Gauge-invariant implementation of the $U(1)$-Higgs model on optical lattices

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with

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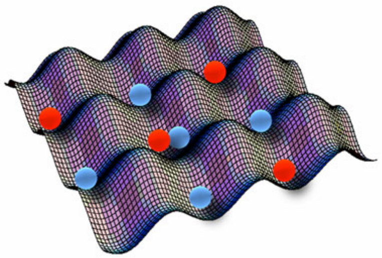
Motivation



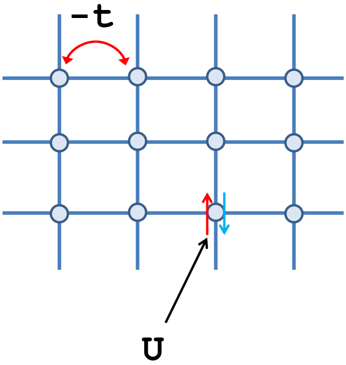
- Technology: Ultra-cold atoms trapped in optical lattices (counter propagating laser beams)[1]
- Possibility of tunable interactions
- Goal: Quantum simulator for lattice gauge theory
  - Explicit gauge fields, inexact gauge invariance (most of the proposals)
  - Gauge-invariant effective actions (our proposal)

[1] Picture courtesy of JILA

Fermi-Hubbard model

$$H = -t \sum_{\langle ij\rangle,\alpha} (c^\dagger_{i,\alpha} c_{j,\alpha} + h.c.) + U \sum_{i=1}^{N} n_{i\uparrow} n_{i\downarrow}$$

- t – tunneling between nearest neighbors
- U – onsite Coulomb repulsion



Fermi-Hubbard model

- In the strong-coupling limit $U \gg t$ at half-filling Fermi-Hubbard model is equivalent to spin-1/2 quantum Heisenberg model

$$H = J \sum_{\langle ij \rangle} \mathbf{S}_i \cdot \mathbf{S}_j, \quad J = 4t^2/U$$

- Introduce new fermion fields $f_{i\alpha}$, $\mathbf{S}_i = \frac{1}{2} f^\dagger_{i\alpha} \vec{\sigma}_{\alpha\beta} f_{i\beta}$ (with a constraint $f^\dagger_{i\alpha} f_{i\alpha} = 1$):

$$H = -\frac{1}{2} J \sum_{\langle ij \rangle} f^\dagger_{i\alpha} f_{j\alpha} f_{j\beta} f^\dagger_{i\beta} + J \sum_{\langle ij \rangle} \left(\frac{1}{2} n_i - \frac{1}{4} n_i n_j \right)$$

- This model has a local $SU(2)$ symmetry[2]

[2] Affleck, Zou, Hsu, Anderson (1988), Dagotto, Fradkin, Moreo (1988)

Fermi-Hubbard model

- After particle-hole transformation in the spin-down operator:

$$f_{i,\uparrow}, f^\dagger_{i,\uparrow} \to \Psi_{x,1}, \Psi^\dagger_{x,1}, \quad f_{i,\downarrow}, f^\dagger_{i,\downarrow} \to \Psi^\dagger_{x,2}, \Psi_{x,2}$$

  the Heisenberg Hamiltonian can be written in terms of "mesons" and "baryons"

$$H = \frac{J}{8} \sum_{x,\hat{i}} \left[M_x M_{x+\hat{i}} + 2 \left(B^\dagger_x B_{x+\hat{i}} + B^\dagger_{x+\hat{i}} B_x \right) \right] - \frac{Jd}{4} \sum_x \left(M_x - \frac{1}{2} \right)$$

$$M_x = \sum_{a=1,2} \Psi^\dagger_{x,a} \Psi_{x,a}, \quad B_x = \sum_{a,b=1,2} \frac{\varepsilon_{ab}}{2} \Psi_{x,a} \Psi_{x,b} = \Psi_{x,1} \Psi_{x,2}$$

