Applications of the Feynman–Hellmann theorem in hadron structure

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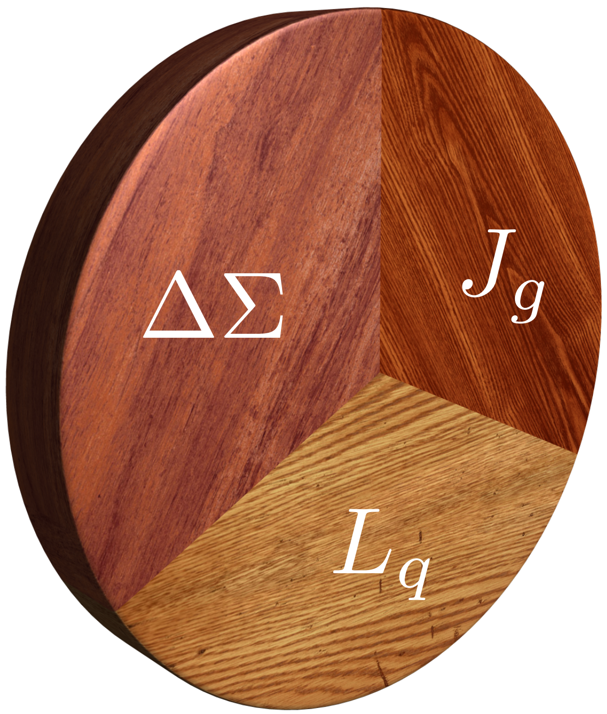
Physics Goals

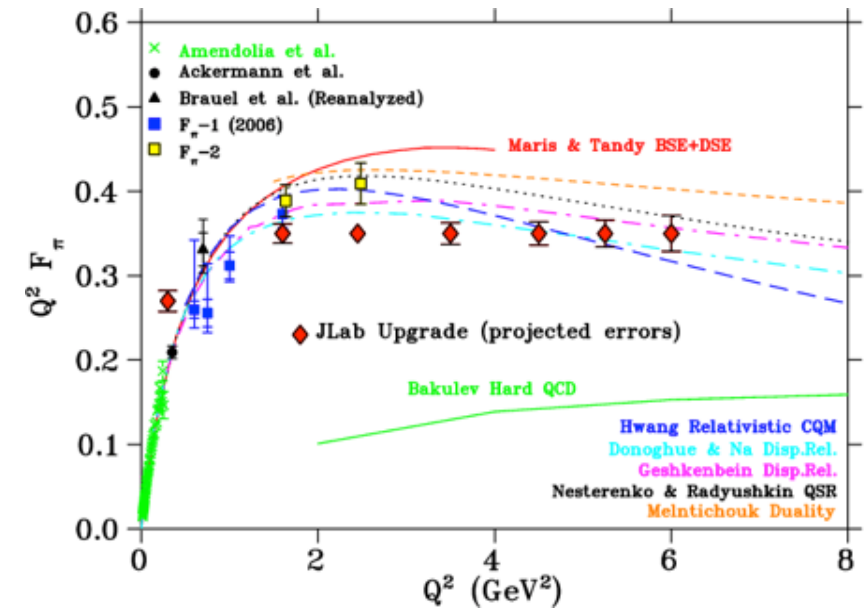
Hadron Spin



What is the nucleon spin decomposition?

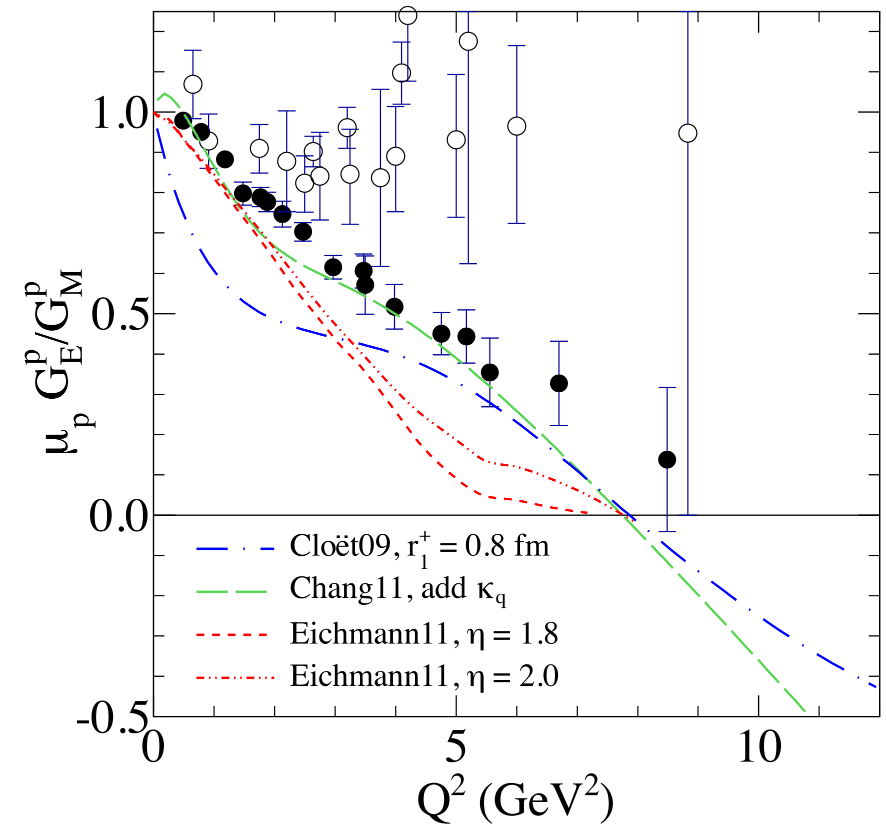
Is quark spin suppression universal among hadrons?

Pion Form Factor



How does the pion form factor transition to perturbative QCD?

Nucleon Form Factor



Is there a “zero-crossing” in the proton electric form factor?

Feynman–Hellmann Theorem

- A method for determining hadronic matrix elements from energy shifts
- Suppose we want $\langle H|\mathcal{O}|H\rangle$
- Proceed by $S \to S + \lambda \int \mathrm{d}^4x\, \mathcal{O}(x)$

  real parameter (λ)

  local operator, e.g. $\bar{q}(x)\gamma_5\gamma_3 q(x)$

- FH tells us $$\frac{\partial E_H(\lambda)}{\partial\lambda} = \frac{1}{2E_H(\lambda)}\left\langle H\left|\frac{\partial S(\lambda)}{\partial\lambda}\right|H\right\rangle$$

$$\Rightarrow \quad \frac{\partial E_H(\lambda)}{\partial\lambda} = \frac{1}{2E_H(\lambda)}\langle H|\mathcal{O}|H\rangle$$

  - Calculation of matrix element $\equiv$ hadron spectroscopy

Feynman–Hellmann Theorem

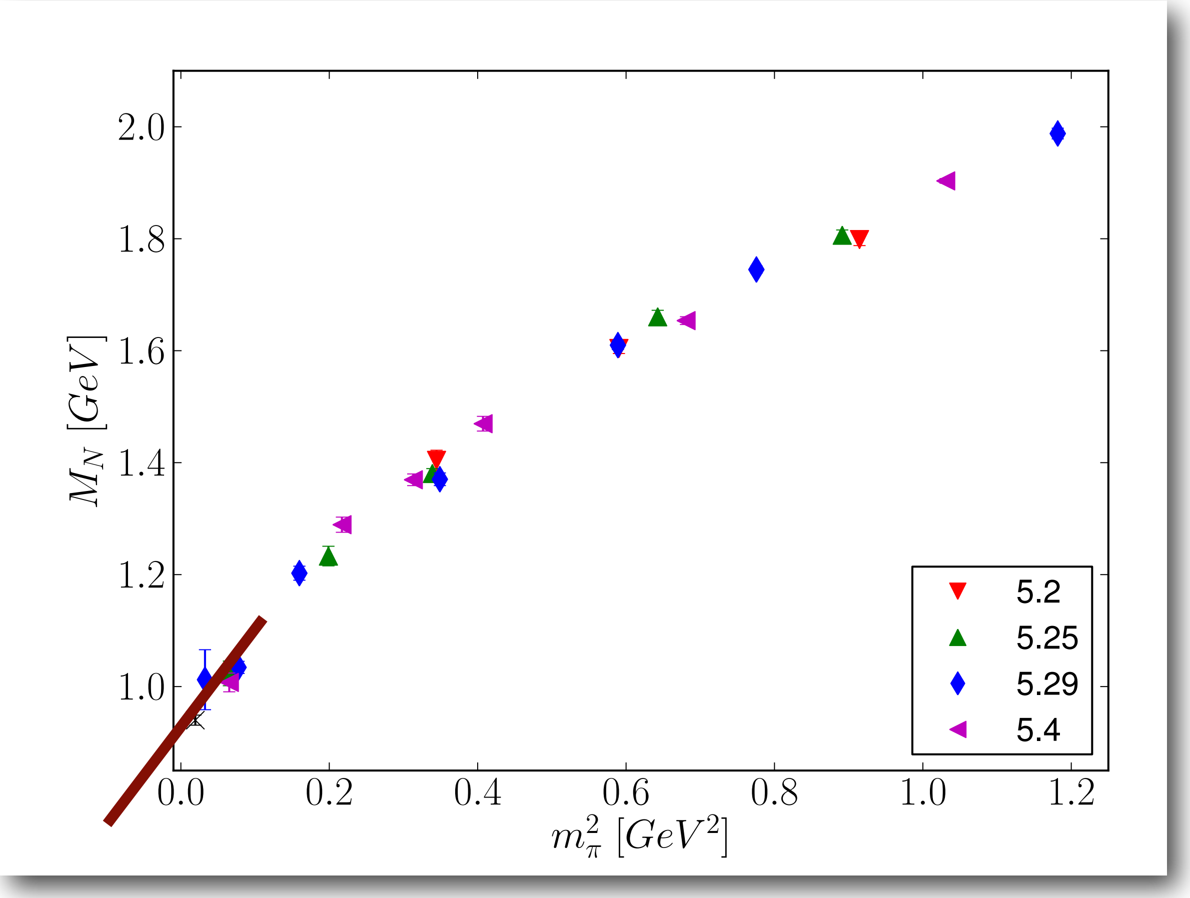
- Most commonly used to determine terms since

$$\sigma_l^H = m_l \langle H | (\bar{u}u + \bar{d}d) | H \rangle \qquad \sigma_s^H = m_s \langle H | \bar{s}s | H \rangle$$

- and: $$S = \sum_q \left[m_q \bar{q}q + \bar{q}Dq \right]$$

plays the role of λ

$$\sigma_{\pi N} \approx m_\pi^2 \frac{dm_N}{dm_\pi^2}\bigg|_{m_\pi = m_\pi^{\text{phys}}}$$



