Tensor renormalization group analysis of CP(N-1) model in two dimensions

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LATTICE2015

Outline

- Introduction
- Tensor Renormalization Group (TRG)
- Higher Order TRG
- CP(N-1) model
- Summary

Introduction

- Lattice QCD calculations use Monte Carlo methods.
- But, we need new methods for the systems suffering from sign problem.

- Tensor renormalization group (TRG) method has no sign problem.

Strong CP problem

- The QCD Lagrangian naturally includes the θ term.

- Why is the term so small ?

But

- The 4D QCD is still difficult for TRG.

- CP(N-1) model in two dimensions

Tensor Renormalization Group (TRG)

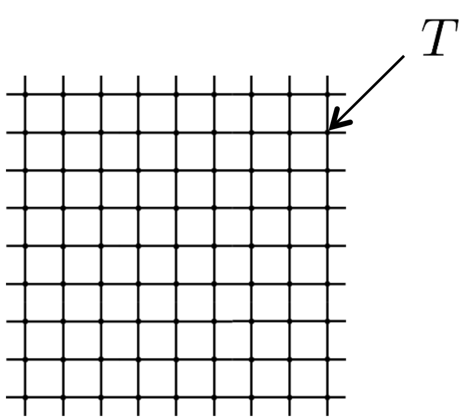
Michael Levin and Cody P. Nave, Phys. Rev. Lett. 99, 120601 (2007).

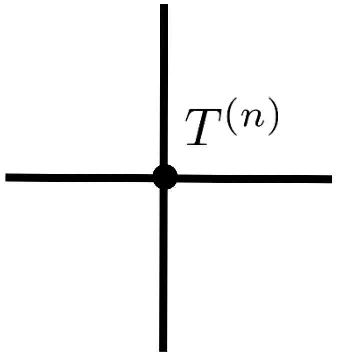
1. Tensor network representation

$$Z = \mathrm{Tr} e^{-\beta H} \quad \Rightarrow \quad Z = \underset{\{x,y\}}{\mathrm{Tr}} \prod_i T_{x_i x_i' y_i y_i'} \cdots$$

2. Reduce the number of tensors

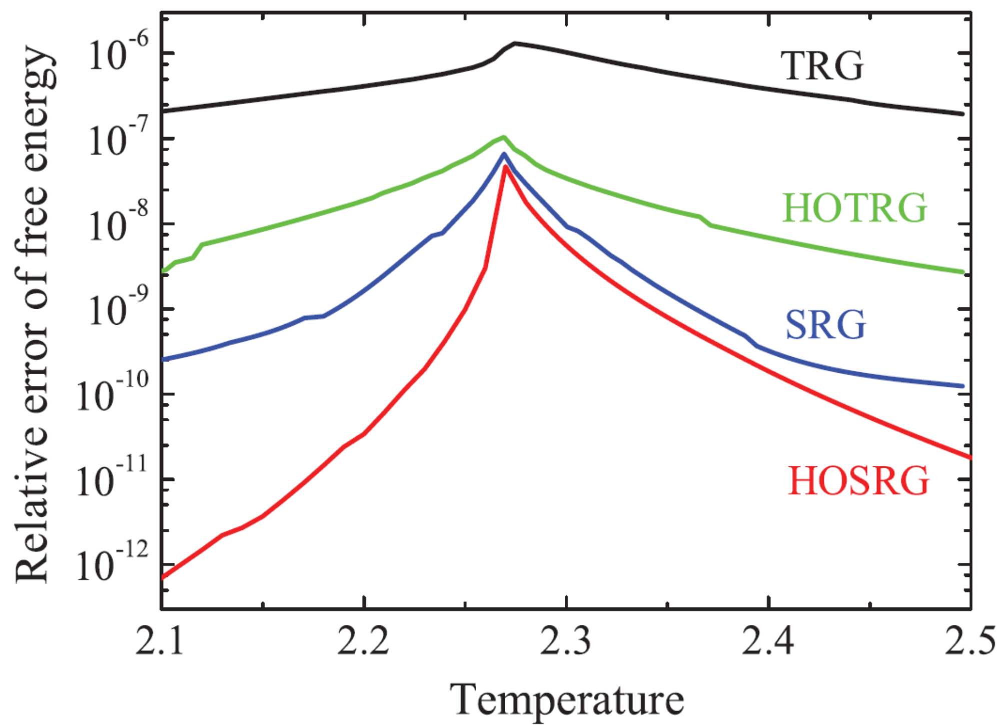
$$Z = \mathrm{Tr}[TT\cdots T] \rightarrow \cdots \xrightarrow{\text{coarse graining}} \cdots \rightarrow Z \simeq \mathrm{Tr}[T^{(n)}]$$





Higher-Order TRG

- TRG method based on the HOSVD
- HOTRG is more accurate than TRG.
- HOTRG can be extended to higher dimensions.