$U(1)$-Higgs model in 1+1D

The partition function:

$$Z = \int D\phi^\dagger D\phi D U e^{-S},$$

$$S = S_g + S_h + S_\lambda,$$

$$S_g = -\beta_{pl} \sum_x \mathrm{Re}\left[U_{pl,x}\right],$$

$$\begin{aligned} S_h = & - \kappa_\tau \sum_x \left[\mathrm{e}^{\mu} \phi_x^\dagger U_{x,\hat{\tau}} \phi_{x+\hat{\tau}} + \mathrm{e}^{-\mu} \phi_{x+\hat{\tau}}^\dagger U_{x,\hat{\tau}}^\dagger \phi_x \right] \\ & - \kappa_s \sum_x \left[\phi_x^\dagger U_{x,\hat{s}} \phi_{x+\hat{s}} + \phi_{x+\hat{s}}^\dagger U_{x,\hat{s}}^\dagger \phi_x \right], \end{aligned}$$

$$S_\lambda = \lambda \sum_x \left(\phi_x^\dagger \phi_x - 1 \right)^2 + \sum_x \phi_x^\dagger \phi_x.$$

$U(1)$-Higgs model in 1+1D

Limiting cases (isotropic, $\kappa_s = \kappa_\tau = \kappa$):

- $\kappa = 0$: $U(1)$ pure gauge theory
- $\lambda < \infty, \beta_{pl} = \infty$: ϕ^4 theory
- $\lambda = \infty, \beta_{pl} = \infty$: $O(2)$ model, Kosterlitz-Thouless transition

$U(1)$-Higgs model in 1+1D

- In the hopping part of the action S_h, we can separate the compact and non-compact variables

$$\begin{aligned} S_h = & - & 2\kappa_\tau |\phi_x||\phi_{x+\hat{\tau}}| \sum_x \cos(\theta_{x+\hat{\tau}} - \theta_x + A_{x,\hat{\tau}} - i\mu) \\ & - & 2\kappa_s |\phi_x||\phi_{x+\hat{s}}| \sum_x \cos(\theta_{x+\hat{s}} - \theta_x + A_{x,\hat{s}}) \end{aligned}$$

and then Fourier transform the Boltzmann weight, *i.e.*

$$\begin{aligned} & \exp[2\kappa_\tau |\phi_x||\phi_{x+\hat{\tau}}| \cos(\theta_{x+\hat{\tau}} - \theta_x + A_{x,\hat{\tau}} - i\mu)] \\ = & \sum_{n=-\infty}^{\infty} I_n(2\kappa_\tau |\phi_x||\phi_{x+\hat{\tau}}|) \exp[in(\theta_{x+\hat{\tau}} - \theta_x + A_{x,\hat{\tau}} - i\mu)] \end{aligned}$$

$U(1)$-Higgs model in 1+1D

- After integrating over the gauge field, the effective action for the gauge and hopping part

$$e^{-S_{eff}} = \sum_{\{m_\Box\}} \left[\prod_{\Box} I_{m_\Box}(\beta_{pl}) \prod_x \left(I_{n_{x,\hat{s}}}(2\kappa_s|\phi_x||\phi_{x+\hat{s}}|) \right.\right.$$
$$\left.\left. \times I_{n_{x,\hat{\tau}}}(2\kappa_\tau|\phi_x||\phi_{x+\hat{\tau}}|)\exp(\mu n_{x,\hat{\tau}}) \right) \right]$$

- Using the hopping parameter expansion for $\kappa = \kappa_s = \kappa_\tau$ and with $M_x \equiv \phi_x^\dagger \phi_x$:

$$S_{eff} = \sum_{\langle xy \rangle} \left(-\kappa^2 M_x M_y + \frac{1}{4}\kappa^4 (M_x M_y)^2 \right)$$
$$-2\kappa^4 \frac{I_1(\beta_{pl})}{I_0(\beta_{pl})} \sum_{\Box(xyzw)} M_x M_y M_z M_w + O(\kappa^6)$$

$$Z = \int D\phi^\dagger D\phi D U e^{-S} \simeq \int DM e^{-S_{eff}(M) - S_\lambda(M)}$$

