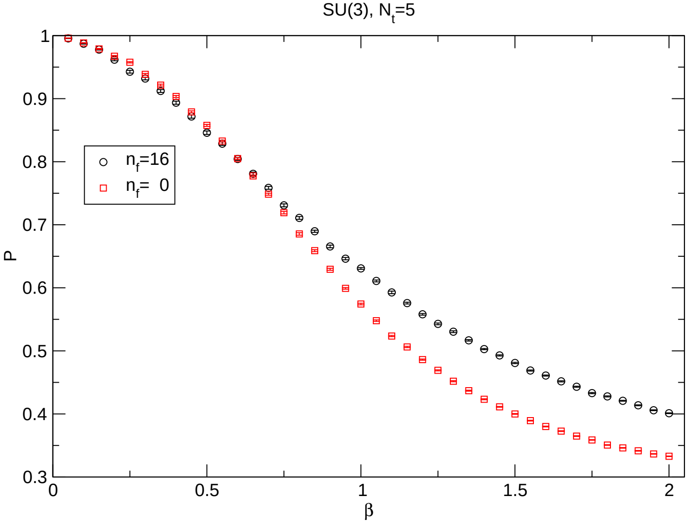
Canonical simulations at fixed n_f

Polyakov loop for $n_f = 16$ (quenched):



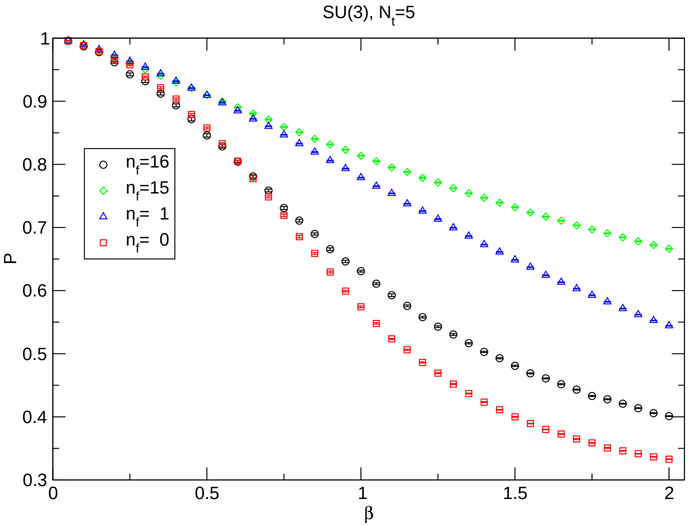
Canonical simulations at fixed n_f

Polyakov loop for $n_f = 16$ and $n_f = 0$:



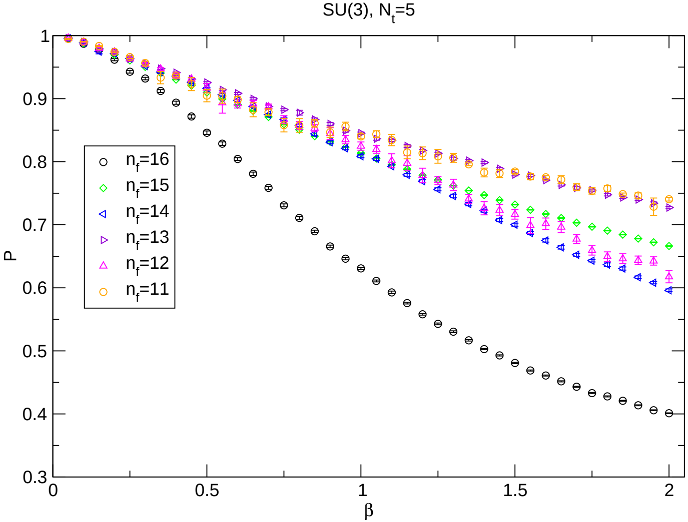
Canonical simulations at fixed n_f

Polyakov loop for $n_f = 16, 15$ and $n_f = 0, 1$:



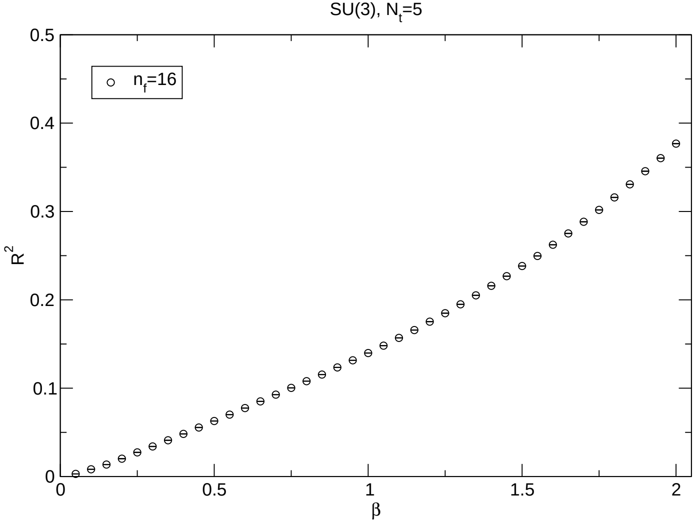
Canonical simulations at fixed n_f

Polyakov loop for $n_f = 16$ to $n_f = 11$:



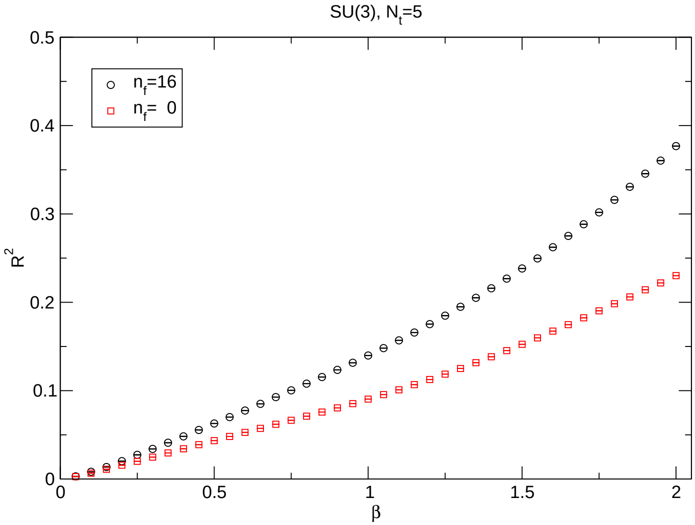
Canonical simulations at fixed n_f

Moduli of X for $n_f = 16$ (quenched):



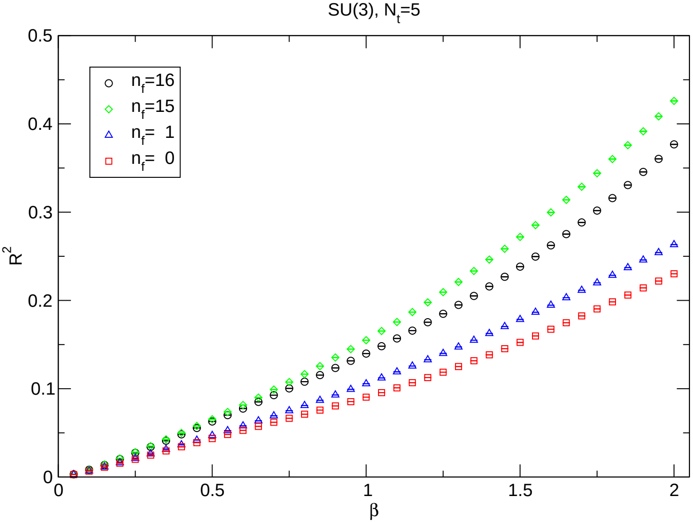
Canonical simulations at fixed n_f

Moduli of X for $n_f = 16$ and $n_f = 0$:



Canonical simulations at fixed n_f

Moduli of X for $n_f = 16, 15$ and $n_f = 0, 1$:



Canonical simulations at fixed n_f

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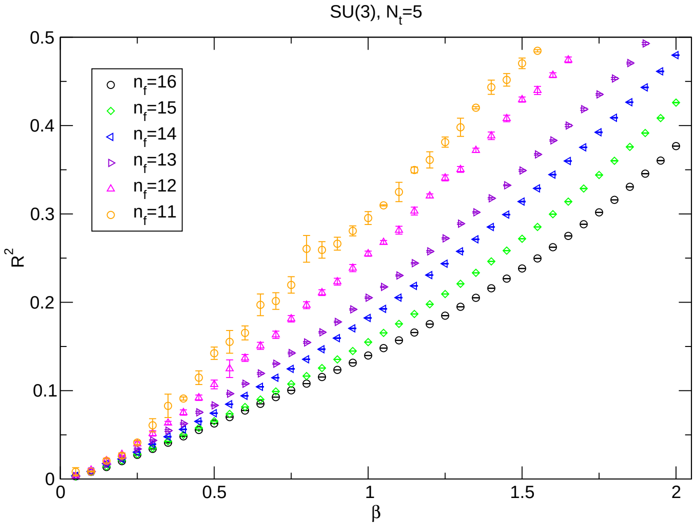
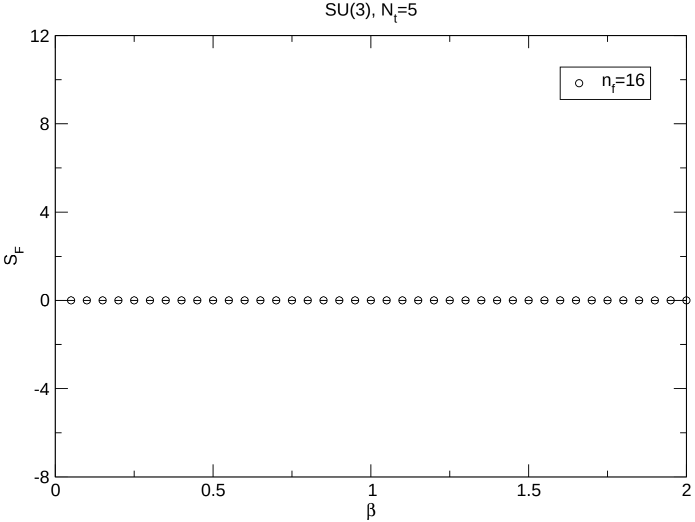


Canonical simulations at fixed n_f

Fermion action $S_F = \langle \ln \det \mathcal{D}_{n_f} \rangle_{n_f}$ for $n_f = 16$ (quenched):



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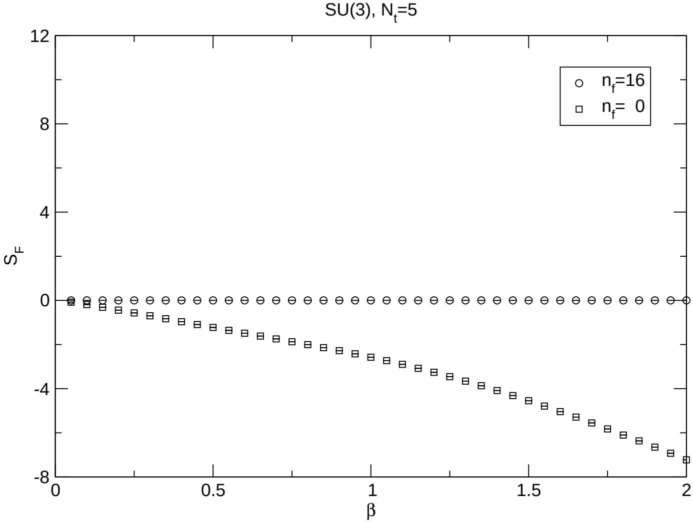
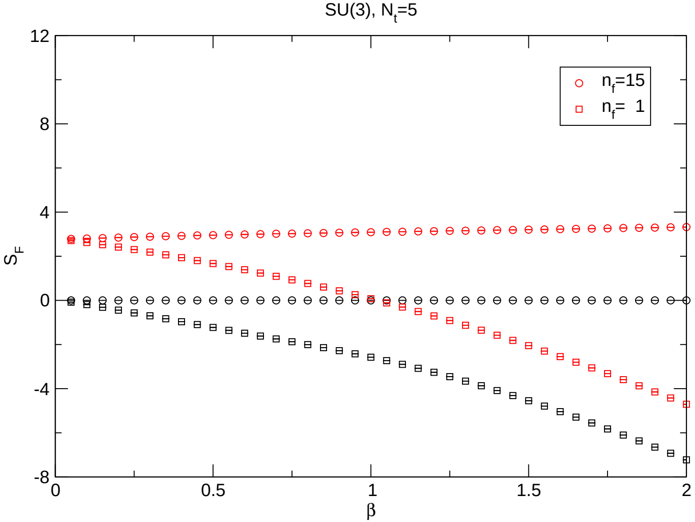
Fermion action $S_F = \langle \ln \det \mathcal{D}_{n_f} \rangle_{n_f}$ for $n_f = 16$ and $n_f = 0$:



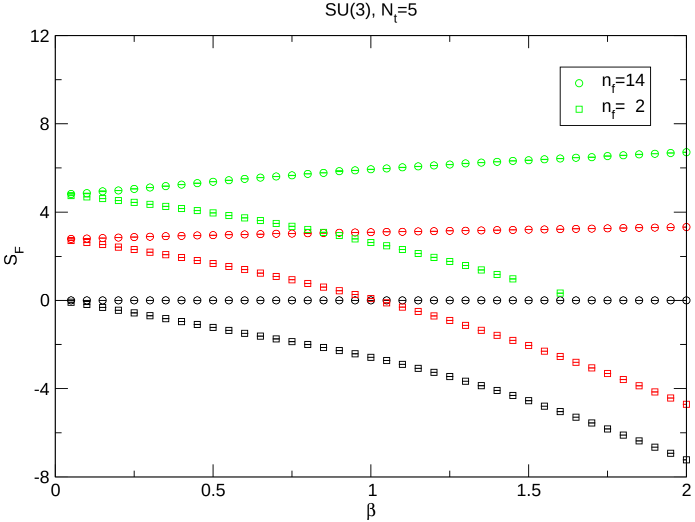
Canonical simulations at fixed n_f

Fermion action $S_F = \langle \ln \det \mathcal{D}_{n_f} \rangle_{n_f}$ for $n_f = 15$ and $n_f = 1$:



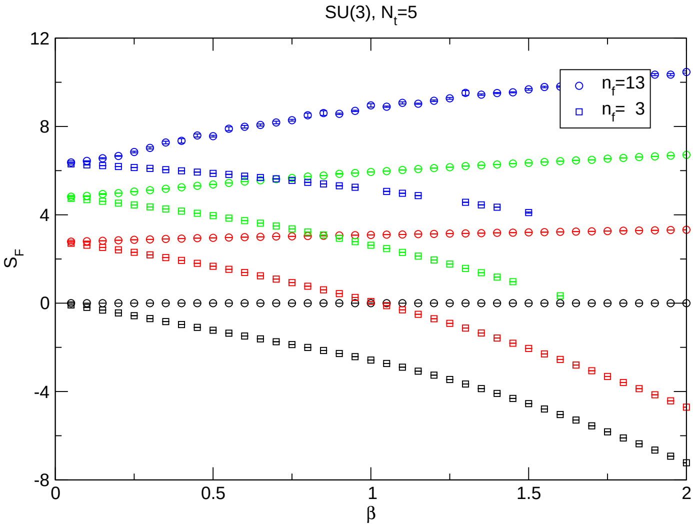
Canonical simulations at fixed n_f

Fermion action $S_F = \langle \ln \det \mathcal{D}_{n_f} \rangle_{n_f}$ for $n_f = 14$ and $n_f = 2$:



Canonical simulations at fixed n_f

Fermion action $S_F = \langle \ln \det \mathcal{D}_{n_f} \rangle_{n_f}$ for $n_f = 13$ and $n_f = 3$:



Summary and Outlook

- Canonical determinants are directly given by transfer matrices

$$\det \mathcal{D}_{n_f}[U, X_i] = \mathrm{Tr}\left[\prod_{t=0}^{L_t-1} T^{\Phi}_{n_f}(t) \cdot T^{W}_{n_f}(t)\right] = \sum_{B} \det \mathcal{T}^{\not{B}\not{B}}$$

  constructed from reduced matrix

$$\mathcal{T} \equiv \prod_{t=0}^{L_t-1} \Phi(t) W(t) \,.$$

- Opens the way to investigate:
  - correlation functions, spectra, phase transition,...
  - large-N limit,
  - $\mathcal{N} = 16$ SYM QM and black hole thermodynamics,
  - reweighting, finite density,...