Z. Y. Xie, J. Chen, M. P. Qin, J. W. Zhu, L. P. Yang, and T. Xiang,
PHYSICAL REVIEW B **86**, 045139 (2012).

Higher-Order TRG

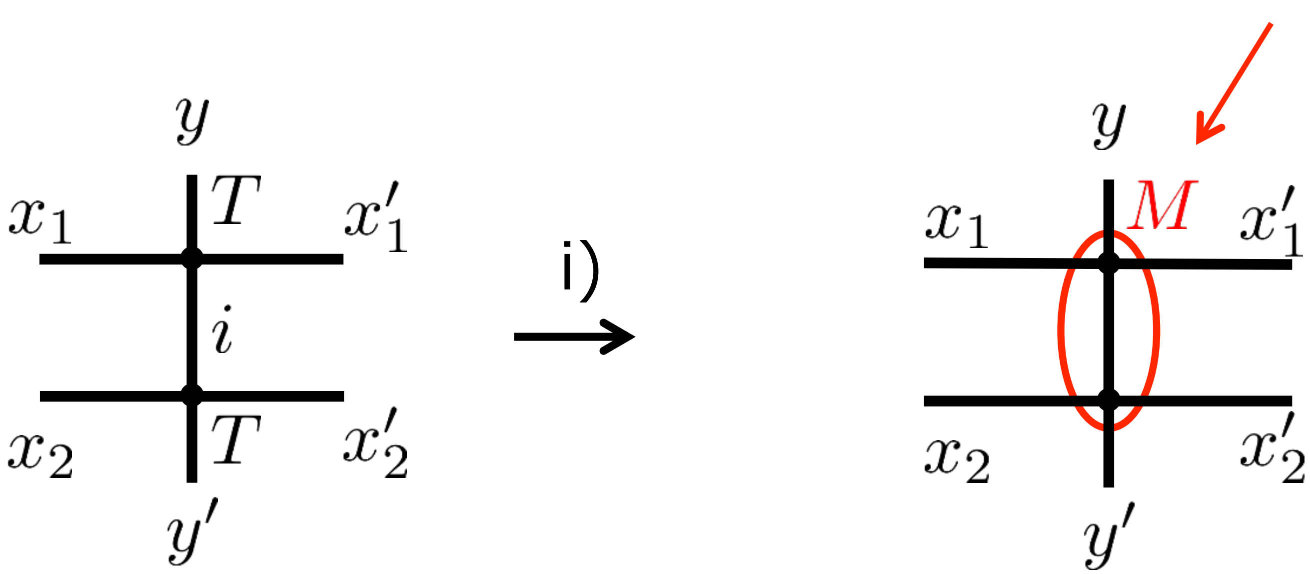
Procedure

Z. Y. Xie, J. Chen, M. P. Qin, J. W. Zhu, L. P. Yang, and T. Xiang, PHYSICAL REVIEW B **86**, 045139 (2012).

i) Combine two T tensors into one M tensor

$x = x_1 \otimes x_2$ $\quad$ $x' = x'_1 \otimes x'_2$

$$M_{x,x',y,y'} = \sum_i T_{x_1,x'_1,y,i} T_{x_2,x'_2,i,y'}$$