Hopping parameter expansion

- The expectation value of the hopping term

$$\langle L_\phi \rangle = \langle \mathrm{Re}\{\phi_x^\dagger U_{x,\hat{\nu}} \phi_{x+\hat{\nu}}\}\rangle$$

- For $\beta_{pl} \to \infty$ one can relate

$$\langle L_\phi \rangle_{Z_{\kappa,\lambda}} = \frac{1}{2V} \frac{d}{d\kappa} \ln\left(\frac{Z_{\kappa,\lambda}}{Z_\lambda}\right)$$

where $Z_\lambda \equiv Z_{\kappa=0,\lambda}$

Hopping parameter expansion

- The hopping parameter expansion[3]

$$
\begin{aligned}
\frac{Z_{\kappa,\lambda}}{Z_\lambda} &= 1 + Vd\gamma_2^2\kappa^2 \\
&+ Vd\left\{\left[\frac{1}{2}(Vd-4d+1)+(d-1)\right]\gamma_2^4+(2d-1)\gamma_2^2\gamma_4+\frac{1}{4}\gamma_4^2\right\}\kappa^4 \\
&+ Vd\left\{\left[\frac{1}{6}(Vd-1)(Vd-2)-\frac{2}{3}(d-1)(2d-1)-(2d-1)^2-(2d-1)(Vd-6d+2)\right.\right. \\
&+ 2(d-1)(2d-3)+(d-1)(Vd-8d+4)]\,\gamma_2^6 \\
&+ (2(d-1)+(2d-1)^2+\frac{1}{4}(Vd-4d+1))\gamma_2^2\gamma_4^2 \\
&+ 8(d-1)^2\gamma_2^2\gamma_4+(2d-1)(Vd-6d+2)\gamma_2^4\gamma_4+\frac{2}{3}(2d-1)(d-1)\gamma_2^3\gamma_6 \\
&+ \left.\frac{1}{2}(2d-1)\gamma_2\gamma_4\gamma_6+\frac{1}{36}\gamma_6^2\right\}\kappa^6
\end{aligned}
$$

where $\gamma_{2k} \equiv \langle \rho^{2k} \rangle_{Z_\lambda}$.

[3] Heitger (1997)

Hopping parameter expansion

- The hopping parameter expansion[4]

$$
\begin{aligned}
\frac{Z_{\kappa,\lambda}}{Z_\lambda} &= 1 + Vd\gamma_2^2\kappa^2 \\
&+ Vd\left\{\left[\frac{1}{2}(Vd-4d+1)+(d-1)\right]\gamma_2^4 + (2d-1)\gamma_2^2\gamma_4 + \frac{1}{4}\gamma_4^2\right\}\kappa^4 \\
&+ Vd\left\{\left[\frac{1}{6}(Vd-1)(Vd-2) - \frac{2}{3}(d-1)(2d-1) - (2d-1)^2 - (2d-1)(Vd-6d+2)\right.\right. \\
&+ 2(d-1)(2d-3) + (d-1)(Vd-8d+4) \\
&+ \left.\frac{4}{3}(d-1)(d-2)\right]\gamma_2^6 + (2(d-1)+(2d-1)^2 + \frac{1}{4}(Vd-4d+1))\gamma_2^2\gamma_4^2 \\
&+ (8(d-1)^2 + (2d-1)(Vd-6d+2))\gamma_2^4\gamma_4 + \frac{2}{3}(2d-1)(d-1)\gamma_2^3\gamma_6 \\
&+ \left.\frac{1}{2}(2d-1)\gamma_2\gamma_4\gamma_6 + \frac{1}{36}\gamma_6^2\right\}\kappa^6
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$$

where $\gamma_{2k} \equiv \langle \rho^{2k} \rangle_{Z_\lambda}$.