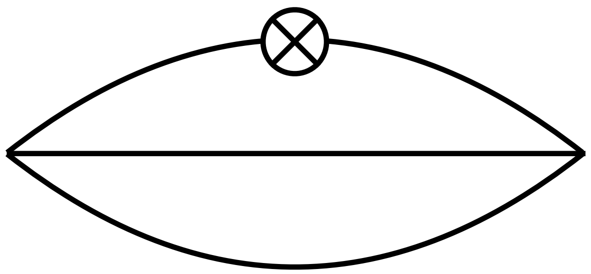
Feynman–Hellmann: Hadron spin

- To access hadron spin fractions, we modify the action to include the axial current
$$S \to S + \lambda \sum_x \bar{q}(x) i\gamma_5\gamma_3 q(x)$$
- FH Theorem then gives
$$\left.\frac{\partial E_H(\lambda)}{\partial \lambda}\right|_{\lambda=0} = \frac{1}{2M_H}\langle H|\bar{q}i\gamma_5\gamma_3 q|H\rangle$$
- but for a spin-*J* hadron with polarisation *m* in the *z*-direction
$$\langle H, Jm|\bar{q}i\gamma_5\gamma_3 q|H, Jm\rangle = 2M_H \Delta q^{Jm}$$
$$\Rightarrow \quad \Delta q = \left.\frac{\partial E_H(\lambda)}{\partial \lambda}\right|_{\lambda=0}$$
- Also note: reversing hadron polarisation $\equiv$ changing sign of λ

Lattice Jargon

- Nf =2+1 O(a)-improved Clover fermions (“SLiNC” action)
  - Tree-level Symanzik gluon action (plaq. + rect.)
- Results from a single lattice spacing (a~0.074fm), and volume (32^3 x 64)
- Most results are at the SU(3)-symmetric point (m_{pi}~470 MeV)
  - Total spin contribution (also m_{pi}~330 MeV)
- ~500 measurements per mass per field strength

Connected Spin Contributions

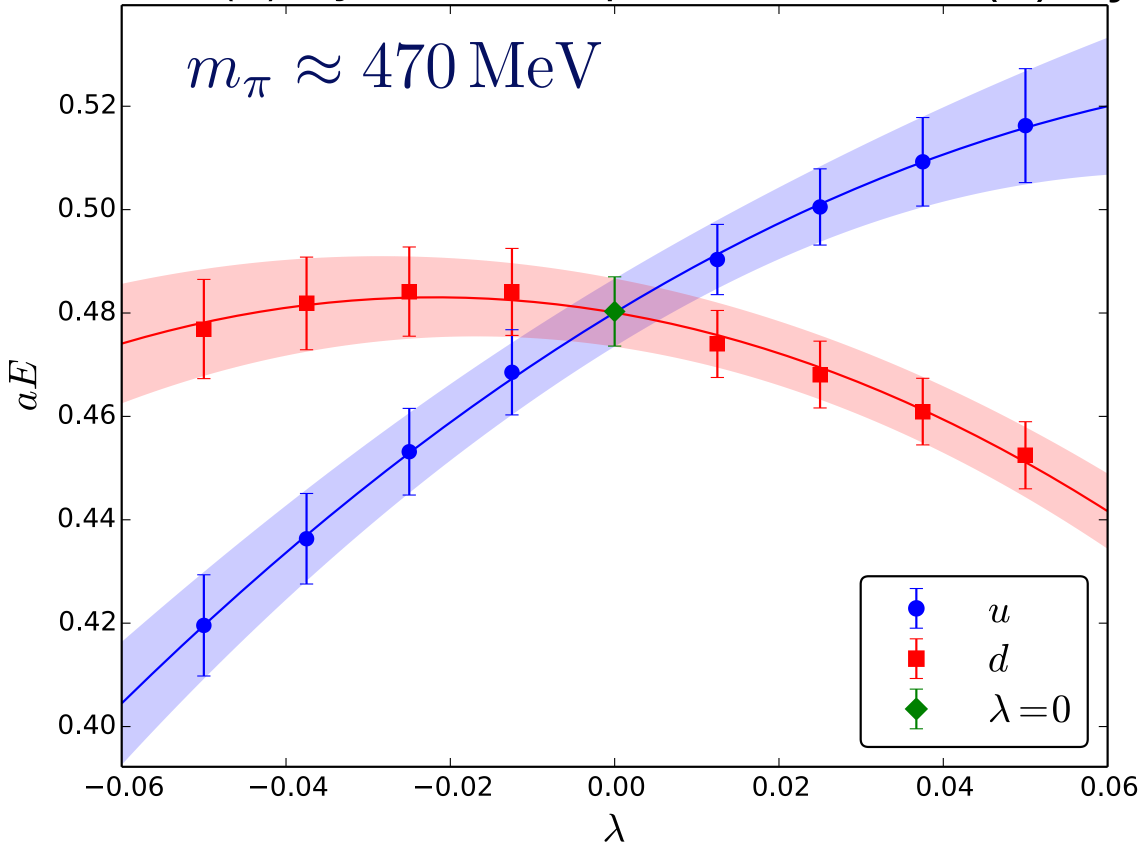


- Use existing N_f=2+1 configurations
- Modify the action of the valence quarks only
- Allows for comparison with results using standard 3-point function methods
- For more details see: A. Chambers *et al*. [CSSM/QCDSF/UKQCD] PRD(2014)

Connected Spin Contributions

- Start with nucleon mass vs. field strength λ

SU(3) symmetric point SU(3) symmetric point,



Fit: quadratic in λ

➡ linear terms give Δu and Δd

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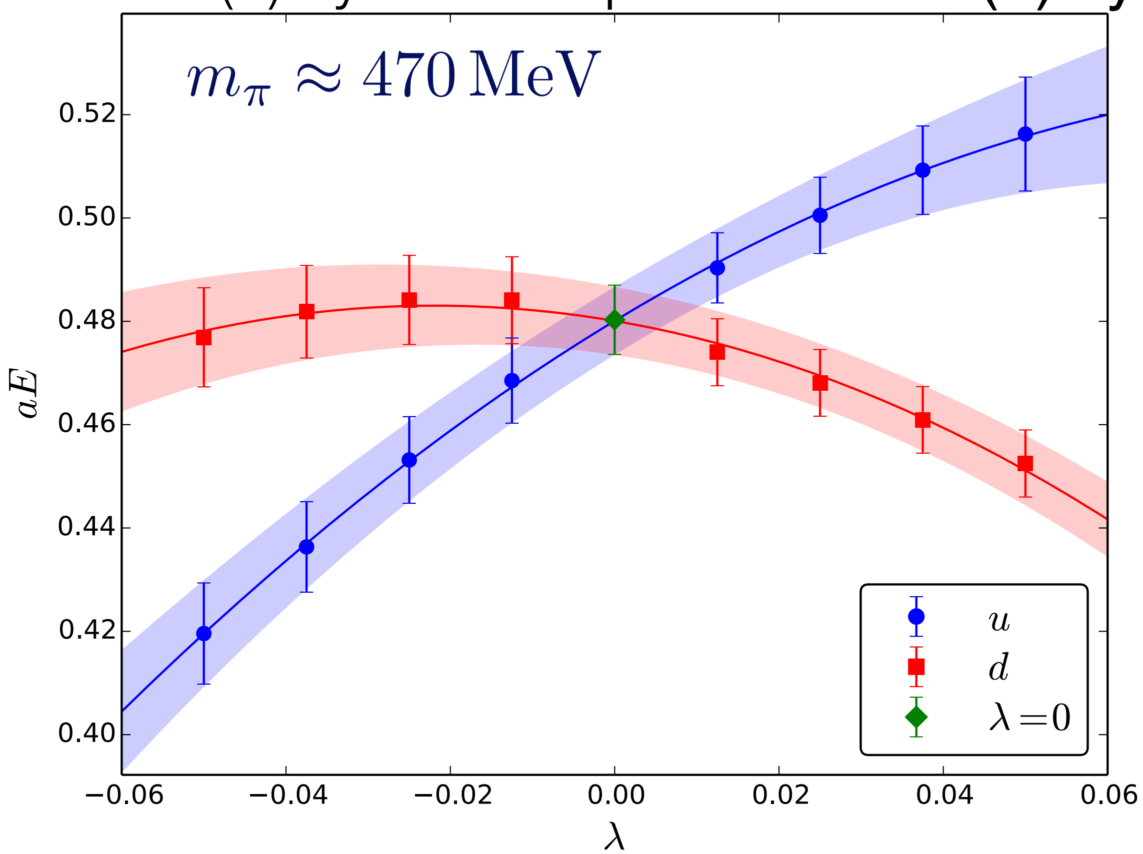
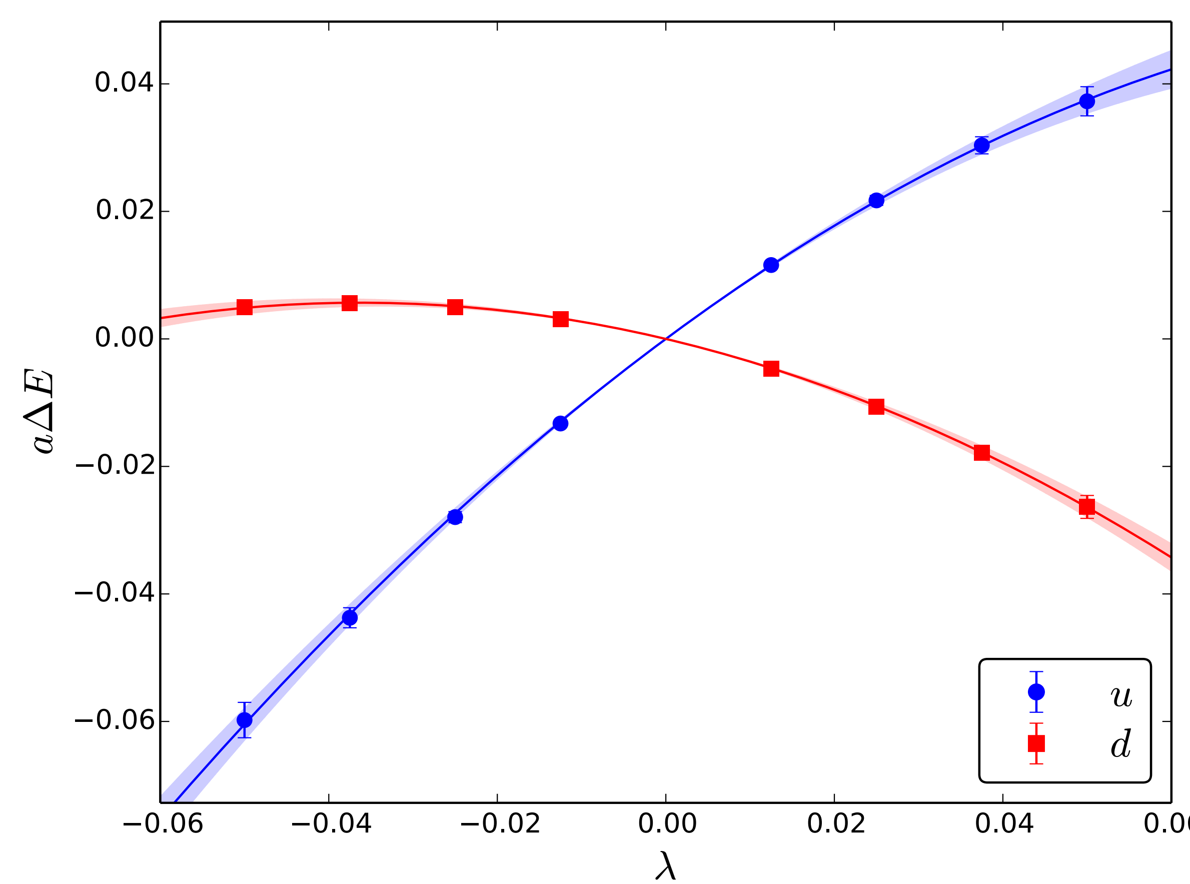


