Lattice and string worldsheet in AdS/CFT: a numerical study

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AdS/CFT and observables

Type IIB strings in $AdS_5 \times S^5$
(g_S, R)

↔

$\mathcal{N} = 4$ super Yang-Mills in 4 d
$(g_{\rm YM}, N)$

- Identification of parameters $\dfrac{R^2}{\alpha'} \equiv \sqrt{g^2_{\rm YM}\, N} = \sqrt{\lambda}$ and $g_S = \dfrac{4\pi\lambda}{N}$

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$\mathcal{N} = 4$ super Yang-Mills in 4 d
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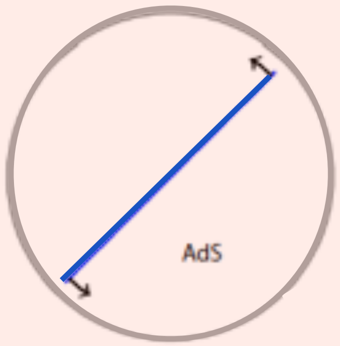
- Identification of parameters: $\frac{R^2}{\alpha'} \equiv \sqrt{g_{\rm YM}^2 N} = \sqrt{\lambda}$ and $g_S = \frac{4\pi\lambda}{N}$
- Dictionary for observables. Example: "**cusp anomaly**" of $\mathcal{N} = 4$ SYM.

Dimension of twist operators $\Delta_{\rm twist} \sim f(\lambda) \ln S\,, \qquad S \gg 1$

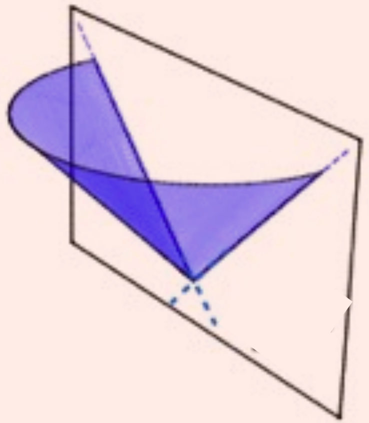
Renormalization of cusped Wilson loops $\langle W_{\rm cusp} \rangle \sim e^{-f(\lambda)\phi \ln \frac{\Lambda}{\epsilon}}$

Energy of a spinning string



$$E_{\rm classical} \sim f(\lambda) \ln S$$

Minimal surface of the string



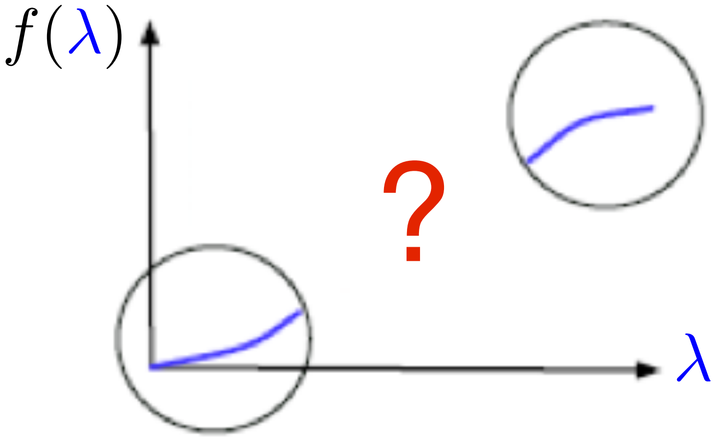
$$Z_{\rm str} = \int D[\phi]\, e^{-S_{\rm str}[\phi]} = e^{-f(\lambda)\, V}$$

AdS/CFT perturbatively

Weak/strong coupling duality: two regimes of controls are opposite. $\frac{R^2}{\alpha'} \equiv \sqrt{g_{\mathrm{YM}}^2 N} = \sqrt{\lambda}$

Solvable for $\lambda \ll 1$

$$f(\lambda) = \lambda\, a_0 + \lambda^2\, a_1 + \cdots$$



Solvable for $\lambda \gg 1$

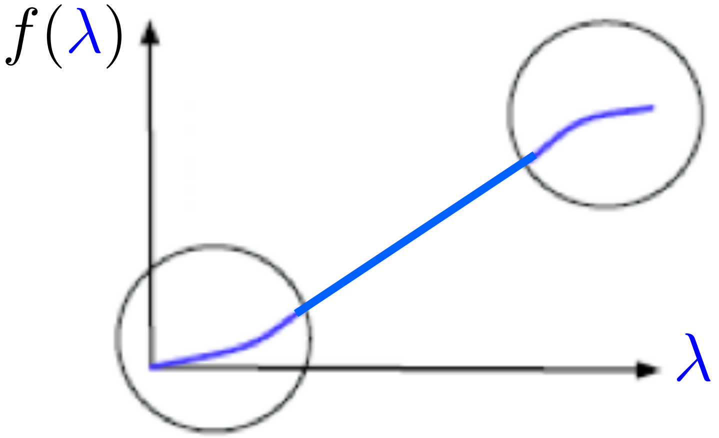
$$f(\lambda) = \sqrt{\lambda}\, b_0 + b_1 + \frac{1}{\sqrt{\lambda}} b_2 + \cdots$$