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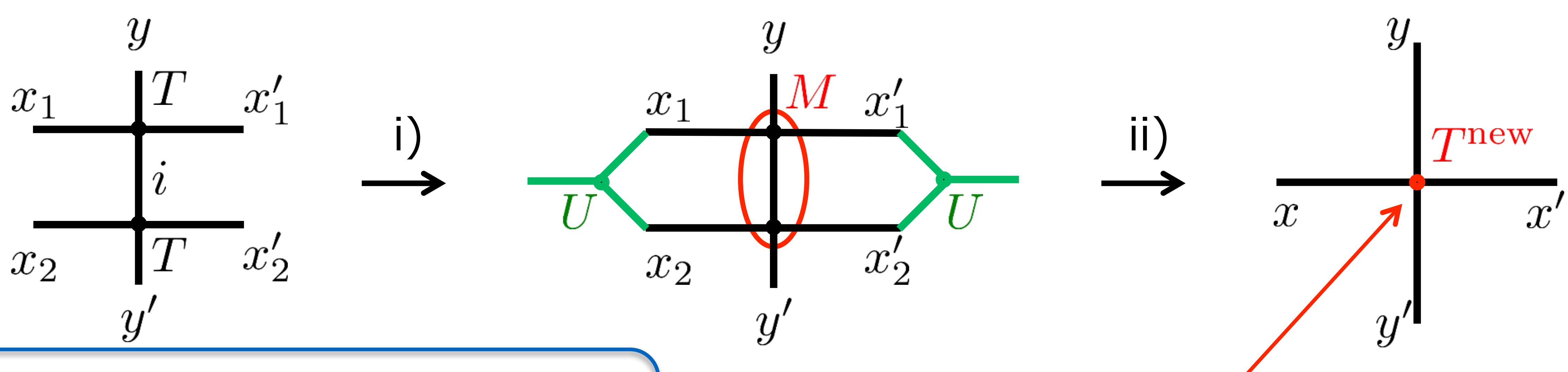
ii) Insert a product of orthogonal matrices $\sum_m^{D_{cut}} U_{im}U_{jm}$ into the two M tensors



HOSVD

$$M_{x,x',y,y'} = \sum_{ijkl} S_{ijkl} U^L_{xi} U^R_{x'j} U^U_{yk} U^D_{y'l}$$

$$T^{\text{new}}_{x,x',y,y'} = \sum_{i,j} U_{i,x} M_{i,j,y,y'} U_{j,x'}$$

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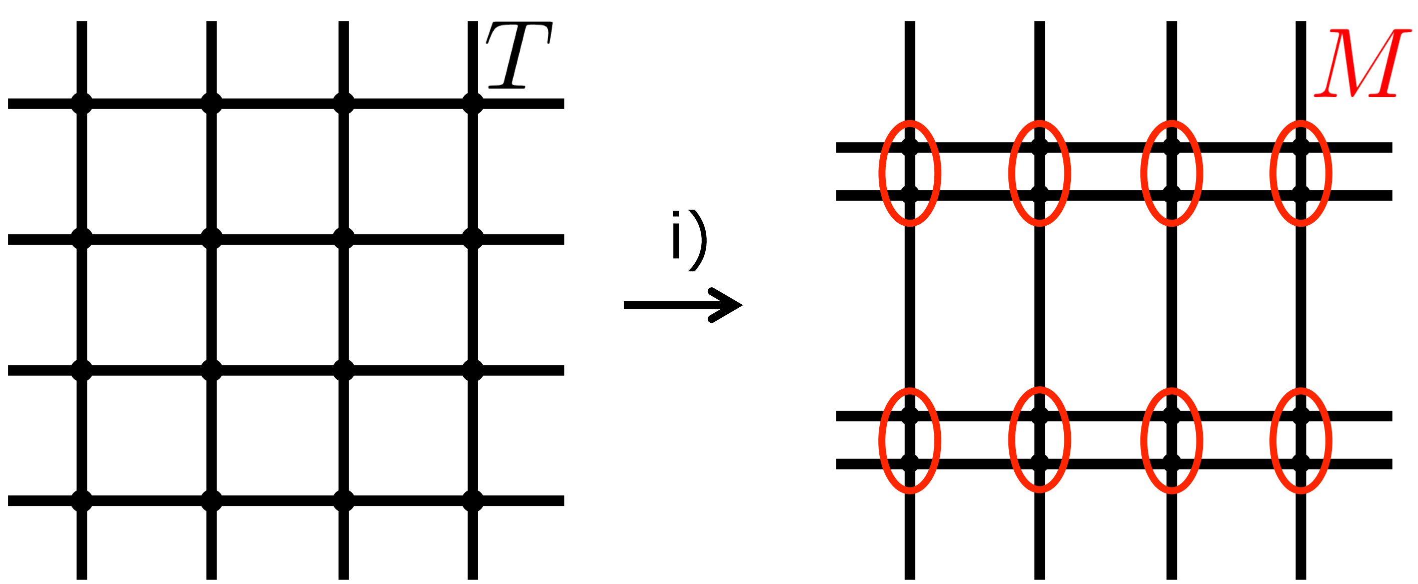