Fit energy differences



Fit: quadratic in λ

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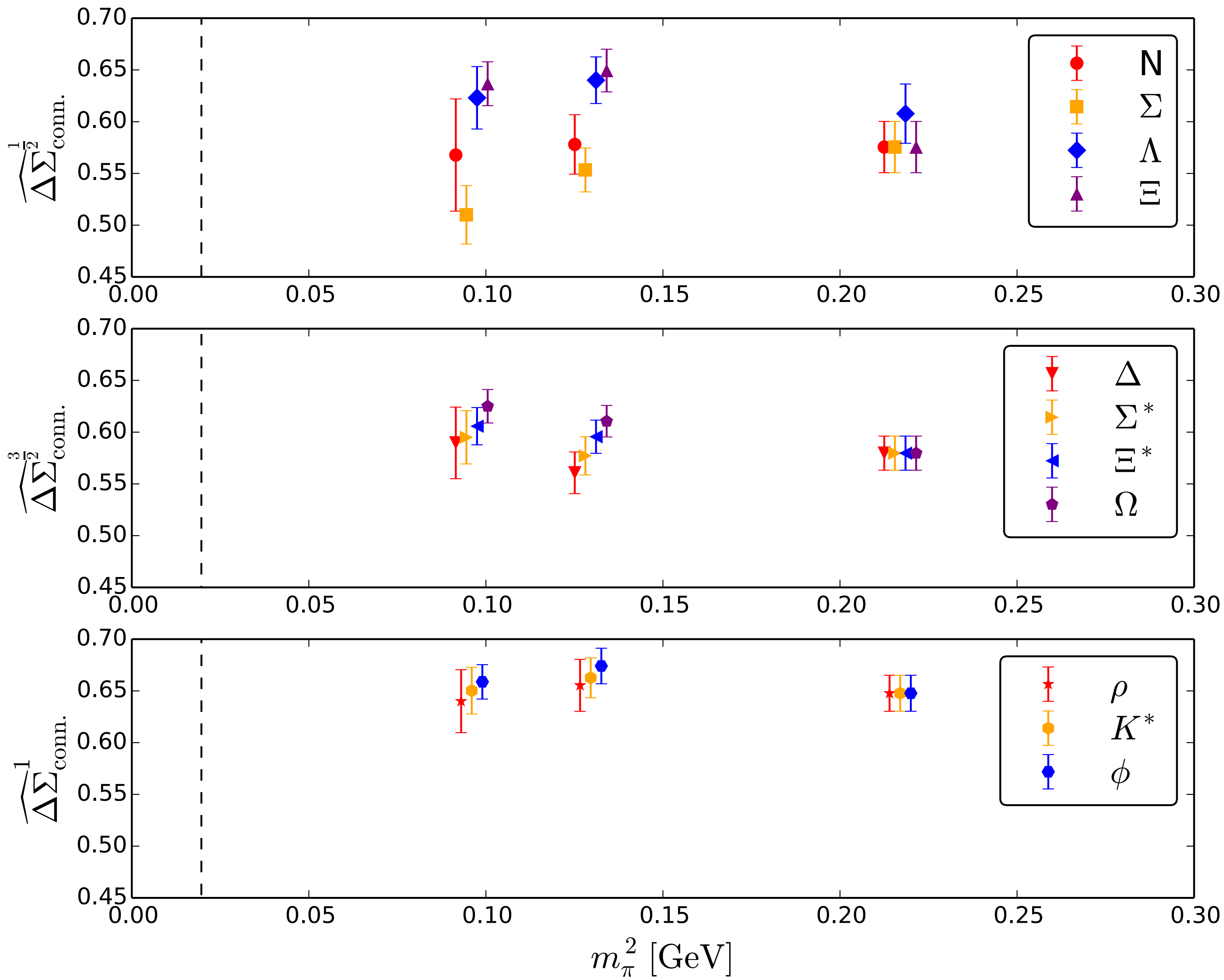
Connected Spin Contributions

- Connected spin factions in various hadrons



(Connected) Spin Fraction Universal ~60%

Disconnected Spin Contributions

- Include operator in HMC

- For Hermitian spin operator, the Fermion matrix is modified by

$$M \to M(\lambda) = M_0 + \lambda\, i\gamma_5\gamma_3$$

  - Does not satisfy γ_5 Hermiticity ⇒ sign problem

- Hence we simulate with γ_5 Hermitian operator

$$M \to M(\lambda) = M_0 + \lambda\, \gamma_5\gamma_3$$

- Correlation function picks up complex phase

$$C(\lambda, t) \xrightarrow{\text{large } t} A(\lambda) e^{-E(\lambda)t} e^{i\phi(\lambda)t}$$

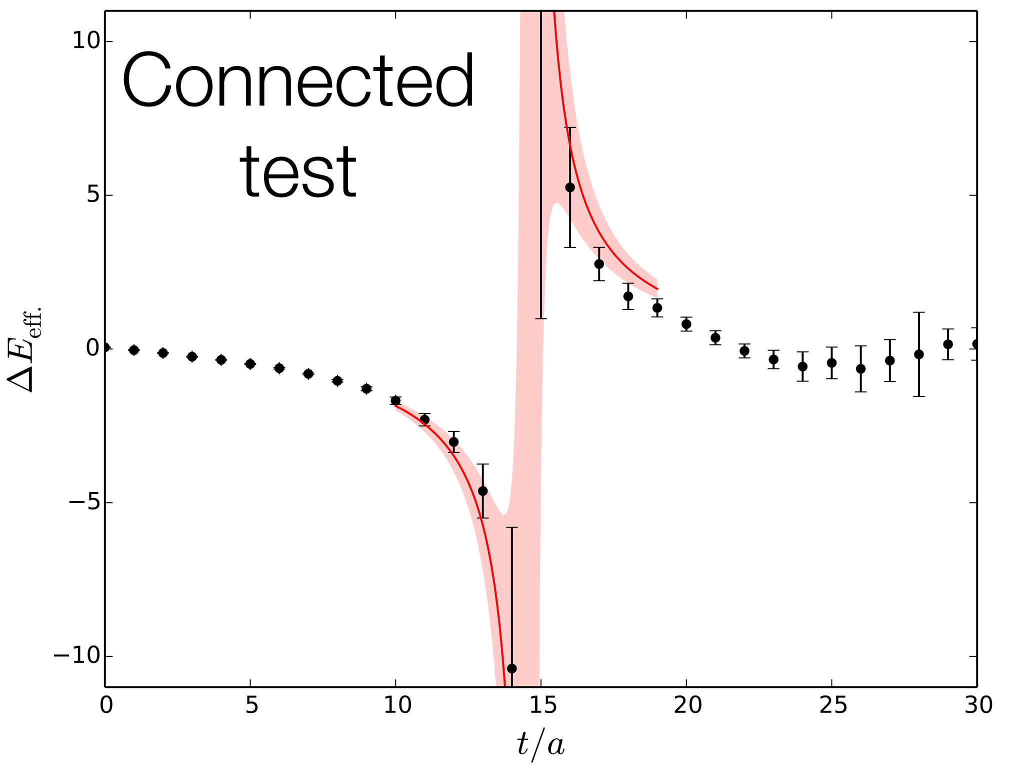
  - Extract matrix element from phase

$$\phi(\lambda) = \lambda \Delta q + \mathcal{O}(\lambda^3)$$

Disconnected Spin Contributions

- Isolate complex phase:
  - "Imaginary spin difference" / "Real spin average"

$$\frac{\mathcal{I}\Big[C_+(\lambda,t)\Big] - \mathcal{I}\Big[C_-(\lambda,t)\Big]}{\mathcal{R}\Big[C_+(\lambda,t)\Big] + \mathcal{R}\Big[C_-(\lambda,t)\Big]} \xrightarrow{\text{large } t} \tan(\phi t)$$