AdS/CFT at finite coupling

Weak/strong coupling duality: two regimes of controls are opposite. $$\frac{R^2}{\alpha'} \equiv \sqrt{g_{\mathrm{YM}}^2\, N} = \sqrt{\lambda}$$

Solvable for $\lambda \ll 1$

$$f(\lambda) = \lambda\, a_0 + \lambda^2\, a_1 + \cdots$$



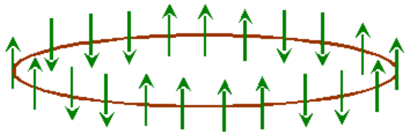
Solvable for $\lambda \gg 1$

$$f(\lambda) = \sqrt{\lambda}\, b_0 + b_1 + \frac{1}{\sqrt{\lambda}} b_2 + \cdots$$

[MInahan Zarembo 2002]

In the large N/planar limit, strong evidence of integrability of the spectral problem.

$$\mathcal{O} = \mathrm{Tr}(\phi_1\phi_1\phi_2\phi_1\phi_2 \ldots) \equiv |\downarrow\downarrow\uparrow\downarrow\uparrow ..\rangle \equiv$$



Anomalous dimensions, perturbatively, are eigenvalues of integrable spin chain hamiltonians.

Assuming this at all loops: a Bethe Ansatz proposed to give the exact spectrum.

Spectacular agreement with perturbative results: [Beisert Eden Staudacher 2006]

[Bern Dixon Kosower 2006, Roiban Tseytlin 2007]

Toward exact solution of a 4-d interacting gauge theory?

String worldsheet sigma-model on the lattice

Lattice investigation of the **string worldsheet sigma-model**: [McKeown Roiban 13]

general, assumptions-free, readily generalizable (AdS_4/CFT_3 and to).

Potentially powerful tool to test integrability (/localization) predictions and AdS/CFT.

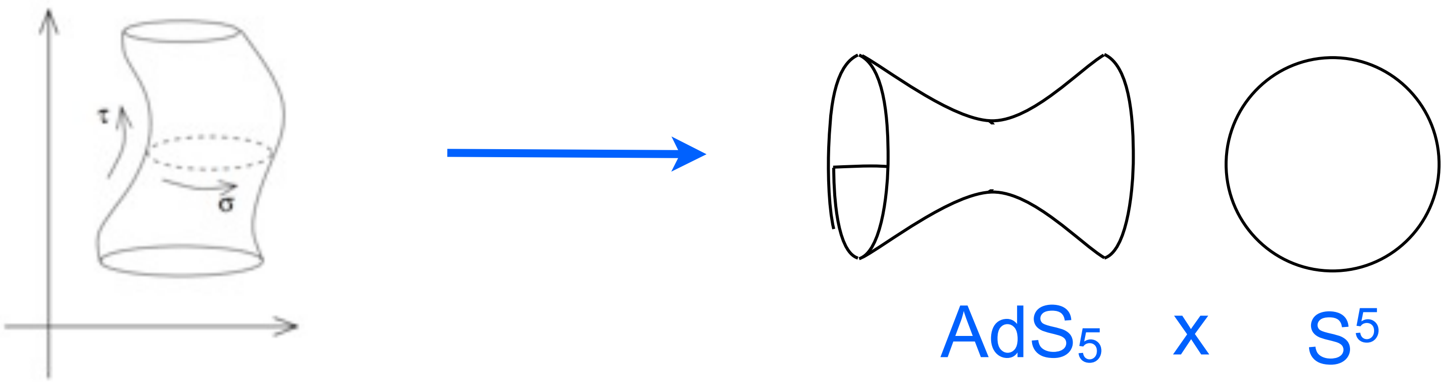
Appealing features:

> 2d: computationally cheap

> no supersymmetry on the world-sheet (Green-Schwarz formulation)

> “strong coupling” analytically known (perturbative $\mathcal{N} = 4$ SYM theory)

The model: Green-Schwarz string in $AdS_5 x S^5$



Non-linear sigma-model:

[Metsaev Tseytlin 1998]

$$S = \frac{\sqrt{\lambda}}{4\pi} \int d\tau d\sigma \, [\, \partial_a X^\mu \partial^a X^\nu G_{\mu\nu} + \bar{\theta}(D + F_5)\theta \, \partial X + \bar{\theta}\theta\bar{\theta}\theta \, \partial_a X \partial^a X + \ldots]$$

Symmetries: global $PSU(2,2|4)$, local bosonic (diffeomorphism) and fermionic (κ-).