[4] Heitger (1997)

Hopping parameter expansion

- The hopping parameter expansion[5]

$$
\begin{aligned}
\frac{Z_{\kappa,\lambda}}{Z_\lambda} &= 1 + Vd\gamma_2^2\kappa^2 \\
&+ Vd\left\{\left[\frac{1}{2}(Vd-4d+1)+(d-1)\frac{I_1(\beta_{pl})}{I_0(\beta_{pl})}\right]\gamma_2^4+(2d-1)\gamma_2^2\gamma_4+\frac{1}{4}\gamma_4^2\right\}\kappa^4 \\
&+ Vd\left\{\left[\frac{1}{6}(Vd-1)(Vd-2)-\frac{2}{3}(d-1)(2d-1)-(2d-1)^2-(2d-1)(Vd-6d+2)\right.\right. \\
&+ 2(d-1)(2d-3)\left(\frac{I_1(\beta_{pl})}{I_0(\beta_{pl})}\right)^2+(d-1)(Vd-8d+4)\frac{I_1(\beta_{pl})}{I_0(\beta_{pl})} \\
&+ \left.\frac{4}{3}(d-1)(d-2)\left(\frac{I_1(\beta_{pl})}{I_0(\beta_{pl})}\right)^3\right]\gamma_2^6+(2(d-1)\frac{I_1(\beta_{pl})}{I_0(\beta_{pl})}+(2d-1)^2+\frac{1}{4}(Vd-4d+1))\gamma_2^2\gamma_4^2 \\
&+ (8(d-1)^2\frac{I_1(\beta_{pl})}{I_0(\beta_{pl})}+(2d-1)(Vd-6d+2))\gamma_2^4\gamma_4+\frac{2}{3}(2d-1)(d-1)\gamma_2^3\gamma_6 \\
&+ \left.\frac{1}{2}(2d-1)\gamma_2\gamma_4\gamma_6+\frac{1}{36}\gamma_6^2\right\}\kappa^6
\end{aligned}
$$

where $\gamma_{2k} \equiv \langle\rho^{2k}\rangle_{Z_\lambda}$.

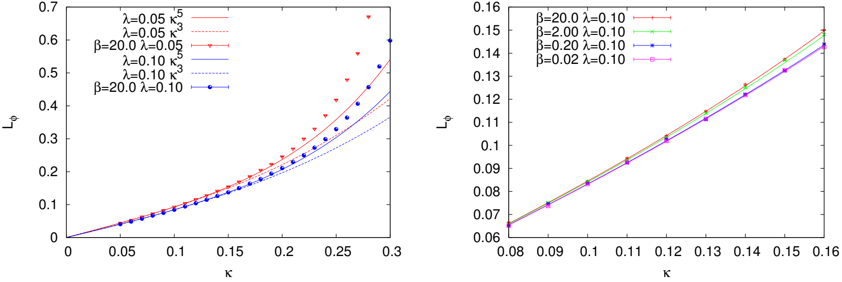
[5] Heitger (1997)

MC tests at small λ

- At small λ the quartic term is a perturbation, Gaussian proposal probability can be used[6]
- To scan arbitrary λ – Biased-Metropolis Heatbath[7] (ongoing work)

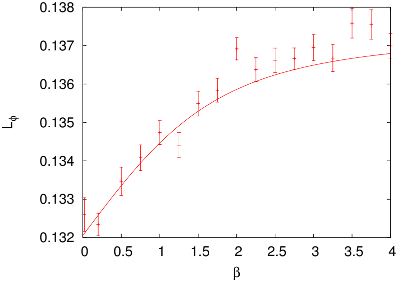
[6] Bunk (1997)
[7] Bazavov, Berg (2005)

MC tests at small λ



- Left: L_ϕ at $\lambda = 0.05$ and 0.1 for $\beta = 20$ compared with the $O(\kappa^3)$ and $O(\kappa^5)$ expansions
- Right: L_ϕ at $\lambda = 0.1$ for $\beta = 0.02 - 20$ compared with the $O(\kappa^5)$ expansion

MC tests at small λ



- L_ϕ for $\kappa = 0.15$ and $\lambda = 0.1$ as function of β_{pl} compared with the hopping expansion with included dependence on β_{pl} up to $O(\kappa^5)$