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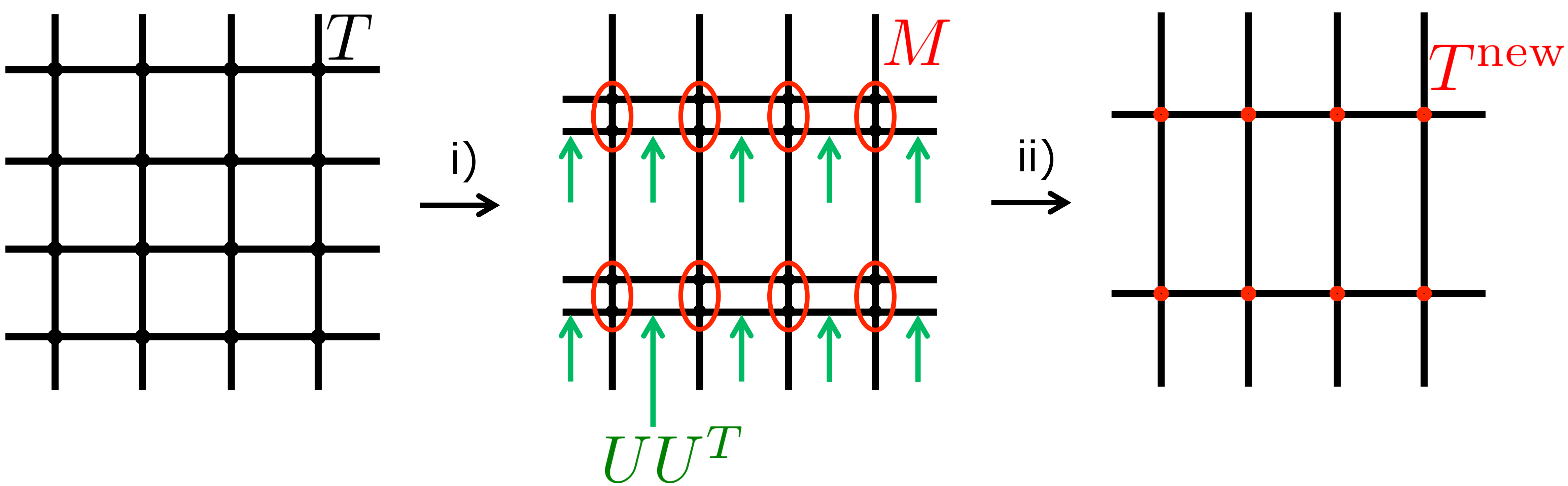

CP(N-1) model in two dimensions

Tensor Network Representation

$$z^a(x)^* z^a(x) = 1, a = 1, \cdots, N$$
$$U_{ij}(x) = \exp\{iaA_{ij}(x)\}$$

$$Z = \int DzDz^*DU \mathrm{e}^{-(S-i\theta Q)}$$
$$= \int \prod_i dz_i dz_i^* \prod_{<i,j>} dU_{i,j} \ \mathrm{e}^{\beta N \sum_{i,j}[z_i^* \cdot z_j U_{i,j} + z_j^* \cdot z_i U_{i,j}^\dagger] + i\frac{\theta}{2\pi}\sum_p q_p}.$$

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Expand the weight with new integers $(s, t, u, v, \cdots \in \mathbb{Z})$

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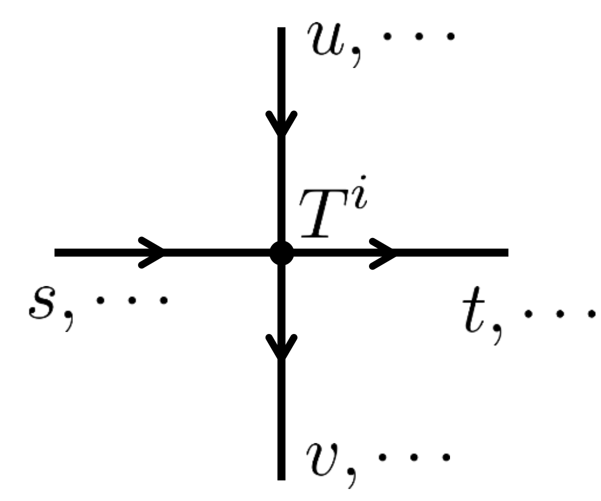
↓

Integrate out old d.o.f

↓

Tensor $T_{s,t,u,v,\cdots}$



CP(N-1) model in two dimensions

We use characterlike expansion.

J. C. Plefka and S. Samuel, Phys. Rev. D **55**, 3966 (1997).

$\theta = 0$

$$\exp\left\{\beta N\left[z_i^* \cdot z_j \exp(iA_{i,j}) + z_i \cdot z_j^* \exp(-iA_{i,j})\right]\right\}$$
$$= Z_0(\beta)\sum_{l,m} d_{(l;m)} \exp[i(m-l)A_{i,j}] h_{(l;m)}(\beta) f_{(l;m)}(z_i, z_j).$$

$Z_0(\beta)$: normalization factor

$d_{(l;m)}$: dimensionalities of characterlike representations

$h_{(l;m)}(\beta)$: characterlike expansion coefficients

$f_{(l;m)}(z_i, z_j)$: characterlike expansion characters

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$$= \frac{I_{N-1+l+m}(2N\beta)}{I_{N-1}(2N\beta)}$$

truncation!