To quantize it use semiclassical methods $\hbar \leftrightarrow 1/g \,, \quad g = \frac{\sqrt{\lambda}}{4\pi} = \frac{R^2}{4\pi\alpha'}$

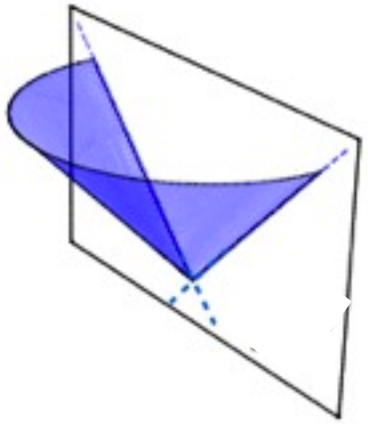
$$X = X_{\rm cl} + \tilde{X} \longrightarrow E = g \left[E_0 + \frac{E_1}{g} + \boxed{\frac{E_2}{g^2}} + \cdots \right]$$

2 loops: current limit

[Giombi Ricci Roiban Tseytlin 2009] [Bianchi Bianchi Bres **VF** Vescovi 2014]

Test observable: cusp anomaly of N=4 SYM

Expectation value of a light-like cusped Wilson loop



$$Z_{\rm cusp} = \langle W[C_{\rm cusp}]\rangle \sim e^{-f(g)\,\phi\ln\frac{L_{\rm IR}}{\epsilon_{\rm UV}}}$$

$$Z_{\rm cusp} = \int [D\delta X][D\delta\theta]\, e^{-S_{\rm IIB}(X_{\rm cusp}+\delta X,\delta\theta)} = e^{-\Gamma_{\rm eff}}$$

String partition function with ``cusp'' boundary conditions

In Poincaré patch (boundary at z=0)

$$ds^2_{AdS_5} = \frac{dz^2 + dx^+\,dx^- + dx^*\,dx}{z^2} \qquad x^\pm = x^3 \pm x^0 \qquad x = x^1 \pm i\,x^2$$

classical solution (τ and σ vary from 0 to ∞) of the string equations of motion:

$$z = \sqrt{\frac{\tau}{\sigma}} \qquad x^+ = \tau \qquad x^- = -\frac{1}{2\sigma}$$

describe a surface bounded by a null cusp, as at the AdS5 boundary $0 = z^2 = -2x^+x^-$.

Test observable: cusp anomaly of N=4 SYM

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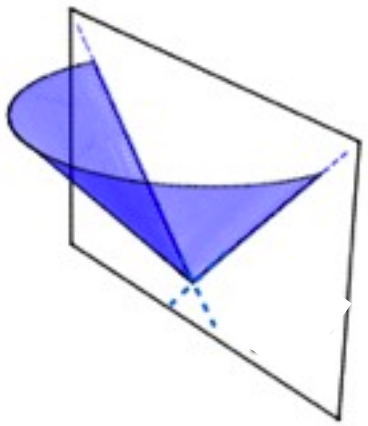

$$Z_{\rm cusp} = \langle W[C_{\rm cusp}]\rangle \sim e^{-f(g)\,\phi\,\boxed{\ln\frac{L_{\rm IR}}{\epsilon_{\rm UV}}}}$$

$$Z_{\rm cusp} = \int [D\delta X][D\delta\theta]\, e^{-S_{\rm IIB}(X_{\rm cusp}+\delta X,\delta\theta)} = e^{-\Gamma_{\rm eff}} = e^{-f(g)\boxed{V}}$$

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$$\Gamma_{\rm eff} = \Gamma^{(0)} + \Gamma^{(1)} + \Gamma^{(2)} + \ldots$$

$$= V\,g\,(a_0 + \frac{a_1}{g} + \frac{a_2}{g^2} + \ldots) \equiv V\,f(g) \qquad V = \int_0^\infty dt \int_0^\infty ds$$

Test observable: cusp anomaly of N=4 SYM

Cusp anomaly formally given by a partition function $f(g) = -\frac{\ln Z}{V}$

or via the **expectation value**

$$\langle S \rangle = \frac{\int [D\delta X][D\delta\theta]\, S\, e^{-S}}{\int [D\delta X][D\delta\theta]\, e^{-S}} = -g\frac{d\ln Z}{dg} = g\frac{df}{dg}$$

This is the object of our simulation

S (action for fluctuations over the cusp) obtained **gauge-fixing** bosonic and fermionic local symmetries - “AdS light-cone gauge”. The gauge-fixing leaves just one symmetry, SO(6). It is “just” quartic in the fermions.

⟶ Introducing auxiliary (complex bosons) fields allows linearization, and Grassmann fields can be formally integrated out. *M:* fermionic operator

$$\det M = (\det M^\dagger M)^{1/2} = \int D\zeta D\bar{\zeta}\, e^{-\int d\tau d\sigma \bar{\zeta}(M^\dagger M)^{-1/2}\zeta}$$

no ambiguities here

The simulation: final lagrangean

The lagrangian to be discretized is

$$\mathcal{L} = |\partial_t \tilde{x} + \frac{1}{2}\tilde{x}|^2 + \frac{1}{\tilde{z}^4}|\partial_s \tilde{x} - \frac{1}{2}\tilde{x}|^2 + (\partial_t \tilde{z}^M + \frac{1}{2}\tilde{z}^M)^2 + \frac{1}{\tilde{z}^4}(\partial_s \tilde{z}^M - \frac{1}{2}\tilde{z}^M)^2$$
$$+ \frac{1}{2}\tilde{\phi}^2 + \frac{1}{2}(\tilde{\phi}_M)^2 + \psi^T M \psi$$

with $\psi \equiv (\tilde{\theta}^i, \tilde{\theta}_i, \tilde{\eta}^i, \tilde{\eta}_i)$ $i = 1, \cdots, 4$ and

