The Wilson flow in scalar field theory

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Motivation

- Renormalisation of energy-momentum tensor for scalar theory
- EMT relates to β-function

$$\langle \int d^D x \, T_{\mu\mu} \, \phi(x_1)...\phi(x_n) \rangle$$
$$= - \left(\sum_k \beta_k \frac{\partial}{\partial g_k} + n(\gamma_\phi + d_\phi) \right) \langle \phi(x_1)...\phi(x_n) \rangle$$

- ϕ^4-theory in 3 dimensions: toy model for theories with IR fixed point

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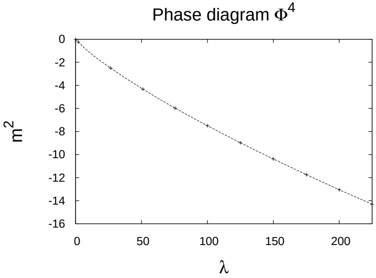
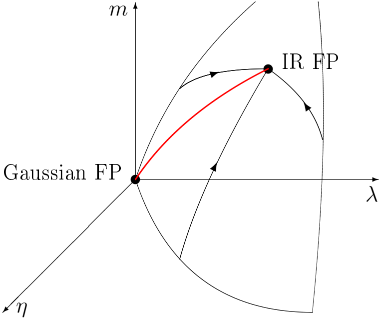
- ϕ^4-theory in 3 dimensions: toy model for theories with IR fixed point





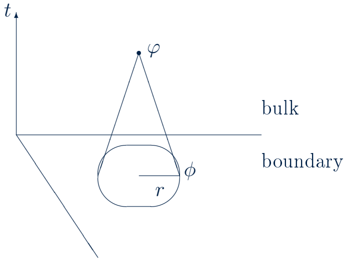
Gradient flow in ϕ^4

- Euclidean action

$$S = \int d^D x \, \left(\frac{1}{2} (\partial_\mu \phi)^2 + \frac{1}{2} m^2 \phi^2 + \frac{\lambda}{4!} \phi^4 \right)$$

- Flow equation [*Mohanan,Orginos 2014*]

$$\partial_t \varphi(t,x) = \partial^2 \varphi(t,x), \qquad \varphi(t,x)|_{t=0} = \phi(x)$$



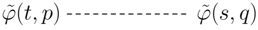
- Smoothing effect, radius $r = \sqrt{8t}$

Field Theory in D+1 dimensions

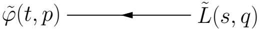
- Action [*Lüscher,Weisz 2011*]

$$S = S_{\text{boundary}} + S_{\text{bulk}}$$
$$S_{\text{bulk}} = \int_0^\infty dt \int d^D x \, L(t,x) \left(\partial_t - \partial^2\right) \varphi(t,x)$$

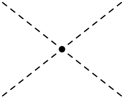
- Feynman rules



$$\frac{1}{p^2+m^2} e^{-(t+s)p^2}$$



$$\theta(t-s) e^{-(t-s)p^2}$$

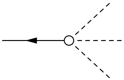


$$-\lambda$$

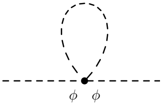
Gradient flow and perturbation theory

- Flow equation as gradient of action

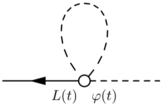
$$\partial_t \varphi = \partial^2 \varphi - m^2 \varphi - \frac{\lambda}{3!} \varphi^3$$

- Additional Feynman rule  $-\lambda$
- Mass in damping factor $e^{-(t+s)(p^2+m^2)}$

- Self energy of the bulk field φ in 4 dimensions



$\propto \frac{1}{\epsilon}$



$\propto \frac{\delta(t)}{\epsilon}$

- Divergence canceled by multiplicative renormalisation of φ, L
- BUT: theory non-renormalisable for non-zero couplings in bulk action

Energy-momentum tensor and Ward identities

- Translational Ward identity

$$\langle\, \delta_{x,\rho} P \,\rangle = -\langle\, P \, \partial_\mu T_{\mu\rho}(x) \,\rangle$$

- Local operator of translation

$$\delta_{x,\rho} P = \frac{\delta P}{\delta \phi(x)} \, \partial_\rho \phi(x)$$

- Comparison gauge theory

$$\delta_{x,\rho} P \;=\; \frac{\delta P}{\delta A^a_\mu(x)} \, F^a_{\rho\mu}(x)$$

Wilson flow - gradient flow on the lattice

- Lattice action

$$\hat{S} = a^D \sum_n \left(\frac{1}{2} (\hat{\partial}_\mu \phi)^2 + \frac{1}{2} m^2 \phi^2 + \frac{\lambda}{4!} \phi^4 \right)$$

- Flow equation

$$\partial_t \varphi(t,x) = \hat{\partial}^2 \varphi(t,x), \qquad \varphi(t,x)|_{t=0} = \phi(x)$$

$$\left(\partial_t + \hat{k}^2 \right) \tilde{\varphi}(t,k) = 0$$

- Flow equation implemented by numerical integration

Algorithm

- Two parts: [*Brower, Tamayo 1989*]
  - Metropolis update
  - Swendsen-Wang update: cluster algorithm, not ergodic
- Autocorrelation time