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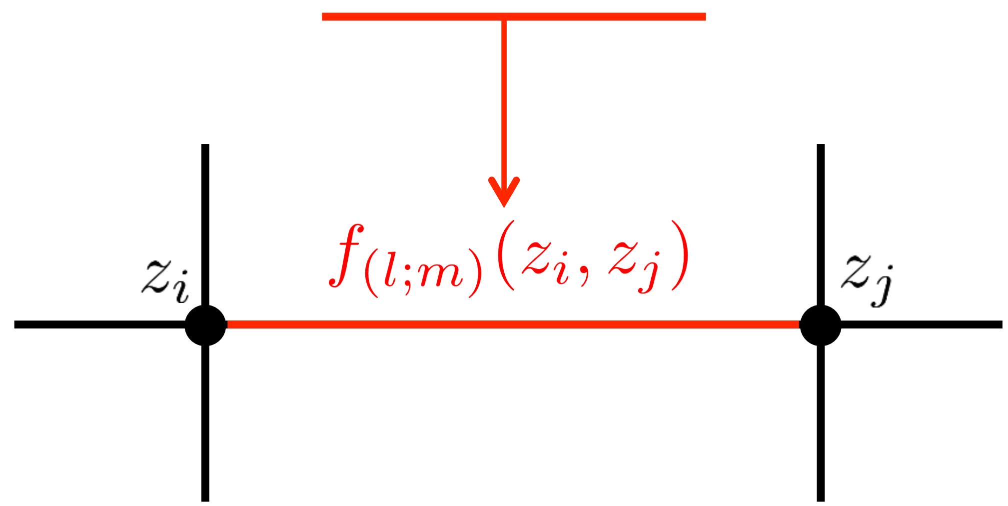

e.g.

$$f_{(0;0)}(z_i, z_j) = 1,$$

$$f_{(1;0)}(z_i, z_j) = \sqrt{N}(z_i \cdot z_j^*),$$

$$f_{(1;1)}(z_i, z_j) = N\sqrt{\frac{N+1}{N-1}}\left[(z_i \cdot z_j^*)(z_i^* \cdot z_j) - \frac{1}{N}\right],$$

$$\vdots$$

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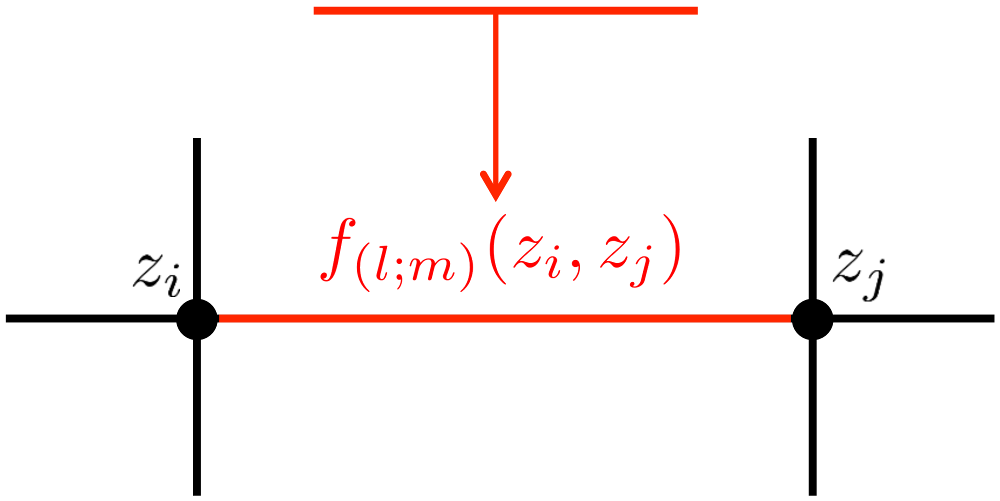
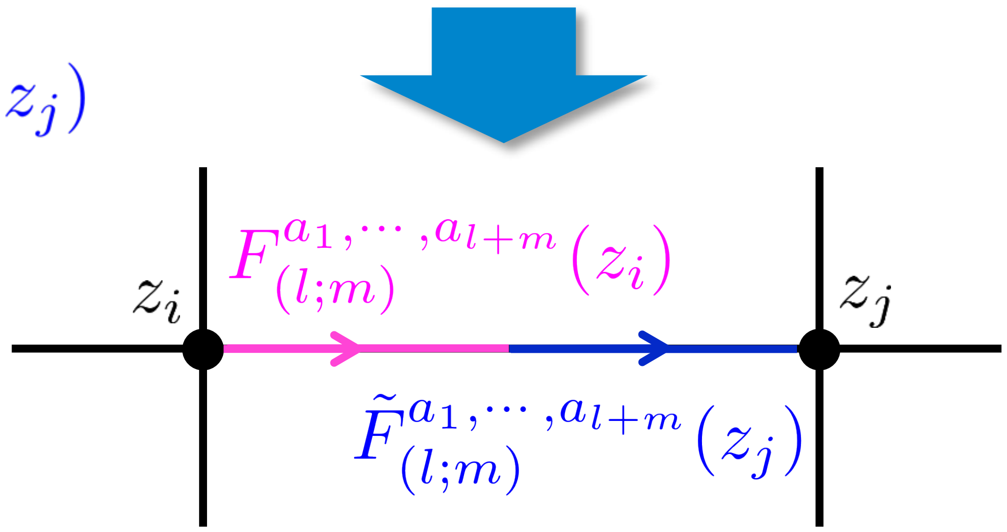