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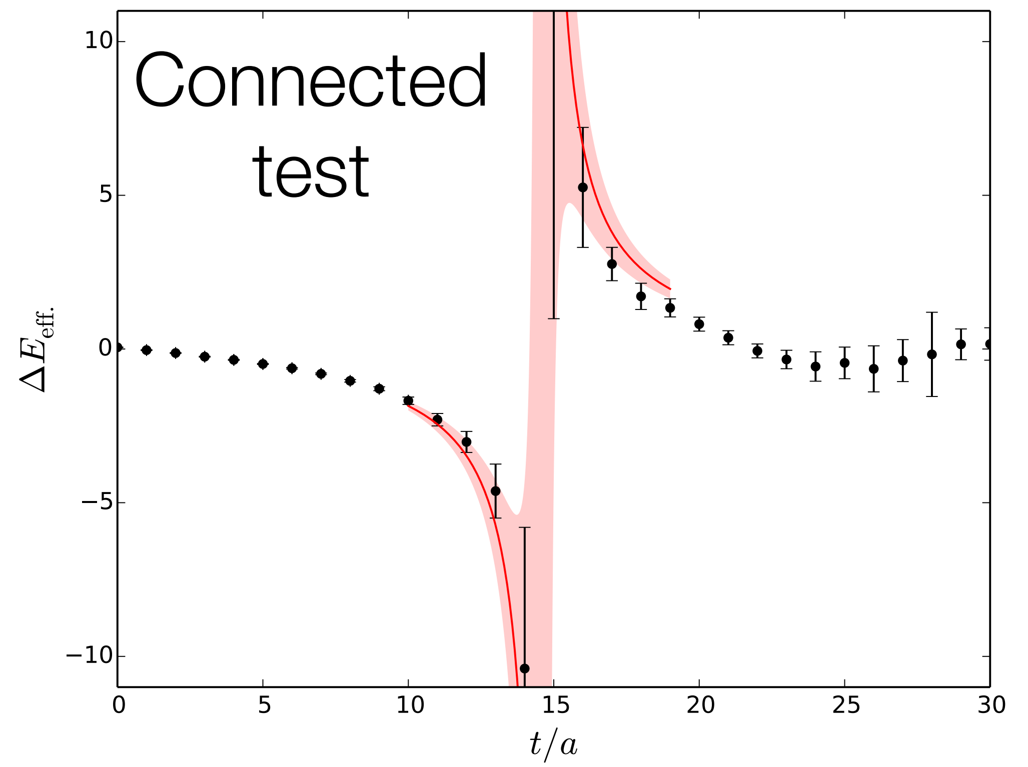
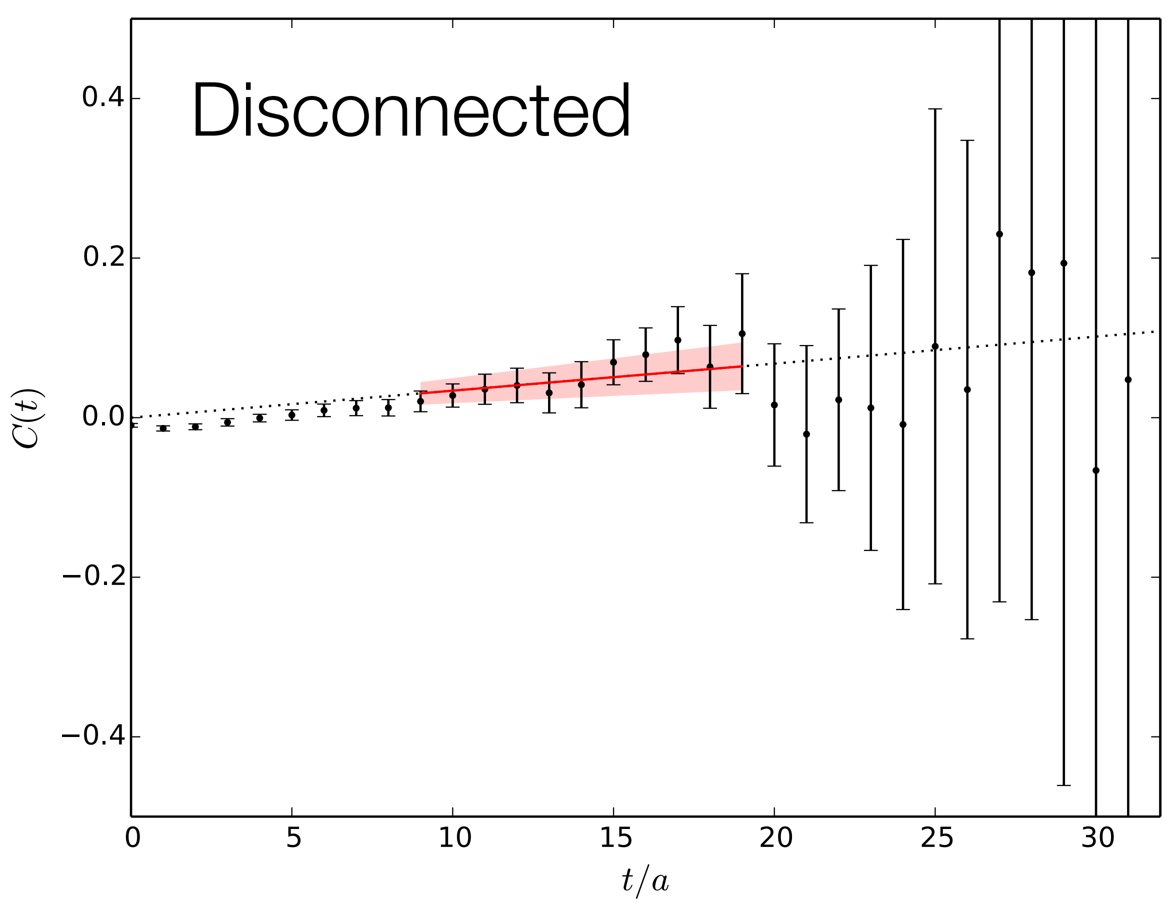
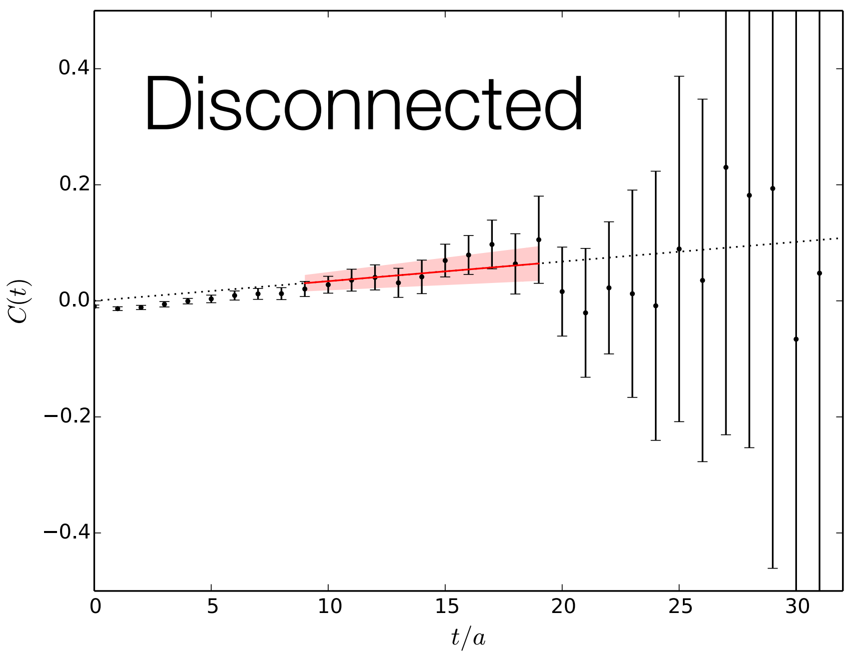




Disconnected Spin Contributions

- Difficult to distinguish tangent behaviour from excited states



$$\mathcal{H}(t) = \frac{\mathfrak{Im}\left[C_+(t) - C_-(t)\right]}{\mathfrak{Re}\left[C_+(t) + C_-(t)\right]} \to \tan(\phi t)$$

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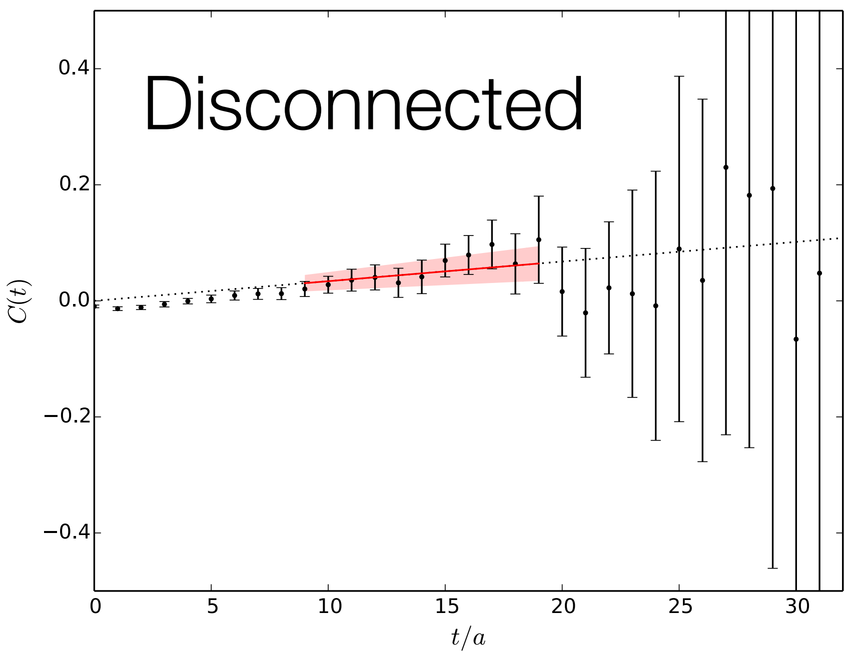