λ	m	L	$\tau(\mathrm{S})$	$\tau(\varphi^2)$
2	0.4726	8	1.02	2.00
2	0.4741	28	1.05	2.00
100	2.7219	8	1.02	1.00
100	2.7245	28	1.02	1.00
175	3.4118	8	1.02	1.00
175	3.4137	28	1.04	1.00

  - Measurements every 200 Metropolis and 40 Cluster
  - Simulation time very short, $L=8$: 1h for 75000 measurements, $L=28$: 8h for 33000 measurements
  - Autocorrelation time monitored for all observables
  - Always small: order 1, measurements uncorrelated

Translational Ward identity on the lattice

- Lattice regularisation breaks translation symmetry explicitly

$$\langle \hat{\delta}_{x,\rho} \hat{P} \rangle = -\langle \hat{P} \left(\hat{\partial}_\mu \hat{T}_{\mu\rho} + \hat{R}_\rho \right) \rangle$$

  - $\hat{R}_\rho$ gives finite contributions in expectation values
  - EMT requires renormalisation

- Renormalised lattice TWI

$$\langle Z_\delta \, \hat{\delta}_{x,\rho} \hat{P} \rangle = -\langle \hat{P} \left(\hat{\partial}_\mu [\hat{T}_{\mu\rho}] + [\hat{R}_\rho] \right) \rangle$$

- Renormalised $\hat{T}_{\mu\rho}$

$$[\hat{T}_{\mu\rho}] = c_1 \hat{\partial}_\mu \phi \hat{\partial}_\rho \phi - \delta_{\mu\rho} \left(\frac{c_2}{2} \sum_\lambda \hat{\partial}_\lambda \phi \hat{\partial}_\lambda \phi \right.$$
$$\left. + \frac{c_3}{2} \hat{\partial}_\mu \phi \hat{\partial}_\mu \phi + \frac{c_4}{4!} \phi^3 \hat{\partial}_\mu \phi + \frac{c_5}{6!} \phi^6 + \frac{c_6}{2} \phi \hat{\partial}_\mu \phi + \frac{c_7}{4!} \phi^4 + \frac{c_8}{2} \phi^2 \right)$$

Comparison of renormalised EMT

$$[\hat{T}_{\mu\rho}(x)] = \sum_i c_i \left\{ \hat{T}^{(i)}_{\mu\rho} - \langle \hat{T}^{(i)}_{\mu\rho} \rangle \right\}$$

- Gauge theory: 3 operators mix

$$\hat{T}^{[1]}_{\mu\rho} = -\delta_{\mu\rho} \frac{1}{2g_0^2} \mathrm{tr} \hat{F}_{\alpha\beta} \hat{F}_{\alpha\beta}$$

$$\hat{T}^{[3]}_{\mu\rho} = \delta_{\mu\rho} \frac{2}{g_0^2} \left[\frac{1}{4} \mathrm{tr} \hat{F}_{\alpha\beta} \hat{F}_{\alpha\beta} - \mathrm{tr} \hat{F}_{\mu\alpha} \hat{F}_{\mu\alpha} \right]$$

$$\hat{T}^{[6]}_{\mu\rho} = (\delta_{\mu\rho} - 1) \frac{2}{g_0^2} \mathrm{tr} \hat{F}_{\mu\alpha} \hat{F}_{\rho\alpha}$$

- Scalar theory: 8 operators $\hat{T}^{(i)}_{\mu\rho}$, proportional to

$$\hat{\partial}_\mu \phi \hat{\partial}_\rho \phi$$

$$\propto \delta_{\mu\rho} : \quad \sum_\lambda \hat{\partial}_\lambda \phi \hat{\partial}_\lambda \phi, \quad \hat{\partial}_\mu \phi \hat{\partial}_\mu \phi, \quad \phi^3 \hat{\partial}_\mu \phi, \quad \phi^6, \quad \phi \hat{\partial}_\mu \phi, \quad \phi^4, \quad \phi^2$$

Renormalisation of the EMT

- Renormalisation condition [*Del Debbio,Patella,Rago 2013*]
  - Choose probe $\hat{P}_T$: function of fields at $T > 0$, then:
  - Coefficients c_i can be tuned such that the EMT is finite
  - $[\hat{R}_\rho] \to 0$

$$Z_\delta \langle \hat{\delta}_{x,\rho} \hat{P}_T \rangle = -\sum_i c_i \langle \hat{P}_T \hat{\partial}_\mu \hat{T}^{(i)}_{\mu\rho}(x) \rangle$$

- Determination Z_δ (test)