$$M = \begin{pmatrix} 0 & i\partial_t & -\mathrm{i}\rho^M \left(\partial_s + \frac{1}{2}\right) \frac{\tilde{z}^M}{\tilde{z}^3} & 0 \\ \mathrm{i}\partial_t & 0 & 0 & -\mathrm{i}\rho^\dagger_M \left(\partial_s + \frac{1}{2}\right) \frac{\tilde{z}^M}{\tilde{z}^3} \\ \mathrm{i}\frac{\tilde{z}^M}{\tilde{z}^3}\rho^M \left(\partial_s - \frac{1}{2}\right) & 0 & 2\frac{\tilde{z}^M}{\tilde{z}^4}\rho^M \left(\partial_s \tilde{x} - \frac{\tilde{x}}{2}\right) & i\partial_t - A^\dagger \\ 0 & \mathrm{i}\frac{\tilde{z}^M}{\tilde{z}^3}\rho^\dagger_M \left(\partial_s - \frac{1}{2}\right) & \mathrm{i}\partial_t + A & -2\frac{\tilde{z}^M}{\tilde{z}^4}\rho^\dagger_M \left(\partial_s \tilde{x}^* - \frac{\tilde{x}}{2}^*\right) \end{pmatrix}$$

$$A^i{}_j = \frac{1}{\sqrt{2}\tilde{z}^2}\tilde{\phi}_M \rho^{MNi}{}_j \tilde{z}_N - \frac{1}{\sqrt{2}\tilde{z}}\tilde{\phi}\,\delta^i{}_j + \mathrm{i}\,\frac{\tilde{z}_N}{\tilde{z}^2}\rho^{MNi}{}_j\,\partial_t \tilde{z}^M$$

where $(\rho^M)_{ij}$ are off-diagonal blocks of SO(6) Dirac matrices $\gamma^M \equiv \begin{pmatrix} 0 & \rho^\dagger_M \\ \rho^M & 0 \end{pmatrix}$

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> A naive regularization leads to doublers

→ “Wilson fermion” procedure.

> Light-cone momentum is typically set to 1

→ Here: reintroduce $m \sim P_+$

The simulation: parameter space

- In the continuum model there are two parameters, $g = \frac{\sqrt{\lambda}}{4\pi}$ and $m \sim P_+$.

  In perturbation theory divergences cancel and dimensionless quantities are pure functions of the (bare) coupling

$$F = F(g) \ .$$

- Our discretization cancels (1-loop) divergences.

  Assume it is true nonperturbatively for lattice regularization, with lattice spacing *a* and box size $L^2 = (N\,a)^2 = V$.

  There are in total three dimensionless parameters

$$g\,, \qquad N \equiv \frac{L}{a}\,, \qquad M \equiv a\,m$$

  Therefore

$$F_{\mathrm{LAT}} = F_{\mathrm{LAT}}(g, N, M)$$

The simulation: continuum limit

Remove the cutoff and compare to other results (here: integrability) or other regularizations.

If there are no divergences (i.e. no terms proportional to 1/a)

$$F_{\mathrm{LAT}}(g,N,M) = F(g) + \underbrace{\mathcal{O}\left(\frac{1}{N}\right) + \mathcal{O}(M)}_{\substack{\text{finite lattice spacing} \\ \text{(\~{}a) effects}}} + \mathcal{O}(e^{-MN})$$

finite volume
(~ m L) effects

Recipe:

> fix g

> fix $MN = mL$, large enough so that finite volume effects are small

> compute F_{LAT} for $N = 6, 8, 10, 12, 16, \ldots$

> extrapolate to $1/N \to 0$

The simulation: the observable

The relation between partition function and cusp anomaly $f(g)$ is

$$Z = \int [D\phi] e^{-S[\phi]} \equiv e^{-\widetilde{V}\, f(g)} \;=\int [D\phi][D\phi_{\mathrm{aux}}]\, e^{-S'[\phi,\phi_{\mathrm{aux}}]} J(g)$$

The action simulated on the lattice is the modified one S' (auxiliary fields and jacobian)

$$\langle S' \rangle = \frac{\int D\phi D\phi_{\mathrm{aux}}\, S'\, J(g)\, e^{-S'}}{\int D\phi D\phi_{\mathrm{aux}}\, J(g)\, e^{-S'}} = \boxed{g\frac{d\ln J(g)}{dg}} - g\,\frac{d\ln Z}{dg}$$

and its relation to $f(g)$ - which goes via $\ln Z$ - picks a constant factor (from now on $S' \to S$)

$$\langle S \rangle = \frac{15}{2} N^2 + \frac{1}{8} m^2 V\, g\, f'(g)$$

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Recall: $m^2 = \dfrac{M^2}{a^2}$, $V = a^2 N^2$