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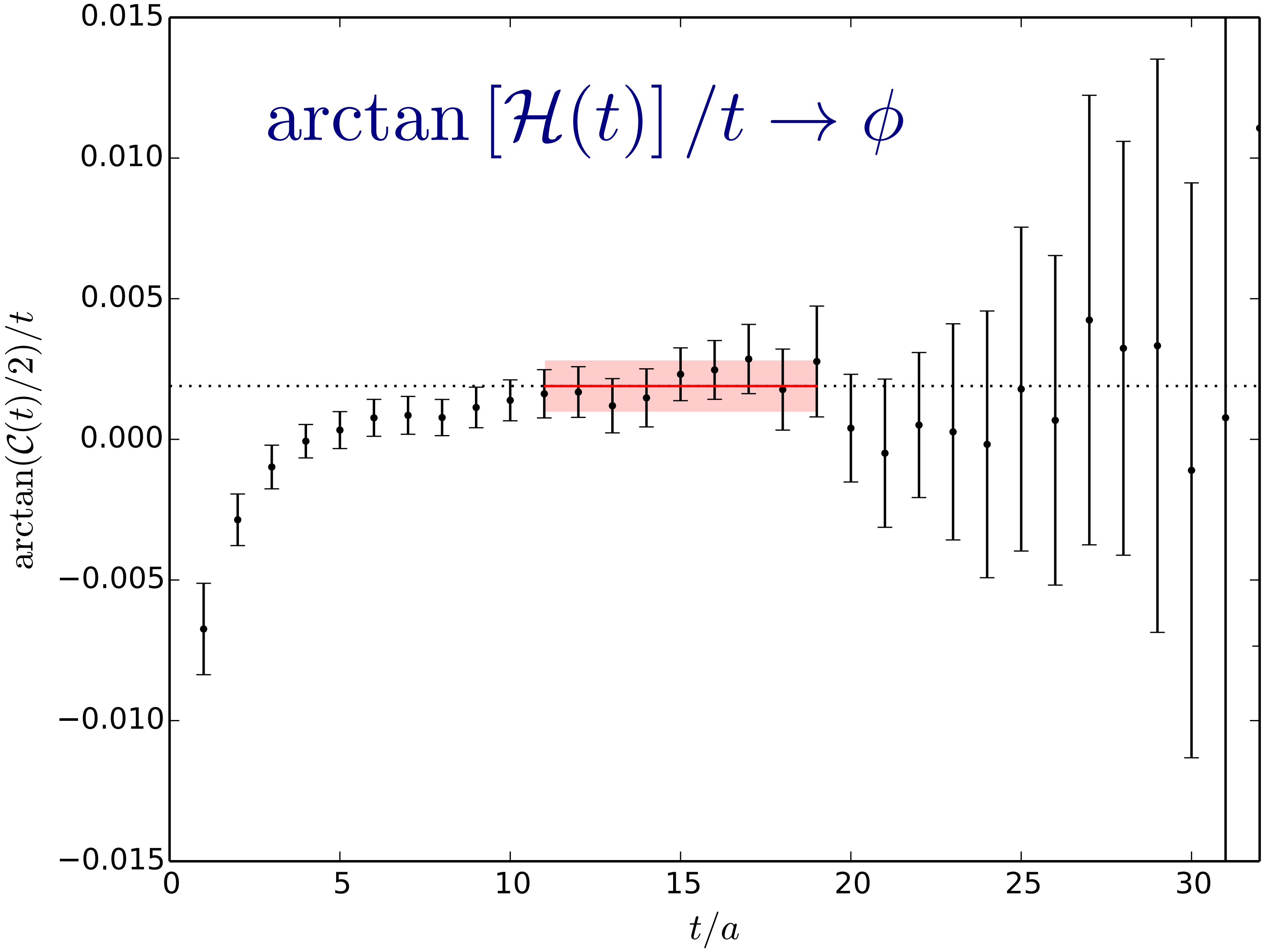
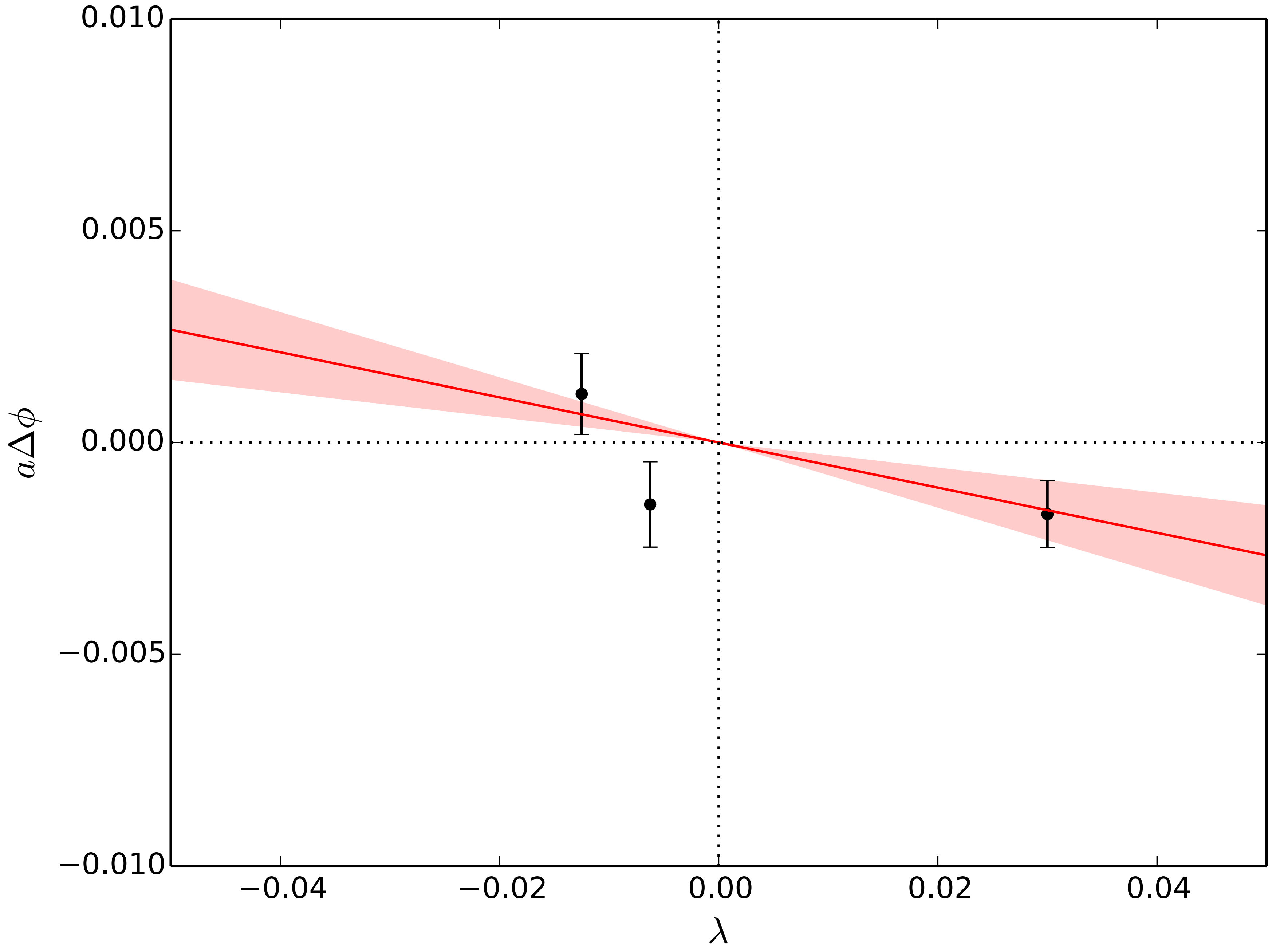
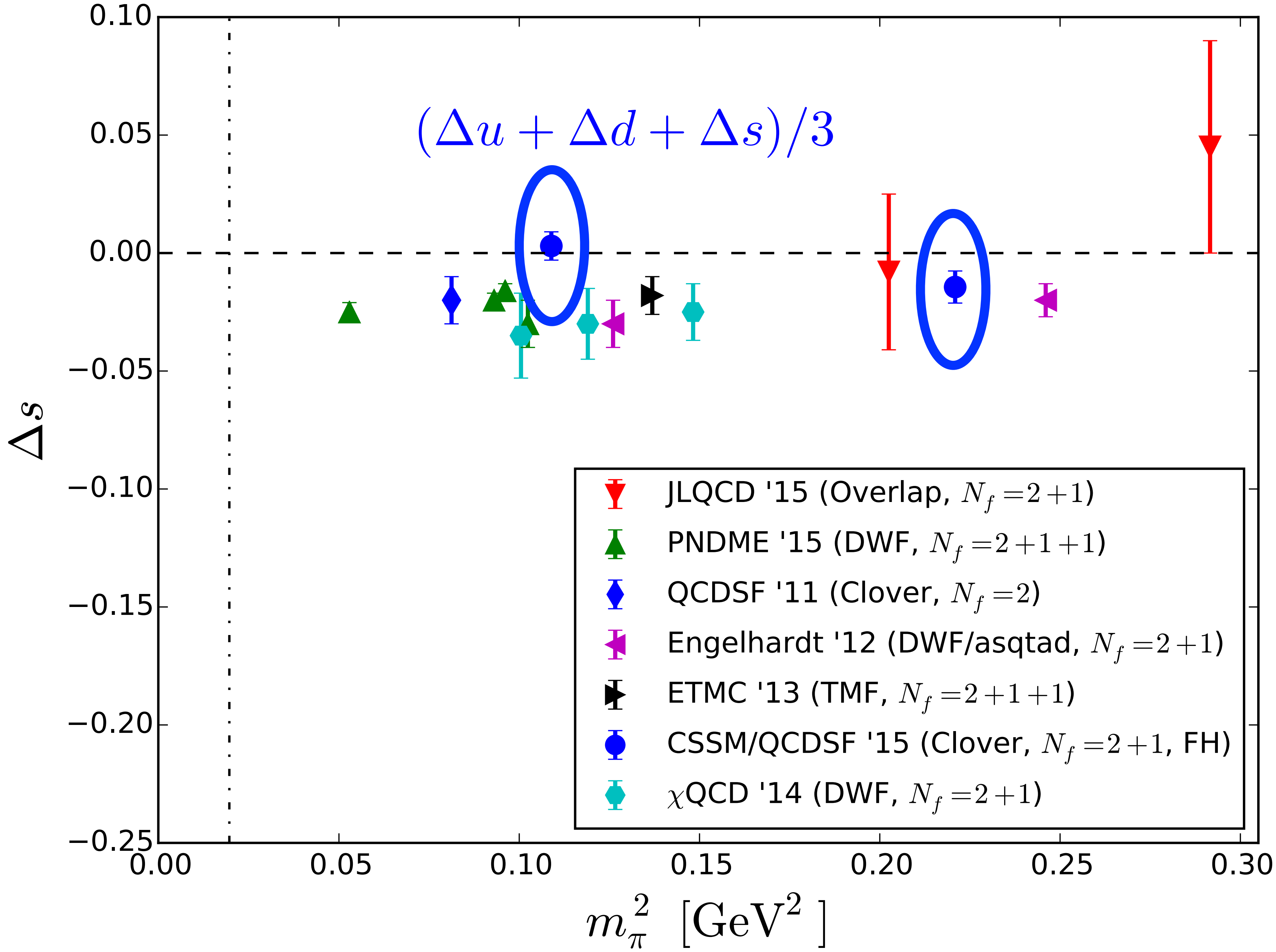


Disconnected Spin Contributions

- SU(3) symmetric point; 3 field strengths





Strangeness spin

Global comparison

Non-forward matrix elements:
Momentum injection from external field

Feynman-Hellmann: Non-Forward

- Modify Lagrangian with external field containing a spatial Fourier transform
    - eg. vector current $$\mathcal{L}(y) \to \mathcal{L}_0(y) + \lambda e^{i\vec{q}.\vec{y}}\overline{q}(y)\gamma_\mu q(y)$$
- Choose Breit frame Fourier projection of hadron state $E(\vec{p}') = E(\vec{p})$
    - Time dependence is trivial (to leading order in λ): “Energy shift”
- Pion form factor

$$\langle \pi(\vec{p}')|\overline{q}(0)\gamma_\mu q(0)|\pi(\vec{p})\rangle = (p+p')_\mu F_\pi(q^2)$$

- “Feynman-Hellmann”:

$$\left.\frac{\partial E}{\partial \lambda}\right|_{\lambda=0} = \frac{(p+p')_\mu}{2E} F_\pi(q^2)$$