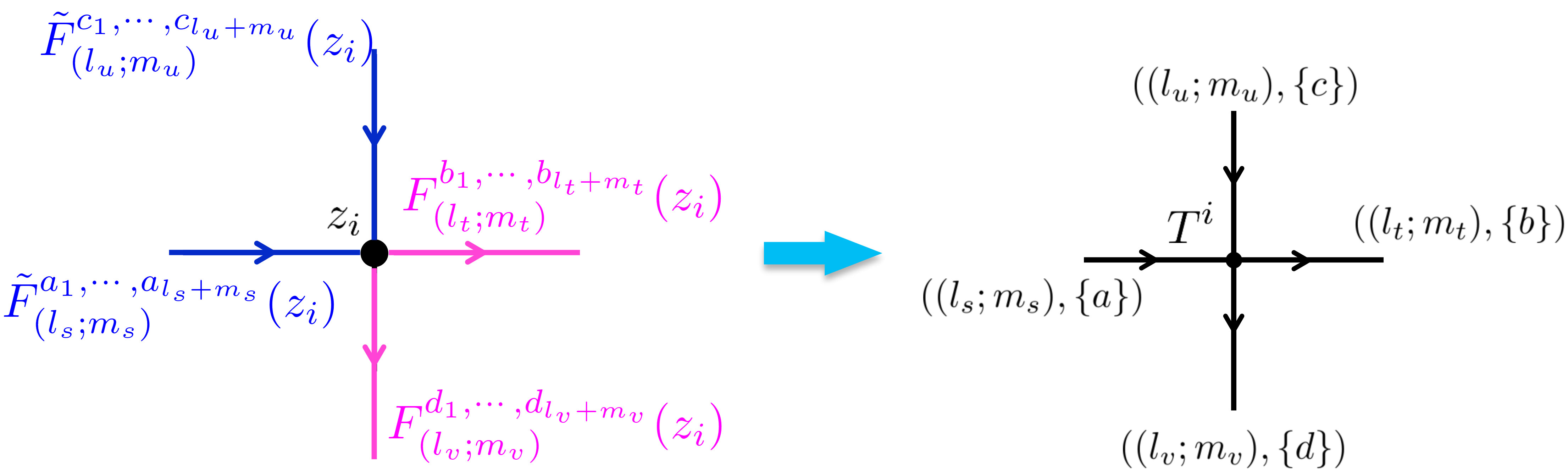
$$f_{(l;m)}(z_i, z_j)$$
$$= \sum_{\{a\}} F_{(l;m)}^{a_1,\cdots,a_{l+m}}(z_i)\tilde{F}_{(l;m)}^{a_1,\cdots,a_{l+m}}(z_j)$$