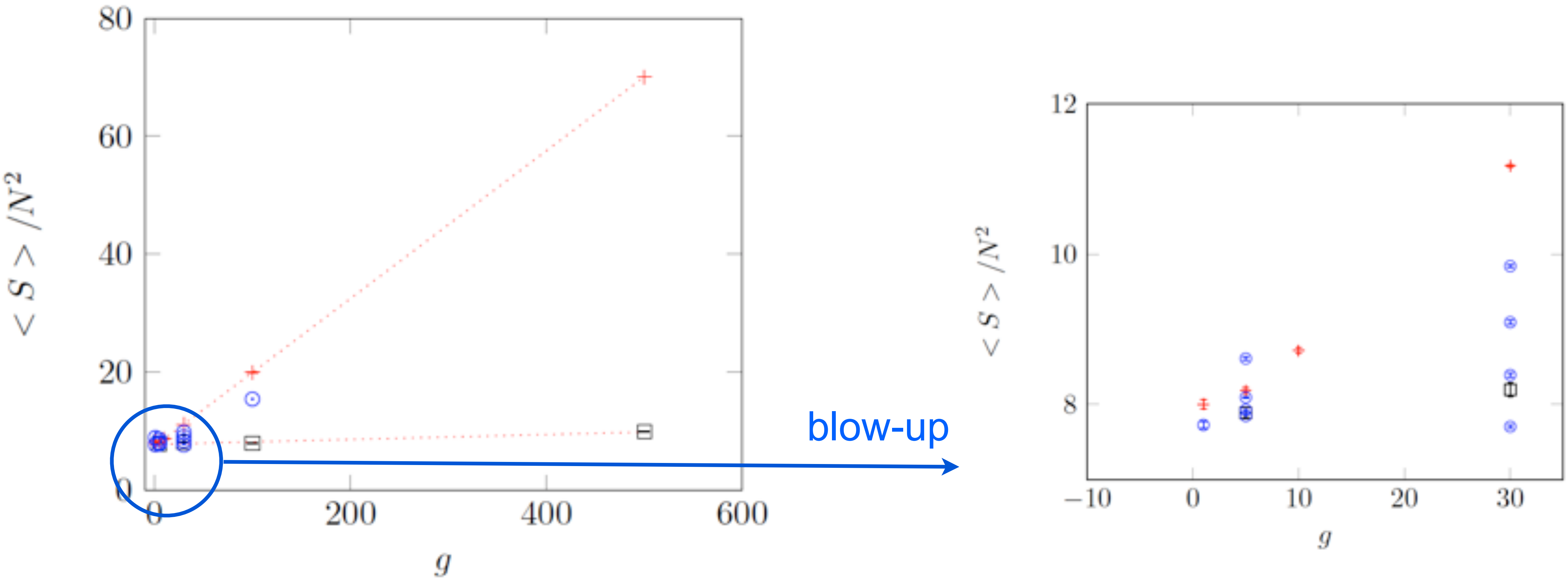
1. Fit $\dfrac{\langle S \rangle}{N^2} = \dfrac{c}{2} + \dfrac{1}{2} M^2 g$ to find c, having in mind $f(g)|_{g \to \infty} = 4g \left[1 - \dfrac{3 \ln 2}{4\pi} \dfrac{1}{g} + \dots \right]$
2. Compute the continuum limit of $\dfrac{\langle S \rangle - cN^2}{\frac{1}{2} M^2 N^2 g} = \dfrac{1}{4} f'(g)$

For both we have predictions.

Status of the simulation - I

Fit $\frac{\langle S \rangle}{N^2} = \frac{c}{2} + \frac{1}{2} M^2 g$ for fixed/different values of M (red: M=0.5, black: M=0.1)