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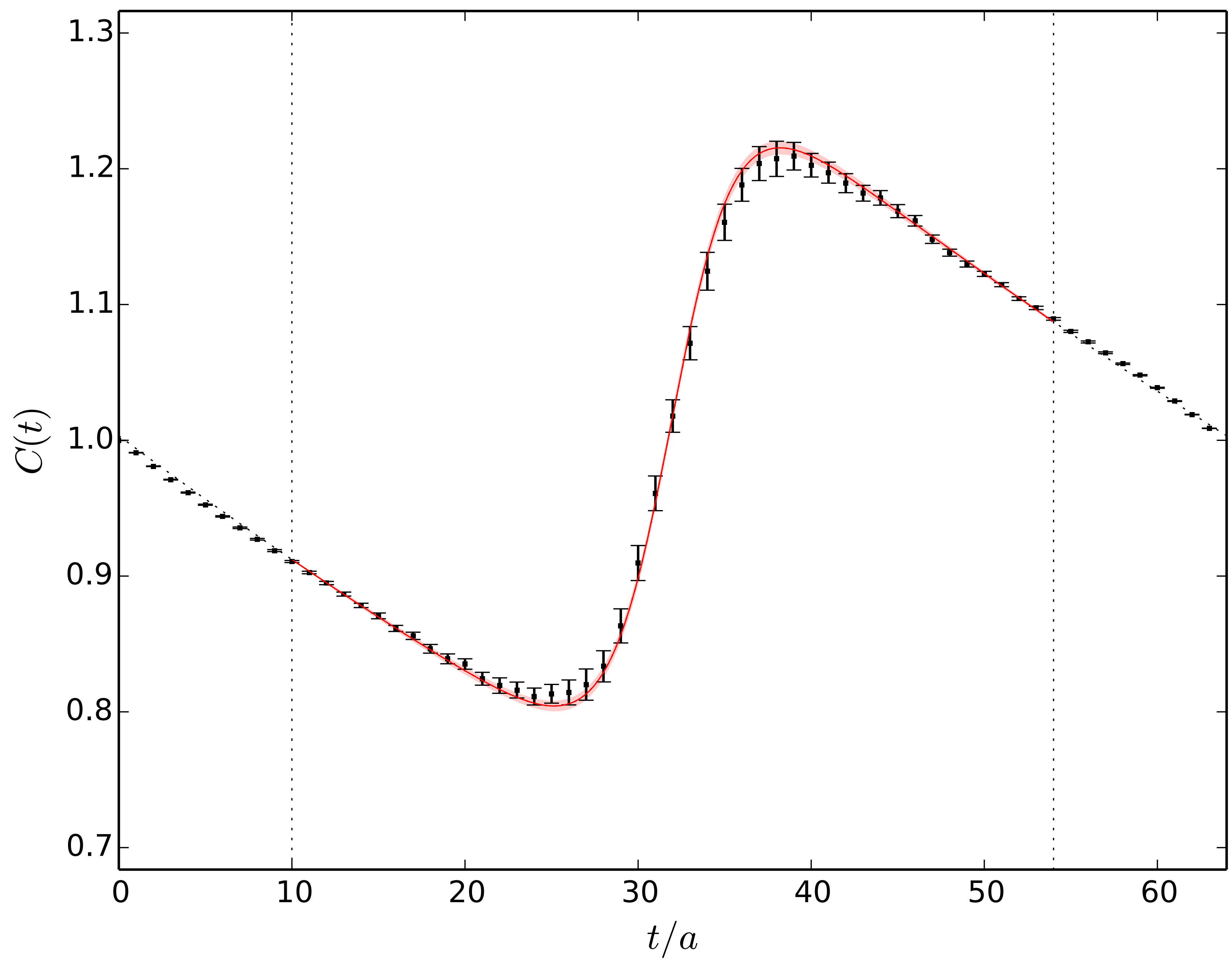
- "Feynman-Hellmann":

$$\left.\frac{\partial E}{\partial \lambda}\right|_{\lambda=0} = \frac{(p+p')_\mu}{2E} F_\pi(q^2) \quad \xrightarrow{\mu=4} \quad \left.\frac{\partial E}{\partial \lambda}\right|_{\lambda=0} = F_\pi(q^2)$$

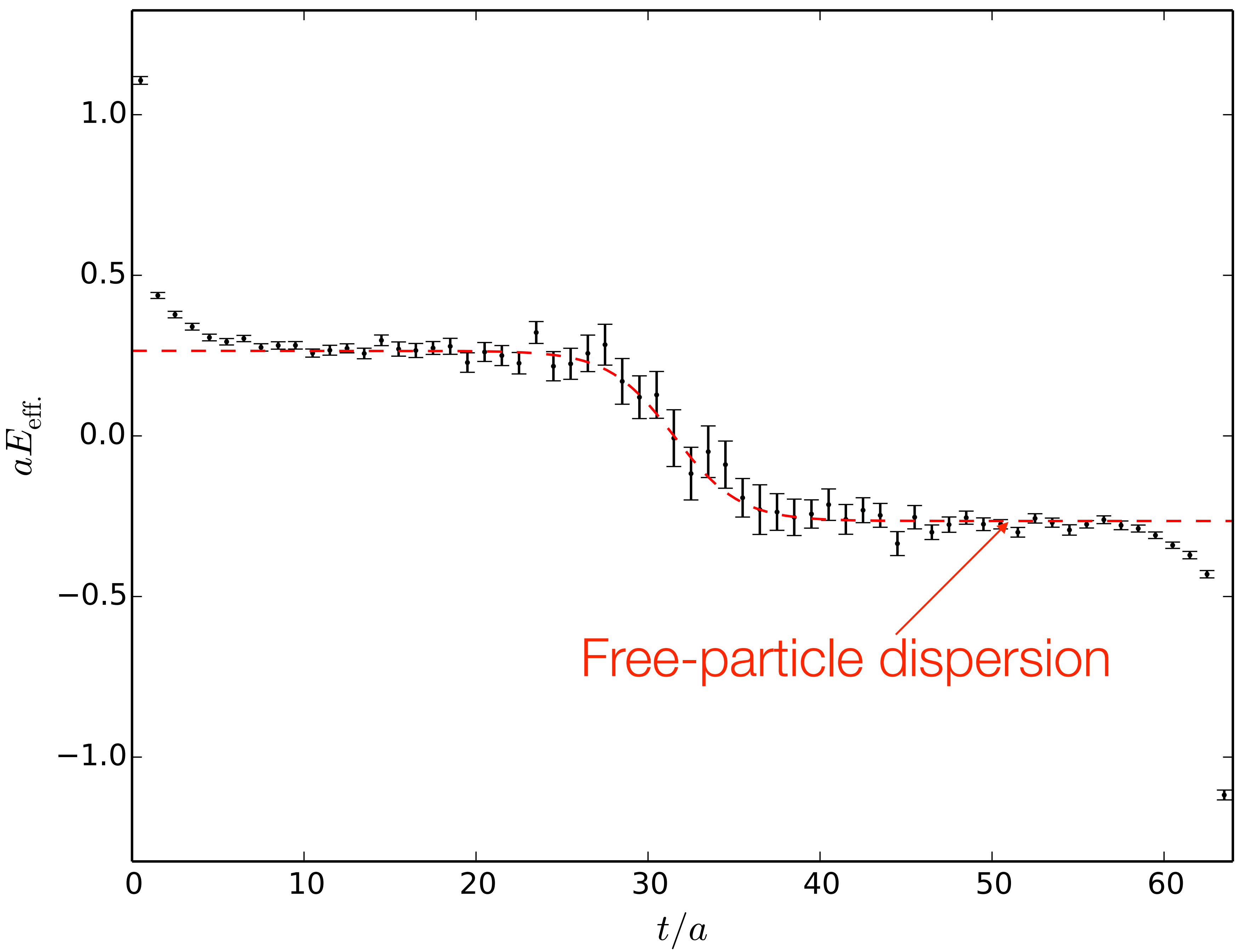
FH Non-forward: Pion

- Ratio of correlators

$$C(t) = \frac{G(\lambda,t)}{G(0,t)} \overset{t\to\infty}{\simeq} \frac{A(\lambda)}{A(0)} \frac{\left[e^{(E+\Delta E)t} + e^{(E-\Delta E)(T-t)}\right]}{e^{-Et} + e^{-E(T-t)}}$$



$$\vec{p} = (-1, 0, 0)$$
$$\vec{q} = (\ \ 1, 1, 0)$$
$$\vec{p}' = (\ \ 0, 1, 0)$$