CP(N-1) model in two dimensions

Integrate out z

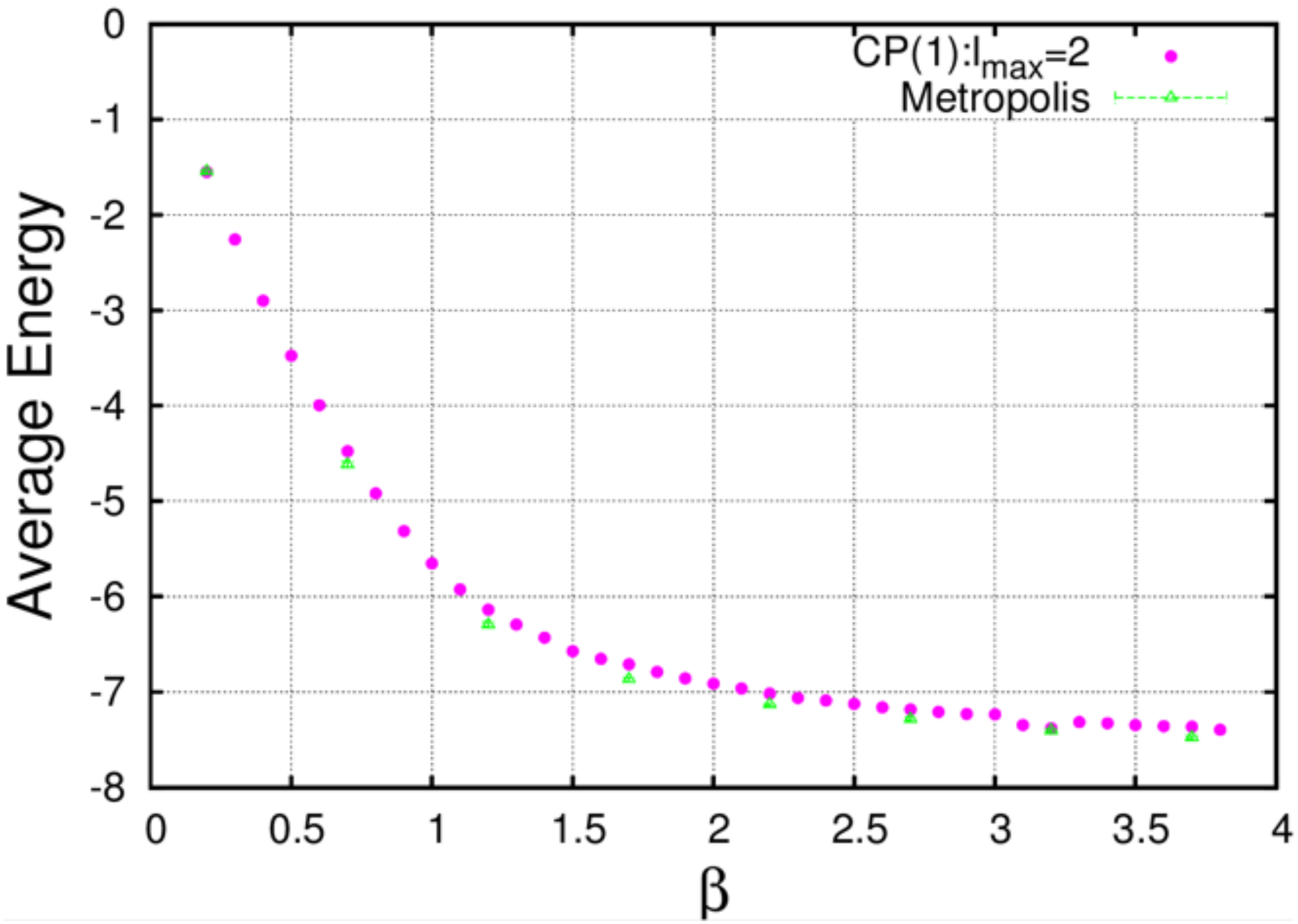
$\theta = 0$



$$T_{((l_s;m_s),\{a\})((l_t;m_t),\{b\})((l_u;m_u),\{c\})((l_v;m_v),\{d\})}$$
$$= \int dz_i dz_i^*$$
$$\sqrt{d_{(l_s;m_s)}d_{(l_t;m_t)}d_{(l_u;m_u)}d_{(l_v;m_v)}h_{(l_s;m_s)}(\beta)h_{(l_t;m_t)}(\beta)h_{(l_u;m_u)}(\beta)h_{(l_v;m_v)}(\beta)}$$
$$\tilde{F}^{a_1,\cdots,a_{l_s+m_s}}_{(l_s;m_s)}(z_i)F^{b_1,\cdots,b_{l_t+m_t}}_{(l_t;m_t)}(z_i)\tilde{F}^{c_1,\cdots,c_{l_u+m_u}}_{(l_u;m_u)}(z_i)F^{d_1,\cdots,d_{l_v+m_v}}_{(l_v;m_v)}(z_i).$$

CP(1) model in two dimensions

HOTRG vs. Monte Carlo (Metropolis)