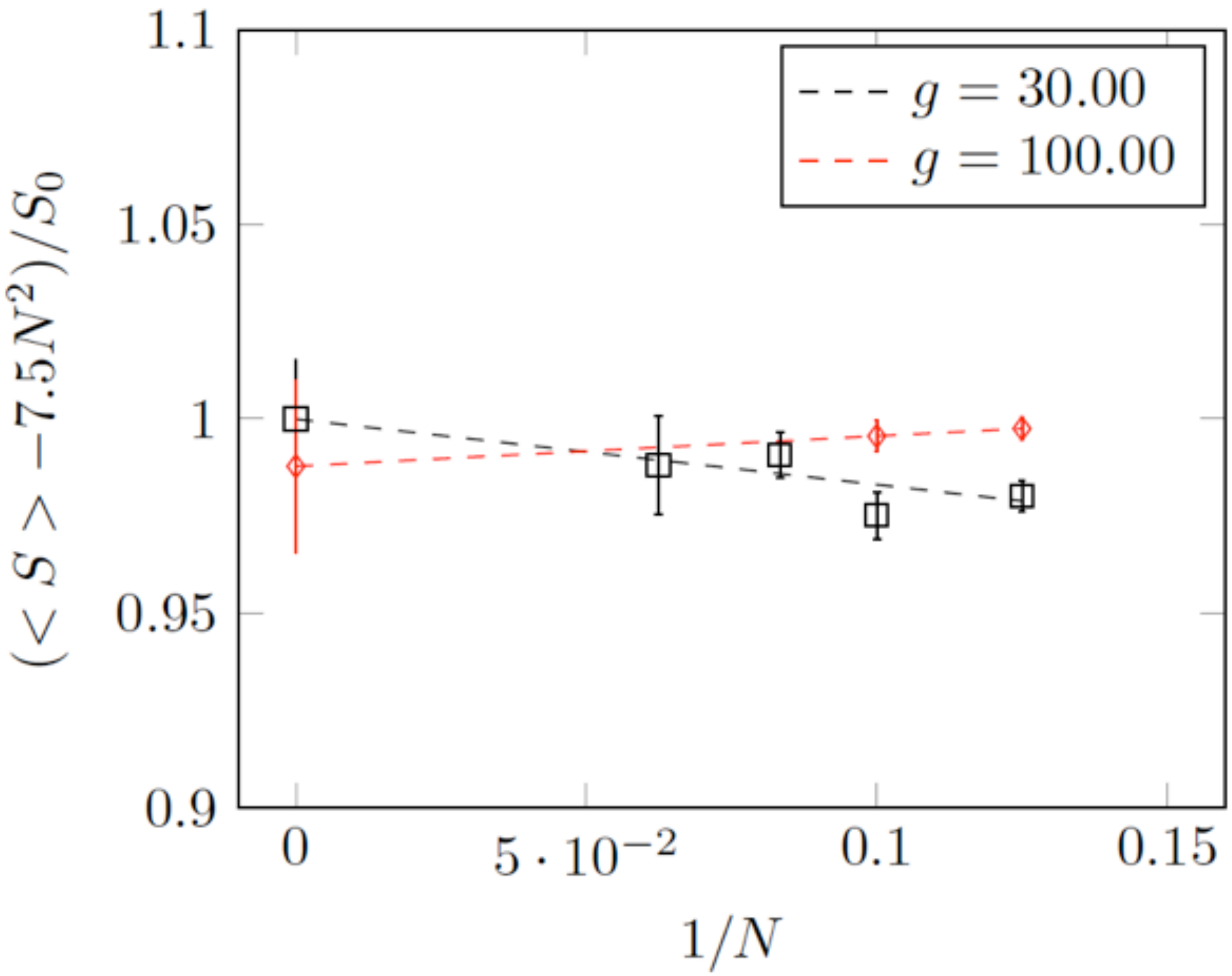
Find $c = 15$ (extrapolating to $g \to 0$), as expected.



Status of the simulation - II

Continuum limit (increasing N) at g=100 and g=30

$$\underbrace{\frac{\langle S\rangle - cN^2}{\frac{1}{2}M^2N^2g}}_{S_0} = \frac{1}{4}f'(g)$$