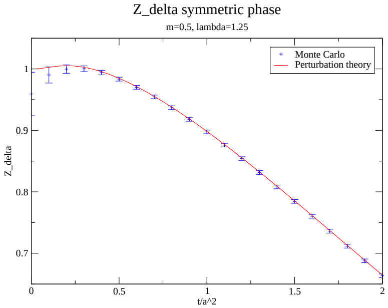
$$Z_\delta \langle \varphi_z a^D \sum_{y \in V} \hat{\delta}_{y,\rho} \varphi_x \rangle = \langle \varphi_z \hat{\partial}_\rho \varphi_x \rangle + \mathcal{O}\left(\mathrm{e}^{-\frac{r^2}{16t}} \right)$$

- Determination c_i: system of 8 equations with 8 different operators $P^{(k)}$

$$Z_\delta V^{(k)} = -\sum_i c_i M^{(k,i)}$$

Calculation of Z_δ

- Agreement MCS and lattice perturbation theory



- Perturbative formula

$$RHS(2t) = -\frac{i}{L^d} \sum_p e^{ip(z-x)} \frac{\hat{p}_\rho}{m^2 + \hat{p}^2} \; e^{-2t\hat{p}^2 - iap_\rho/2} \; (1 + \mathcal{O}(\lambda))$$

$$LHS = Z_\delta \, a^d \sum_{y \in D} J(t, x; 0, y) \, RHS(t)$$

Summary

- Powerful tool for non-perturbative study of QFTs
- High precision measurements
- Define renormalised EMT on the lattice in a meaningful way
- Scaling behaviour, in particular of theories with an IR fixed point

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Thank you!

Susceptibility

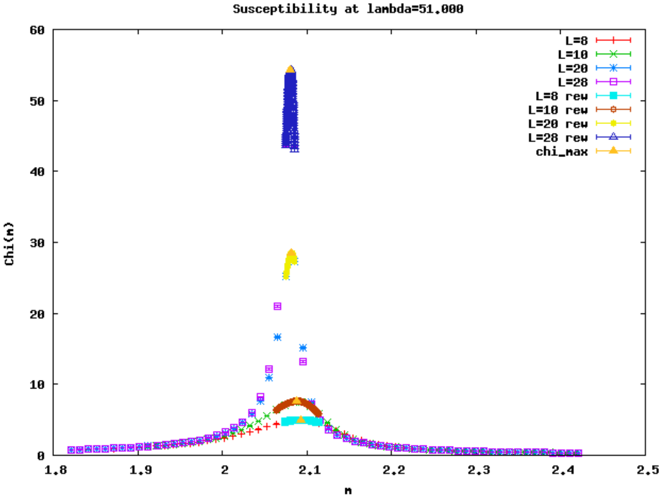

Figure: Susceptibility at $\lambda = 51$ for different lattice sizes

Susceptibility

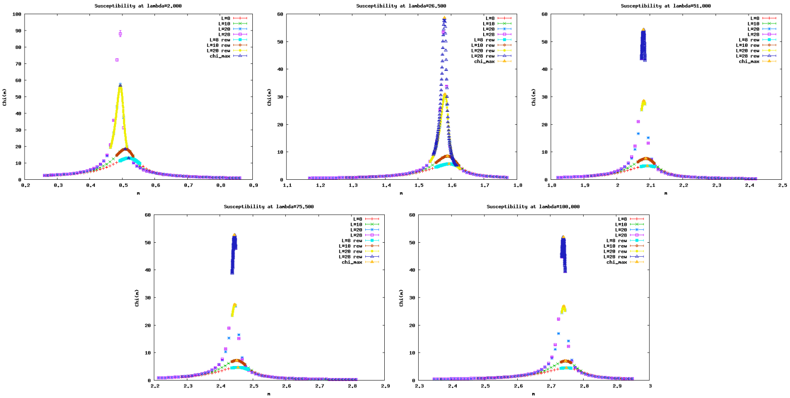

Figure: Susceptibility at $\lambda = 2,\ 26.5,\ 51,\ 75.5,\ 100$ for different lattices sizes

Susceptibility

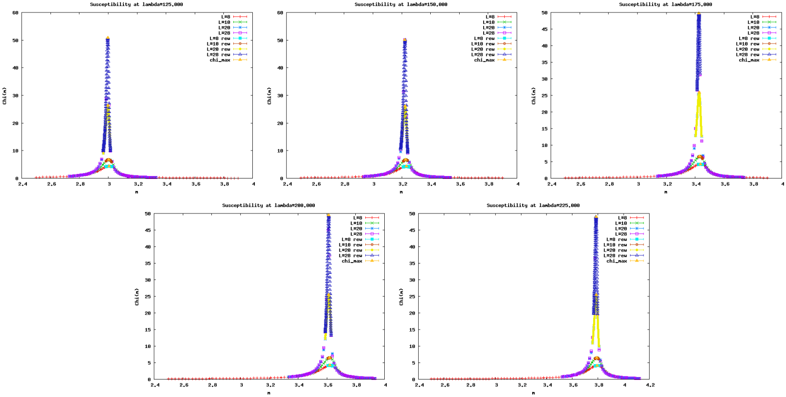

Figure: Susceptibility at $\lambda = 125,\ 150,\ 175,\ 200,\ 225$ for different lattice sizes