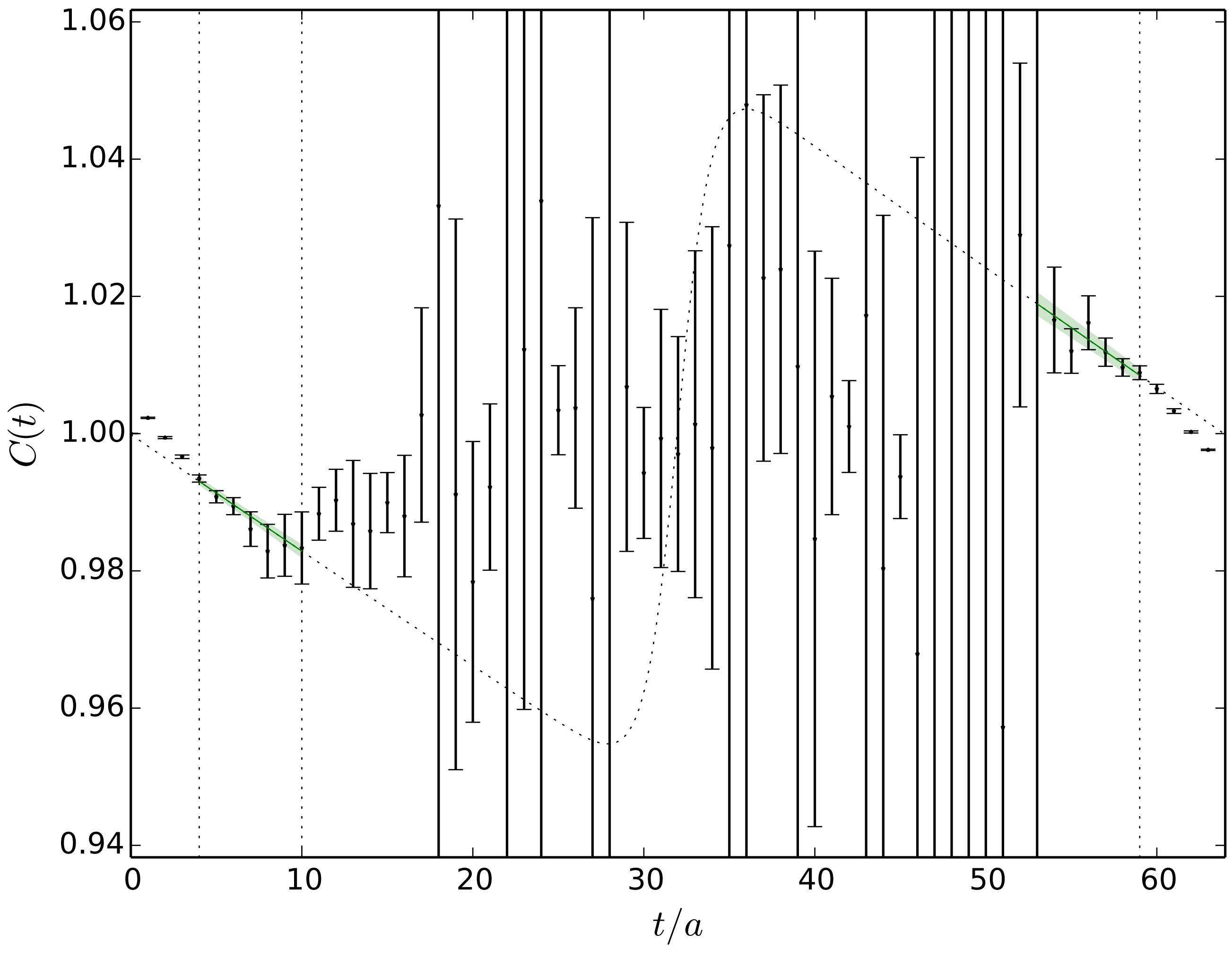


Effective "Mass" plot

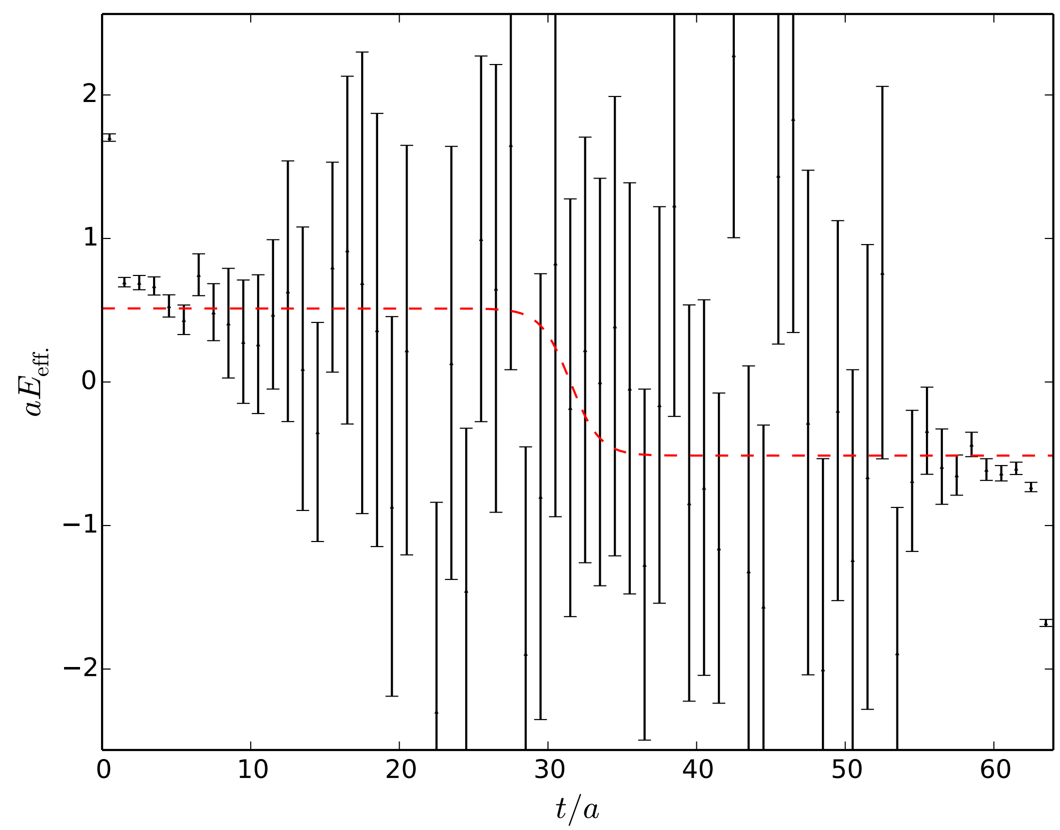
Only determine energy shift where ground state saturated

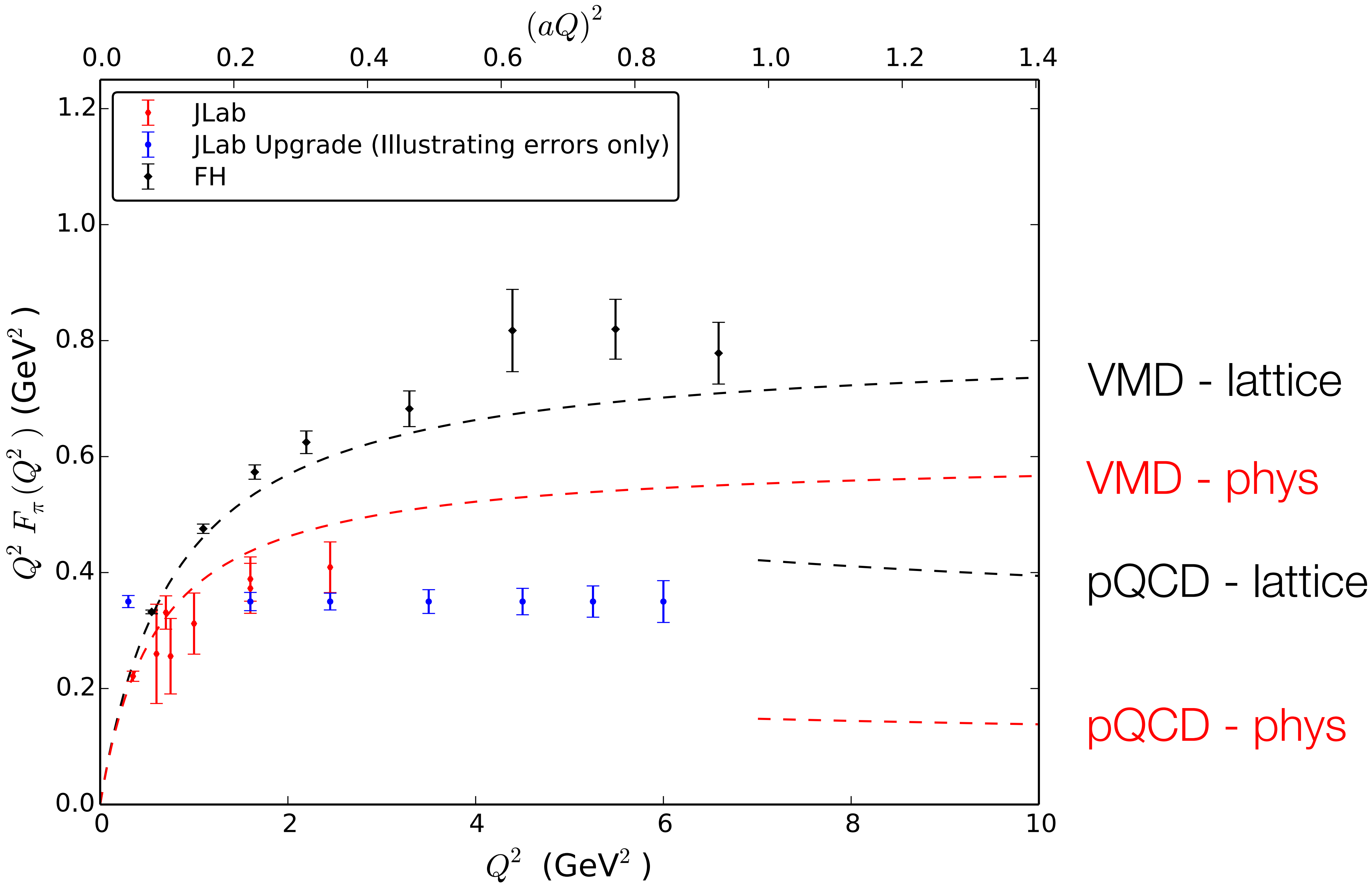
Pushing the limits

- “Extreme” momentum injection



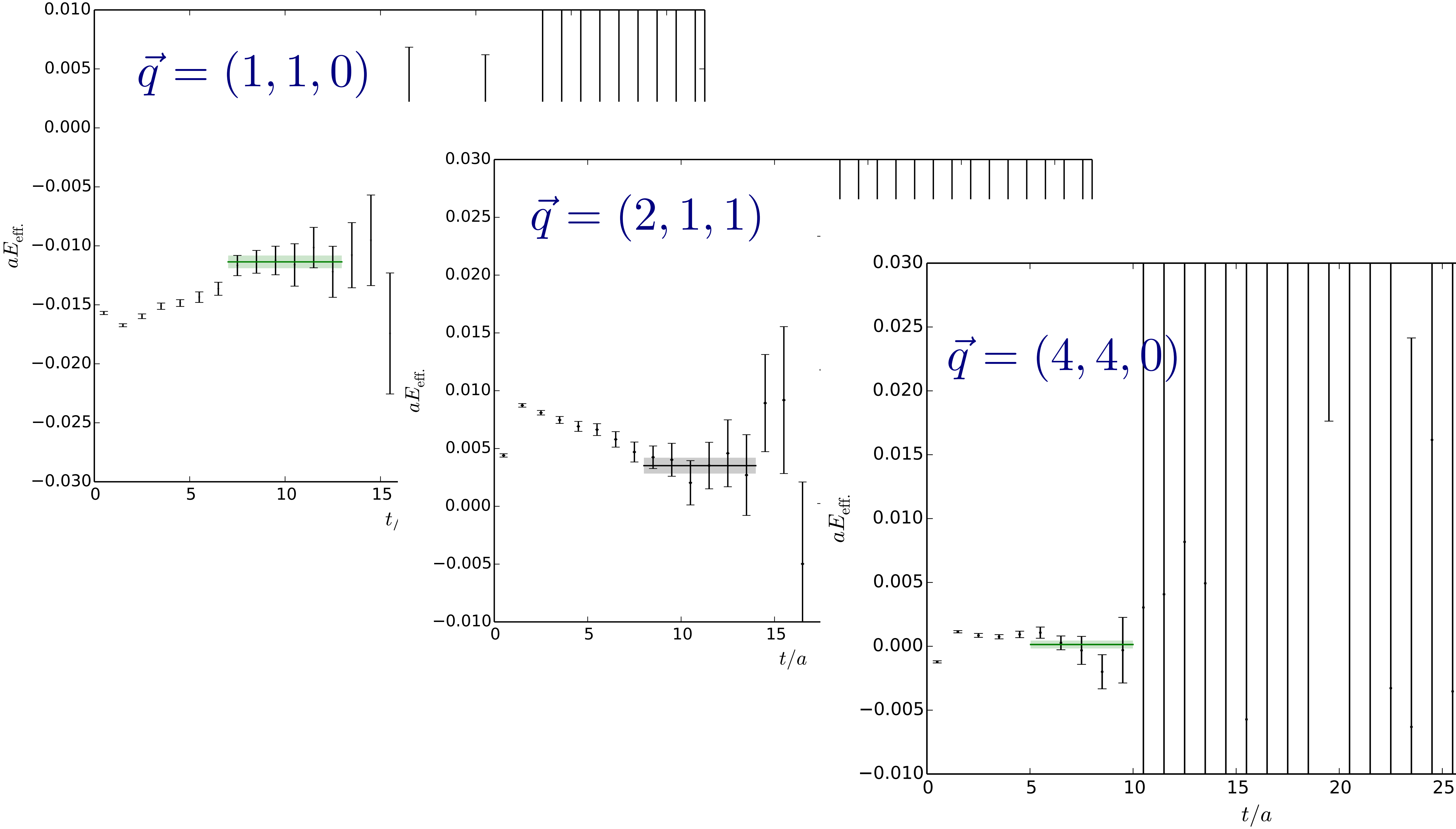
$$\vec{p} = (-2, -1, -1)$$
$$\vec{q} = (\ \ 4,\ \ 2,\ \ 2)$$
$$\vec{p}' = (\ \ 2,\ \ 1,\ \ 1)$$