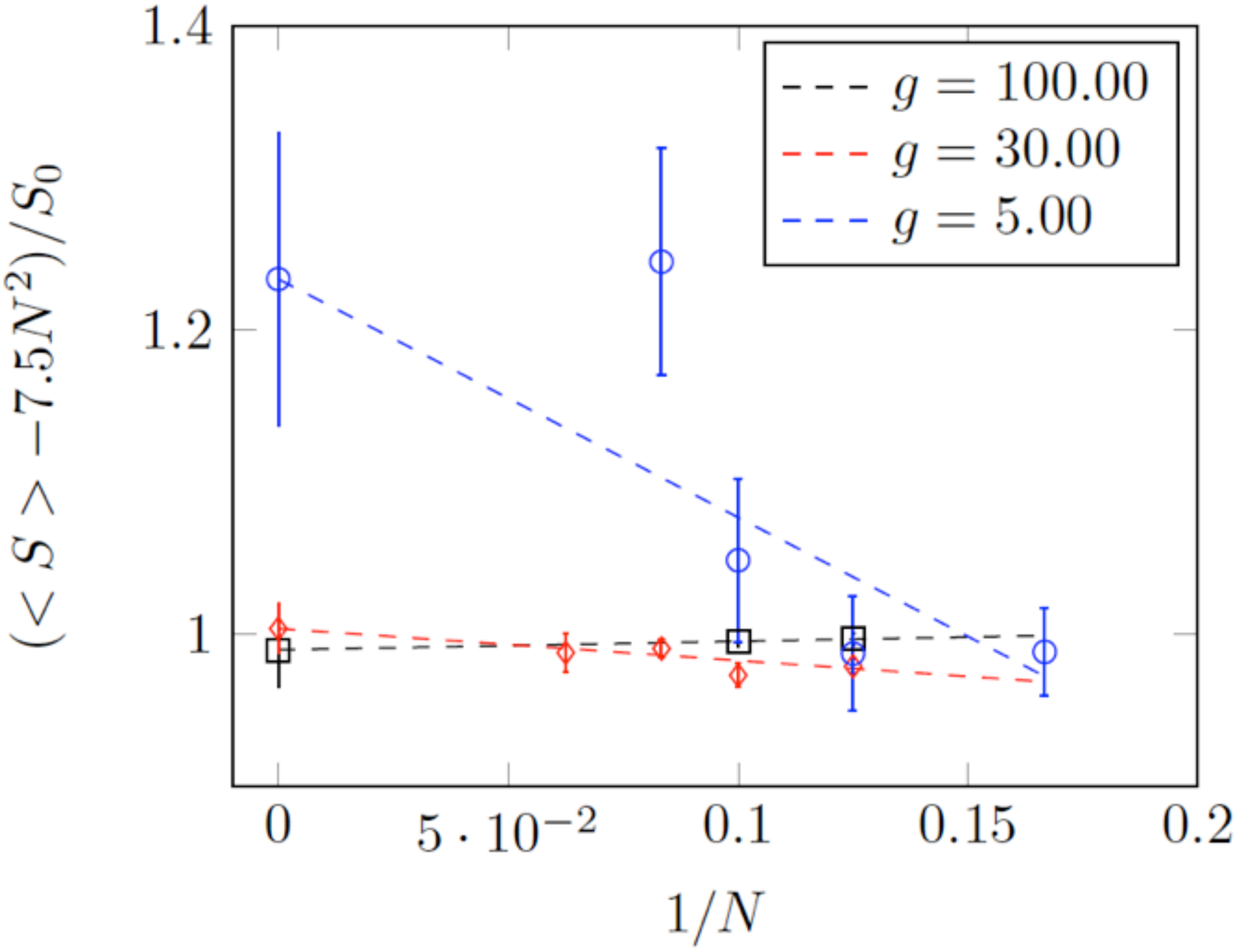


Since $f(g)|_{g\to\infty} = 4g\left[1 - \frac{3\ln 2}{4\pi}\frac{1}{g} + \ldots\right]$ this is good.

Status of the simulation - III

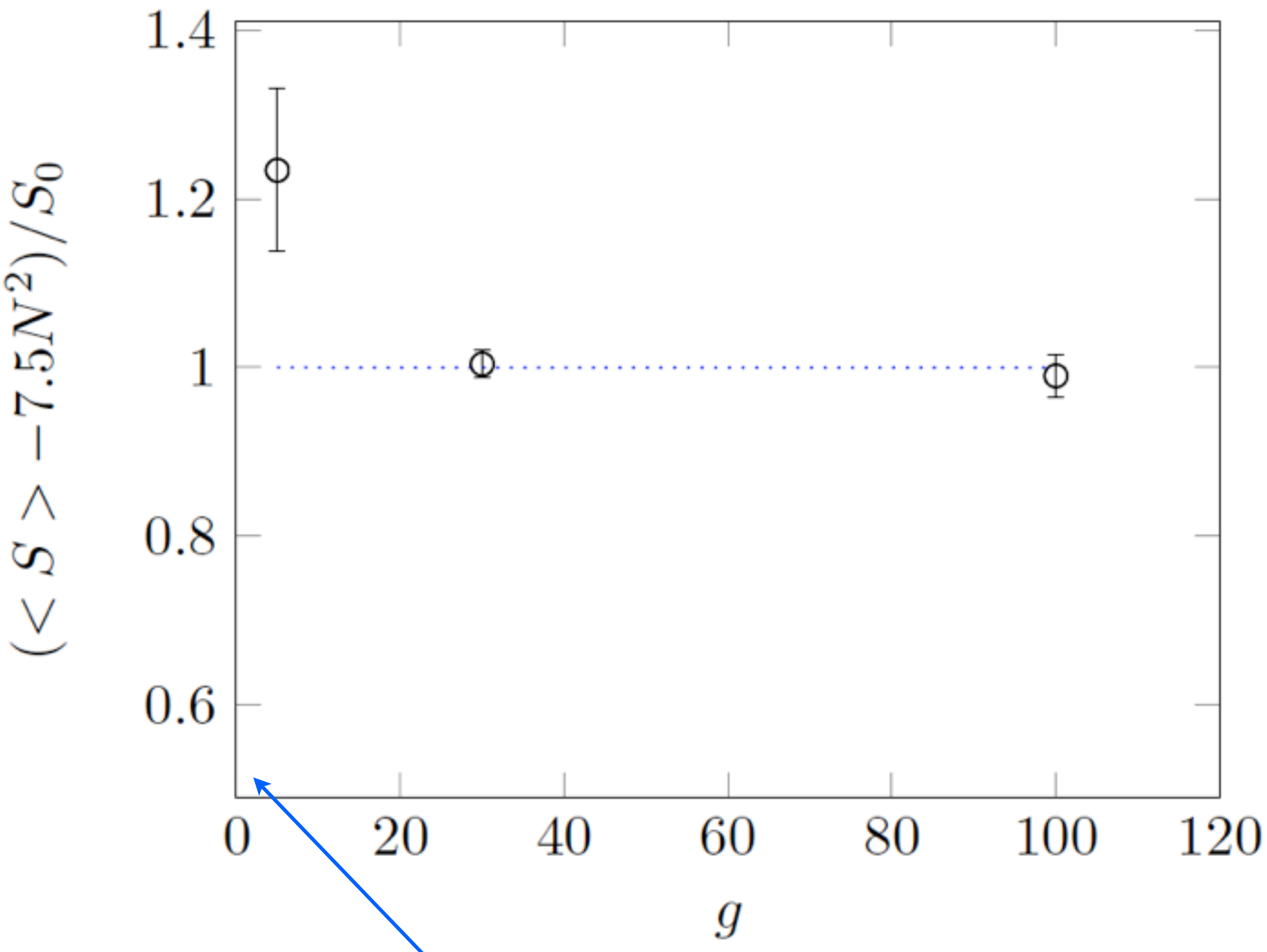
Continuum limit at g=5.

$$\underbrace{\frac{\langle S\rangle - cN^2}{\frac{1}{2}M^2N^2g}}_{S_0} = \frac{1}{4}f'(g)$$



Compatible with the “weak coupling” analysis.