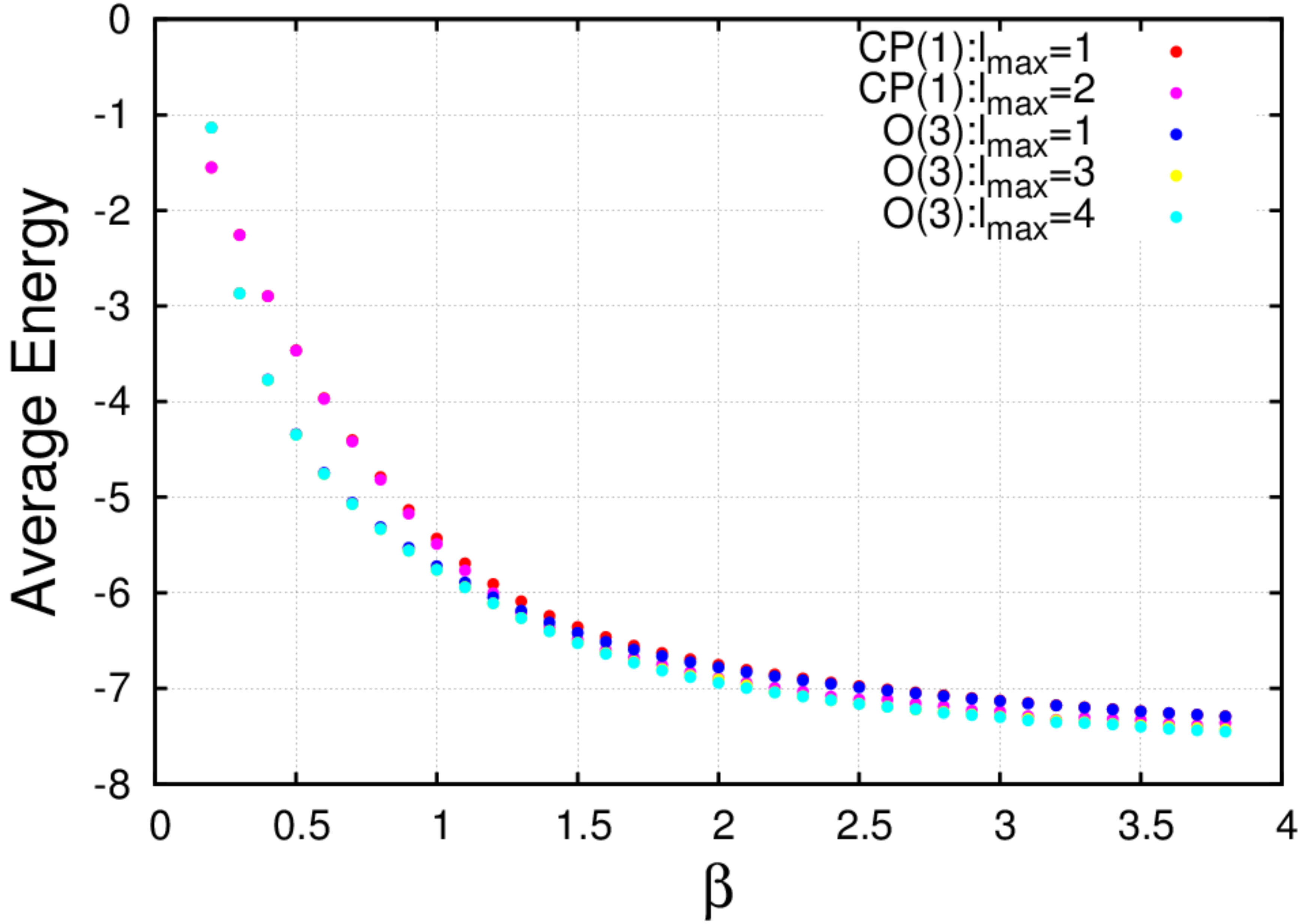
$L = 4$

CP(1) model vs. O(3) model

J. Unmuth-Yockey, Y. Meurice, J. Osborn and H. Zou, arXiv:1411.4213 [hep-lat].

In the limit $\beta \to \infty$,

$$\frac{1}{\beta} + E_{O(3)}(\beta) = E_{CP(1)}(\beta) + 6$$



$$L = 2^{20}$$

Summary

- We derive a tensor network representation of CP(N-1) model (with the θ term).

TRG CP(1) ⟷ TRG O(3) ($\beta \gg 1$)

↕

MC CP(1)

- Next, we will do implementation including the θ term.