$O(2)$ limit

- For $\kappa_s \ll \kappa_\tau$, $\lambda = \infty$ one can obtain the time continuum limit and the Hamiltonian for the quantum rotors is[8]

$$\hat{H} = \frac{\tilde{U}}{2} \sum_x \hat{L}_x^2 - \tilde{\mu} \sum_x \hat{L}_x - \tilde{J} \sum_{\langle xy \rangle} \cos(\hat{\theta}_x - \hat{\theta}_y) ,$$

with $\hat{L} = -i\partial/\partial\theta$, $\tilde{U} = 1/(2\kappa_\tau a)$, $\tilde{\mu} = \mu/a$ and $\tilde{J} = \kappa_s/a$

- The commutation relations are

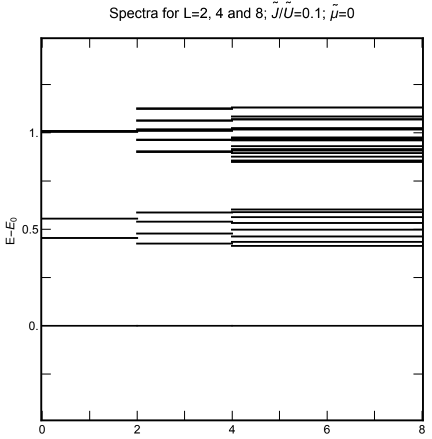
$$[L, \mathrm{e}^{\pm i\hat{\theta}}] = \pm \mathrm{e}^{\pm i\hat{\theta}}$$

- We consider spin-1 truncation and represent this algebra with the angular momentum algebra

$$[\hat{L}^z, \hat{L}^\pm] = \pm \hat{L}^\pm$$

[8] Zou et al., PRA 90, 063603 [1403.5238]

$O(2)$ limit



- The spectrum of the spin-1 Hamiltonian can be calculated with the tensor renormalization group (TRG) blocking
- The spectra for 2, 4 and 8 site chain are shown

$U(1)$-Higgs model

- In the $1 << \beta_{pl} << \kappa_\tau$, $\lambda = \infty$ limit the Hamiltonian for the $U(1)$-Higgs model can also be constructed with TRG[9]:

$$\bar{H} = \frac{\tilde{U}_P}{2} \sum_i \left(\bar{L}^z_{(i)} \right)^2 + \frac{\tilde{Y}}{2} {\sum_i}' (\bar{L}^z_{(i)} - \bar{L}^z_{(i+1)})^2 - \tilde{X} \sum_i \bar{L}^x_{(i)} \ ,$$

where $\tilde{U}_P \equiv 1/a\beta_{pl}$, $\tilde{Y} \equiv 1/2\kappa_\tau a = \beta_{pl}/(2\kappa_\tau)\tilde{U}_P$ and $\tilde{X} \equiv \sqrt{2}\beta_{pl}\kappa_s \tilde{U}_P$

[9] Bazavov, Meurice, Tsai, Unmuth-Yockey, Zhang, 1503.08354

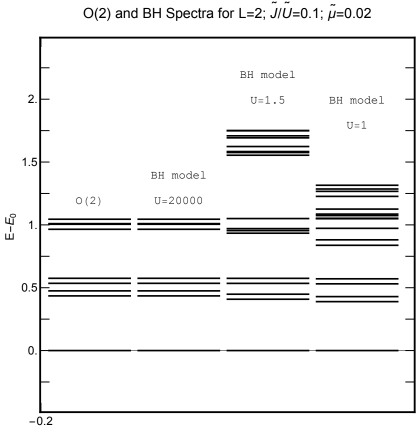
Bose-Hubbard realization for the $O(2)$ model