Status of the simulation - IV



Plot of our observable in the continuum limit as a function of g

Errors are just statistical, and compared with the computational effort (minimal) very good.

at g=1 continuum limit is problematic

Conclusions

- Preliminary results on Green-Schwarz string worldsheet model on the lattice:

  > good control on the “weak coupling” region, continuum limit problematic in lowering *g*.
  > good (Fortran, Matlab) implementations (standard RHMC), internal consistency checks.

  Possible change of discretization needed.

  Important to have further observables for a non-trivial check of the code, and of the continuum limit! For example correlation functions of the fields.

- Future prospects:

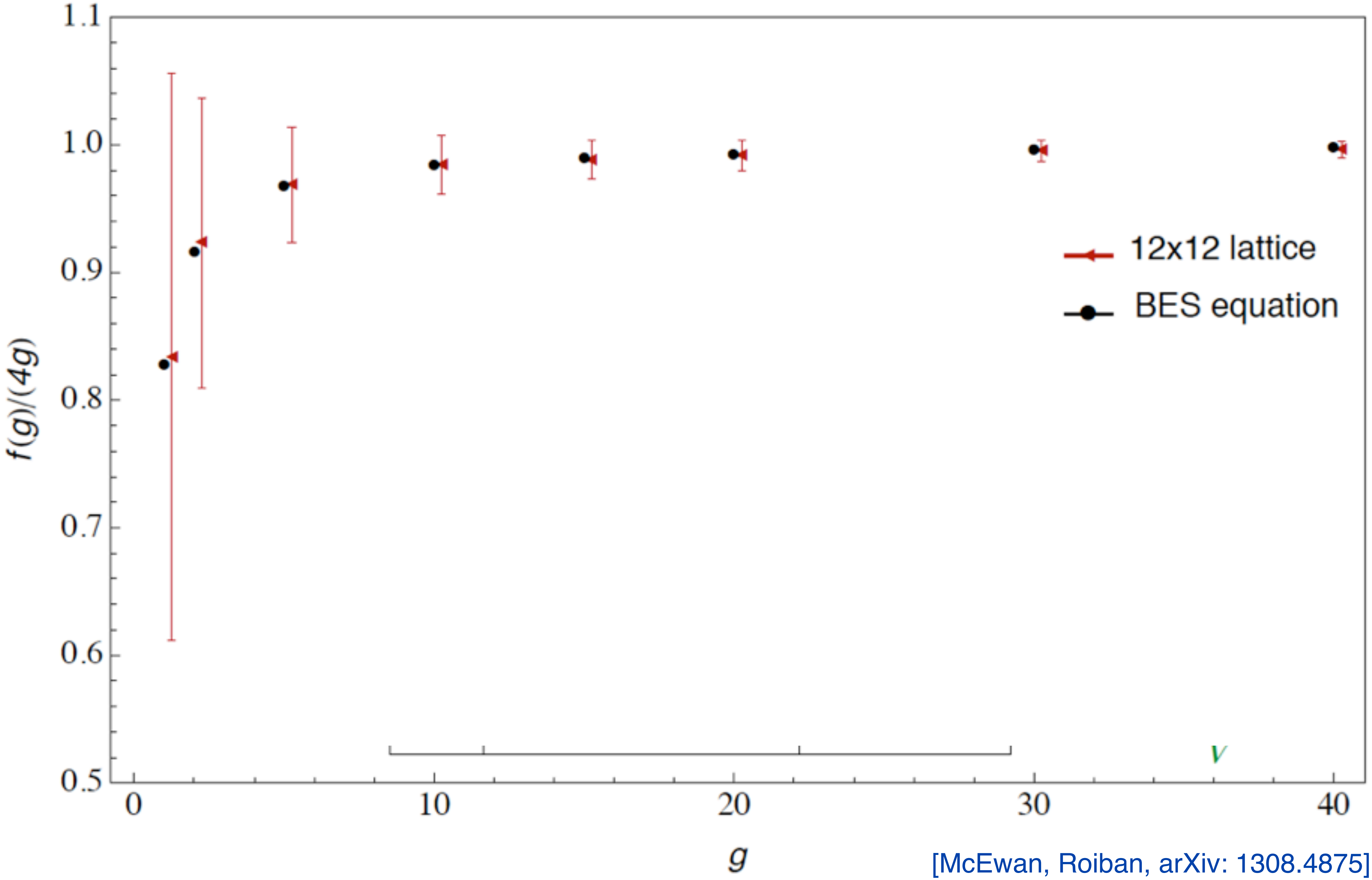
  > cusp anomaly of AdS_4/CFT_3
  > correlators of string vertex operators (three-point functions in gauge theory)

Solving a 4d qft is **hard** → Reduce the problem via AdS/CFT, and “solve a (non-trivial) 2d qft”: Green-Schwarz string in AdS_5xS^5.

An efficient analysis in this context might become crucial device in numerical holography!

Extra slides

Roiban McKeown 2013



[McEwan, Roiban, arXiv: 1308.4875]