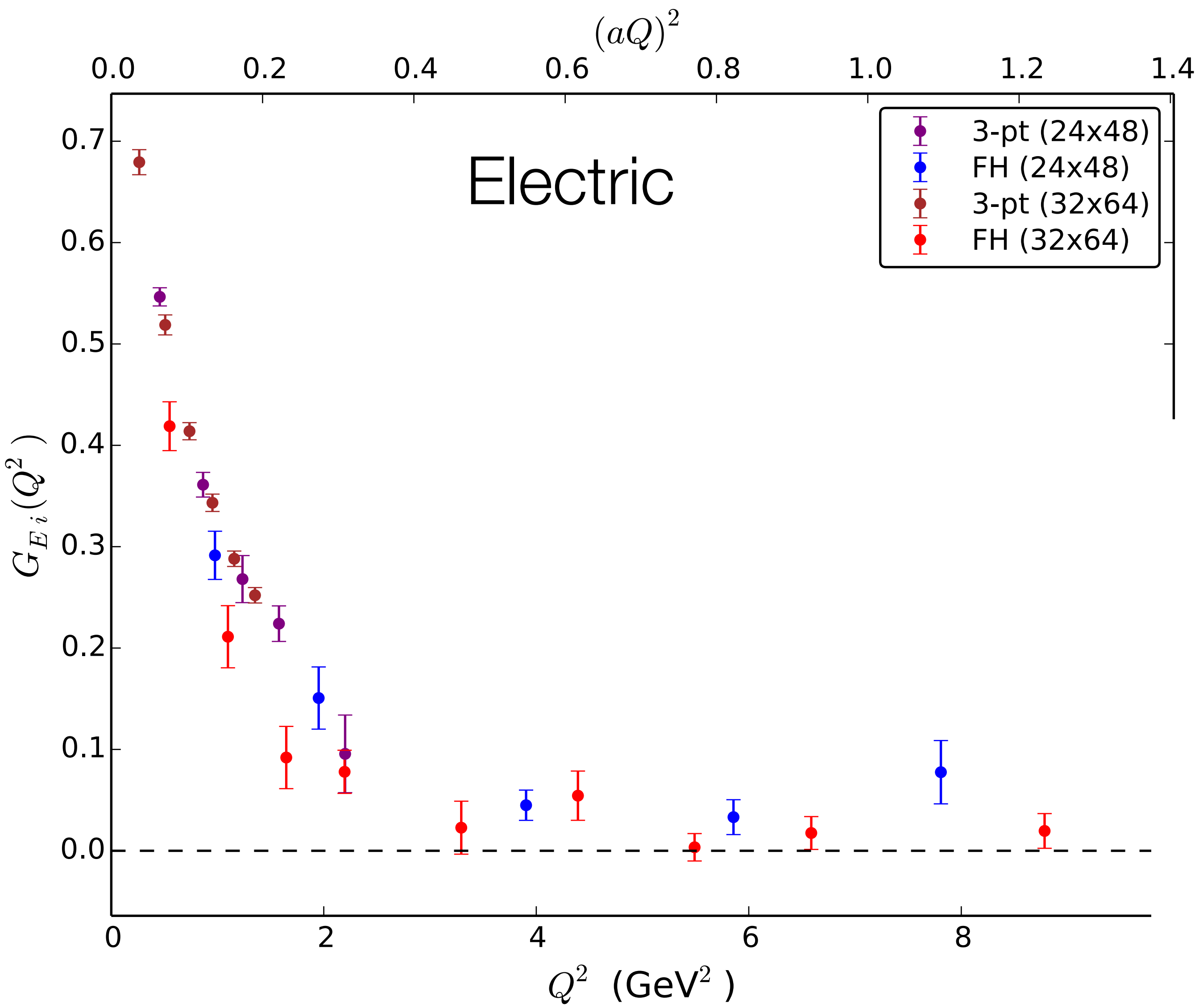
Pion Form Factor

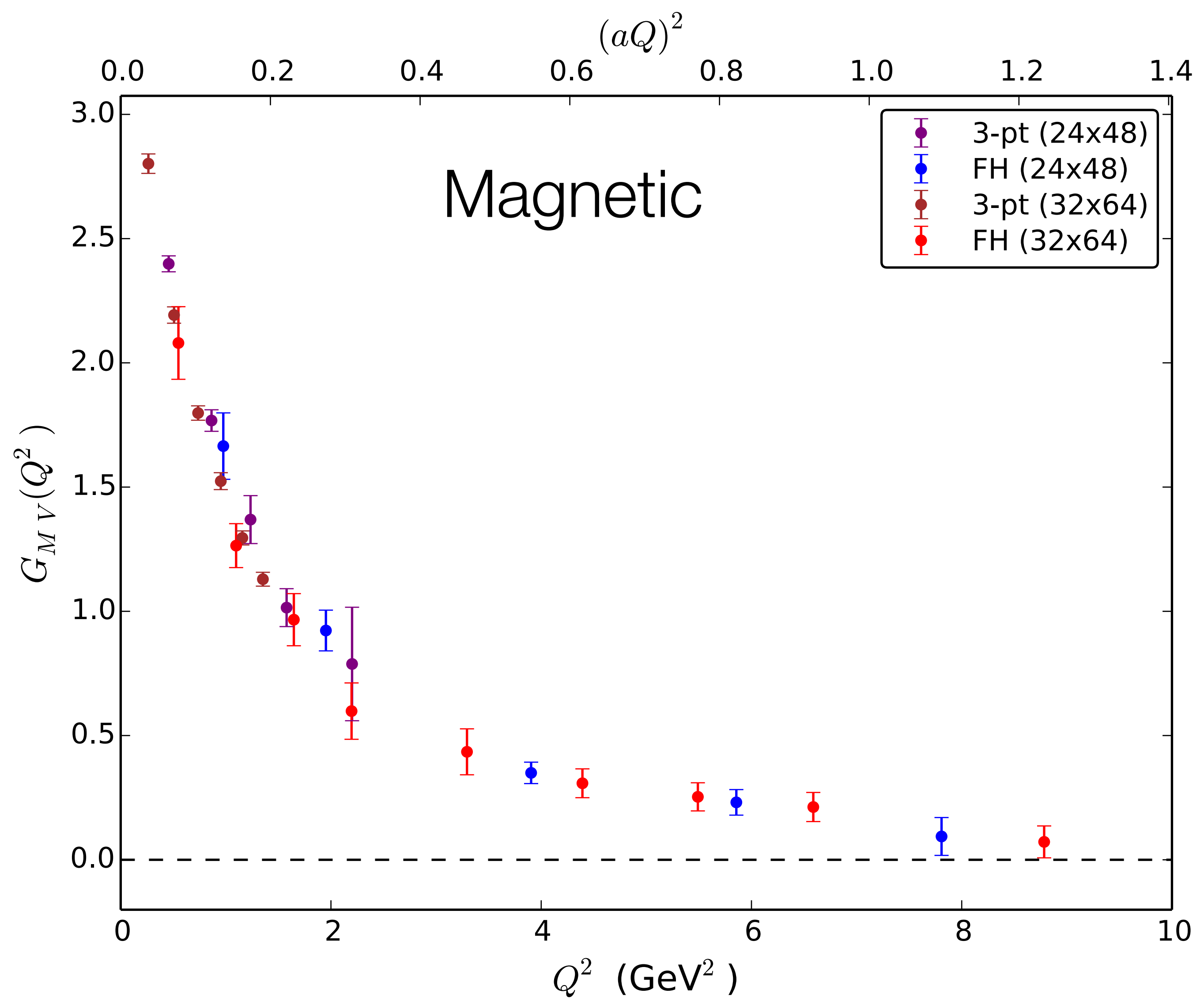
Can probe momentum range of JLab 12 GeV proposal



Nucleon Form Factors

"Up quark" temporal current energy shifts





Nucleon Form Factors

Isovector

Summary

- Feynman-Hellmann technique offers alternative strategy for extracting hadron matrix elements
    - Only requires analysis of 2-pt correlators
- Resolving the nucleon quark spin contribution
    - Competitive precision in the disconnected sector
    - Able to study other hadrons with no additional inversions
        - At SU(3) symmetric point, quark spin of ~60–65% rather universal
- New application of non-forward matrix elements
    - Systematics still to be addressed
- Statistical signals that allow us to probe momentum transfers of interest to upcoming experimental programs

Nucleon Form Factors

Dirac and Pauli form factors defined by

$$\langle \vec{p}' \; \vec{s}' \,|\, \bar{q}(0)\gamma_\mu q(0) \,|\, \vec{p} \; \vec{s} \rangle = \bar{u}(\vec{p}', \sigma') \left[\gamma_u F_1(Q^2) + \sigma_{\mu\nu} \frac{q_\nu}{2m} F_2(Q^2) \right] u(\vec{p}, \sigma)$$

Make idential modification to the action as for the pion case

$$\mathcal{L}(y) \to \mathcal{L}(y) + \lambda \, e^{i\vec{q}\cdot(\vec{y}-\vec{x})} \, \bar{q}(y) \, \gamma_\mu \, q(y)$$

Feynman-Hellmann relation gives

$$\left.\frac{\partial E}{\partial \lambda}\right|_{\lambda=0} = \frac{F_3(\Gamma, \mathcal{J}_{\mathcal{O}}; \vec{p}, \vec{p}, m)}{F_2(\Gamma; \vec{p}, m)}$$

Need to make choice of projection matrix Γ

Nucleon Form Factors

For temporal current, choose projection matrix

$$\Gamma_{\text{unpol.}} = \frac{1}{2}(1+\gamma_4)$$

then the Feynman-Hellmann relation gives

$$\left.\frac{\partial E}{\partial \lambda}\right|_{\lambda=0} = \frac{m}{E}\left[\left(1+\frac{(\vec{p}+\vec{p}\,')^2}{4m(E+m)}\right)F_1(Q^2) - \frac{Q^2}{4m^2}F_2(Q^2)\right]$$

For $\vec{p}\,' = -\vec{p}$ we have simply

$$\left.\frac{\partial E}{\partial \lambda}\right|_{\lambda=0} = \frac{m}{E}G_E(Q^2)$$

Nucleon Form Factors

For spatial current, choose $\Gamma_\pm$ as in axial charge calculation

$$\Re \left.\frac{\partial E}{\partial \lambda}\right|_{\lambda=0} = \pm \frac{1}{2mE} \left[\vec{s} \times \vec{q}\, F_1(Q^2) \right.$$
$$\left. \left[\left[1 + \frac{(\vec{p} + \vec{p}\,')^2}{4m(E+m)} \right] \vec{s} \times \vec{q} - \frac{\vec{s} \cdot (\vec{p} + \vec{p}\,')\vec{p} \times \vec{p}\,'}{2m(E+m)} \right] F_2(Q^2) \right]$$

For $\vec{p}\,' = -\vec{p}$ we have simply

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