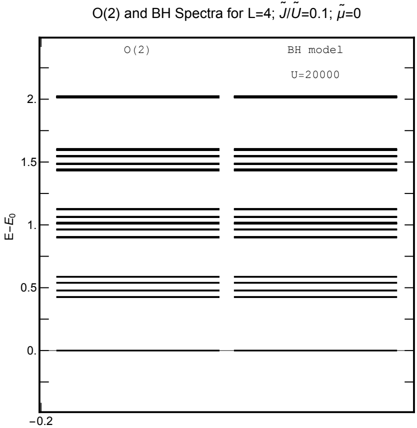
- The two-species Bose-Hubbard Hamiltonian

$$
\begin{aligned}
\mathcal{H} &= -\sum_{\langle ij\rangle}(t_a a_i^\dagger a_j + t_b b_i^\dagger b_j + h.c.) - \sum_{i,\alpha}(\mu_{a+b} + \Delta_\alpha) n_i^\alpha \\
&+ \sum_{i,\alpha} \frac{U_\alpha}{2} n_i^\alpha (n_i^\alpha - 1) + W \sum_i n_i^a n_i^b + \sum_{\langle ij\rangle \alpha} V_\alpha n_i^\alpha n_j^\alpha
\end{aligned}
$$

- When $U_a = U_b = U$, W and $\mu_{a+b} = (3/2)U$ are much larger than any other energy scale, the condition $n_i^a + n_i^b = 2$ is satisfied and the three states $|2,0\rangle$, $|1,1\rangle$ and $|0,2\rangle$ correspond to the three states of the spin-1 projection.
- By using the second-order perturbation theory we can match the BH Hamiltonian with the $O(2)$ one with $t_\alpha = \sqrt{V_\alpha U/2}$, $\tilde{J} = 4\sqrt{V_a V_b}$, $\tilde{U} = 2(U - W)$, and $\tilde{\mu} = -(\Delta_a - V_a) + (\Delta_b - V_b)$

Bose-Hubbard realization for the $O(2)$ model





- Left: The $O(2)$ (spin-1 truncation) and BH spectra for a 2 site chain
- Right: The $O(2)$ (spin-1 truncation) and BH spectra for a 4 site chain

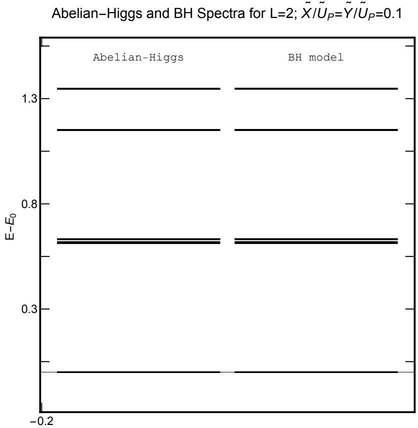
Bose-Hubbard realization for the $U(1)$-Higgs model

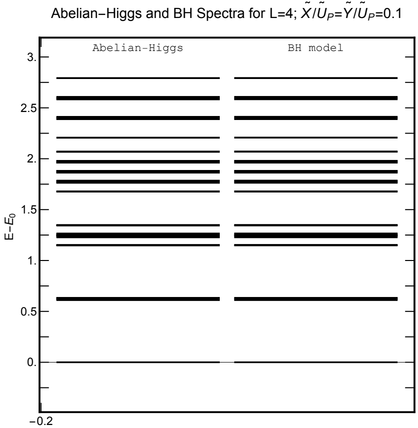
- To include the effects of $\hat{L}^x$ we need to add

$$\Delta H = -\frac{t_{ab}}{2} \sum_i (a_i^\dagger b_i + b_i^\dagger a_i)$$

- The matching can be achieved by imposing $t = 0$, $V_a = V_b = -\tilde{Y}/2$ and $t_{ab} = \tilde{X}$

$U(1)$-Higgs model





- Left: The $U(1)$-Higgs (spin-1 truncation, $\lambda = \infty$) and BH spectra for a 2 site chain
- Right: The $U(1)$-Higgs (spin-1 truncation, $\lambda = \infty$) and BH spectra for a 4 site chain

Conclusion

- The $U(1)$-Higgs model in two limits
  - $\lambda = \infty$, $\beta_{pl} = \infty$
  - $\lambda = \infty$

  can be mapped onto a two-species Bose-Hubbard model
- These two realizations are different and we made two proposals for optical lattice experiments (1403.5238, 1503.08354)
- Next steps:
  - fermions (Schwinger model)
  - non-Abelian gauge field