Numerical study of complex instantons in the Gross-Witten U(N) matrix model

S. Valgushev, P. Buividovich

Universität Regensburg

Lattice 2015, Kobe, Japan

Gross-Witten matrix model is defined as follows:

$$\mathcal{Z} = \int\limits_{U(N)} dW \exp\left[\frac{N}{\lambda}\mathrm{Tr}(W + W^{-1})\right] \qquad \lambda = g^2 N$$

Model describes one-plaquette world in 2 dimension.

There is a 3rd order phase transition in the limit of infinitely large N:

Phys.Rev.D 21 446

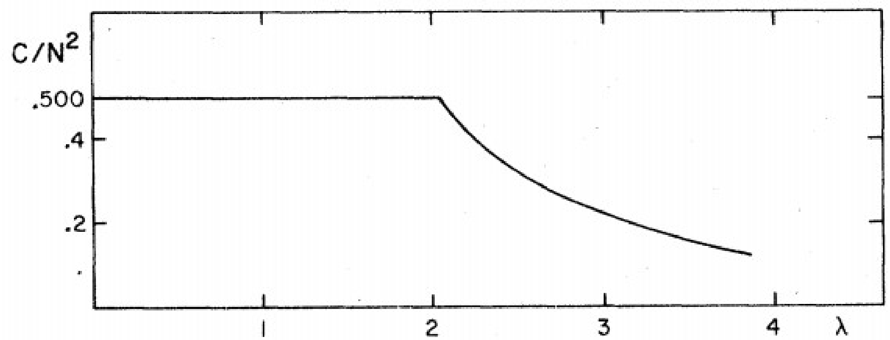


FIG. 2. The specific heat per degree of freedom, C/N^2, as a function of λ (temperature).

There are weak and strong couping regimes separated by transition point $\lambda = 2$.

The 1/N expansion was extensively studied.

It was argued that phase transition is caused by condesation of instantons.
(Neuberger, Nucl.Phys. B 179 253-282)

Model has rich physical content and it is exactly solvable:

$$\mathcal{Z} = \det(I_{j-i}(2N/\lambda))|_{i,j=1\ldots N}$$

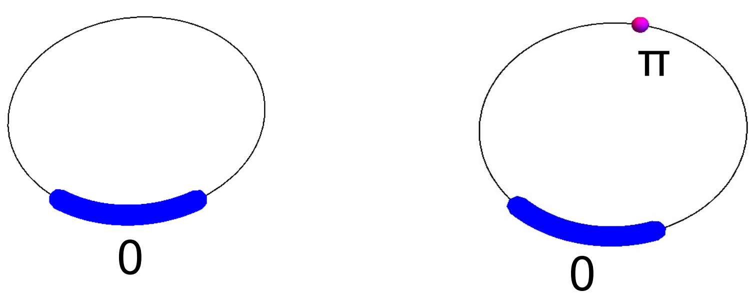
Instantons in the Gross-Witten model

One can reduce partition function of the Gross-Witten model to the integral over phases of matrix eigenvalues ρe^{iz} :

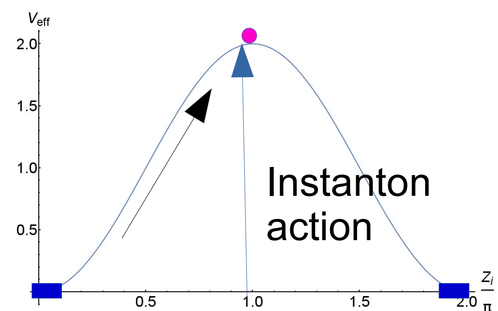
$$\mathcal{Z} = \prod_i \int_{-\pi}^{\pi} dz_i \prod_{i<j} \sin^2\left(\frac{z_i - z_j}{2}\right) \exp\left(\frac{2N}{\lambda}\cos(z_i)\right)$$

$$\mathcal{Z} = \int dz \exp(S(z)) \qquad S(z) = \frac{2N}{\lambda}\sum_i \cos(z_i) + \sum_{i<j} \ln\sin^2\left(\frac{z_i - z_j}{2}\right)$$

In the weak coupling phase there are 2 saddle points: one stable and one unstable. Instanton can be associated with tunneling between them:



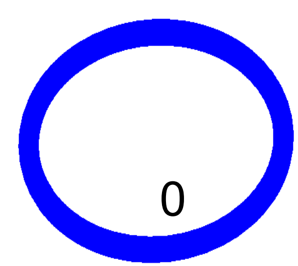
stable unstable



Effective potential

$\lambda < 2$

In contrast, in the strong coupling regime eigenvalues cover entire circle, so there is only one saddle point and no instantons:



$\lambda > 2$

Gross, Witten, Phys.Rev.D 21 446

However, let us consider a double scaling limit in this model which is defined as:

$$g_s \to 0, \lambda \to 2, \kappa = g_s^{-2/3}(2-\lambda) \text{ fixed,}$$

then asymptotic behavior of free energy will be given by a solution of the string equation:

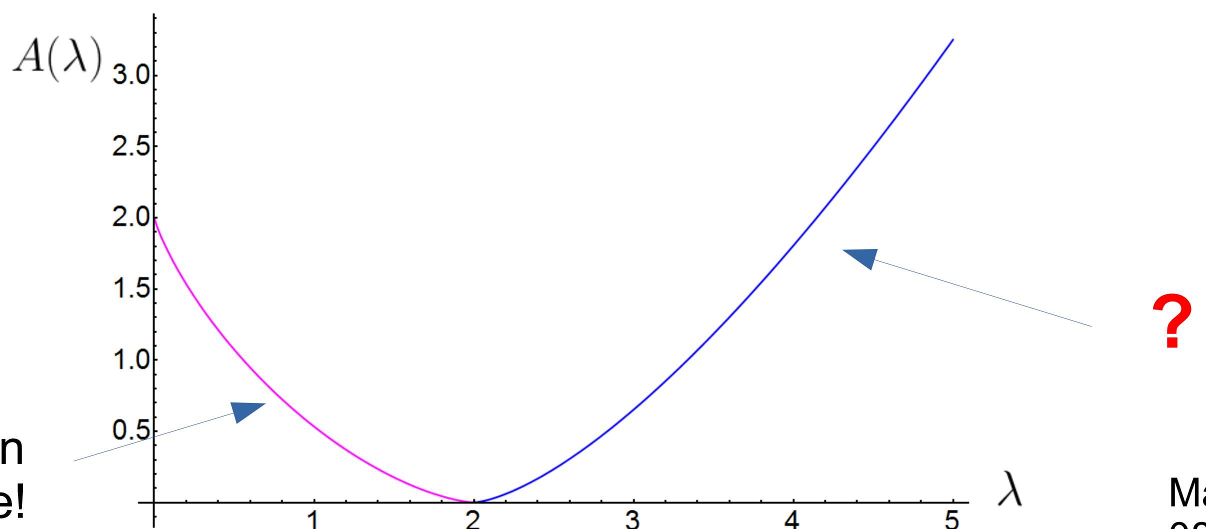
$$F''_{ds}(\kappa) = u^2(\kappa)$$

$$u''(\kappa) - 2u^3(\kappa) + 2\kappa u(\kappa) = 0 \qquad \text{(Painleve II)}$$

Solutions have a formal trans-series form:

$$u(\kappa) = \sqrt{\kappa} \sum_{l=0}^{\infty} C^l \kappa^{-3l/4} e^{-lA\kappa^{3/2}} \epsilon^{(l)}(\kappa)$$

It is possible to fix the shape of the function A(k) from both sides of the phase transition:



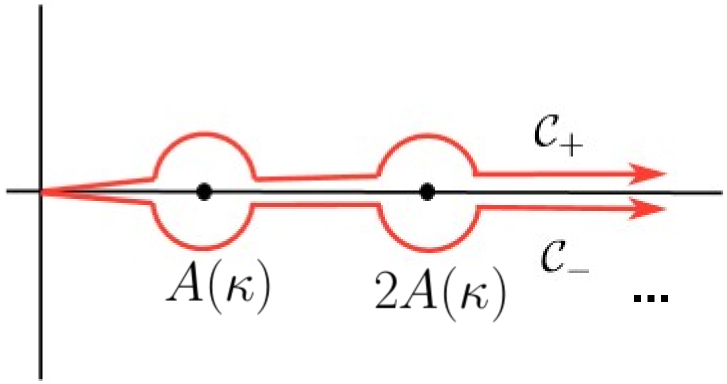
Precisely instanton action from prev. slide!

Marino, JHEP, 0812:114, 2008

In fact, the 1/N expansion of the Gross-Witten model is factorially divergent:

$$F_g \sim (2g)!$$

and its Borel transformation produces many poles along the real axis:



Poles on the Borel plane

Ambiguities of imaginary part caused by these poles are related to the instanton action.

This phenomenon is known as resurgence phenomenon and it was actively studied previously in many quantum mechanical problems, CP^N model and other models (G. Dunne, M. Unsal, ...)

This relation can be explicitly shown in the weak coupling regime, but in the strong couling regime it remains unclear what instantons are?

We would like to answer this question.

Lefcshetz thimbles

It is natural to study contribution of instantons using Morse theory. By virtue of this theory, partition functions can be expressed as a sum over all saddle points of complexified action S(z):

$$\mathcal{Z} = \sum_\sigma n_\sigma \mathcal{Z}_\sigma \quad \mathcal{Z}_\sigma = \int_{\mathcal{J}_\sigma} dz \exp(NS(z)) \sim \frac{\exp(NS(z_\sigma))}{\sqrt{-\det \partial^2 S(z)/\partial z^2}} \sum_{k=0}^{\infty} c_k N^{-k}$$

where $\mathcal{J}_\sigma$ **is the steepest descent contour (Lefcshetz thimble) in the complex plane originating from a given saddle point:**

$$\mathcal{J}_\sigma : \frac{dz(t)}{dt} = -\frac{\delta \bar{S}(z)}{\delta \bar{z}}$$

and n_σ **is a number of intersections of upward flow** $\mathcal{K}_\sigma$ **with the real axis:**

$$\mathcal{K}_\sigma : \frac{dz(t)}{dt} = +\frac{\delta \bar{S}(z)}{\delta \bar{z}} \qquad n_\sigma = < \mathcal{K}_\sigma, \mathbb{R} > \qquad n_\sigma \in \mathbb{Z}$$

Downward/Upward flow is defined in such a way that real part ReS(z) is decreasing/increasing along the flow and imaginary is constant.

Witten, arXiv:1001.2933

Therefore, our program is:

1. **Find and study all saddle points in the complex plane.**

2. **Inspect eigenvalues of Hessian matrix $H_{ij} = \partial^2 S(z)/\partial z_i \partial z_j$ at these points.**

3. **Try to count intersection numbers n_σ.**

Critical point equation: $\partial S(z)/\partial z_i = 0$

It might be quiet tricky to solve this equation analytically in the complex domain without some input guess.

Let's solve it numerically!

Simple Newton iterations:

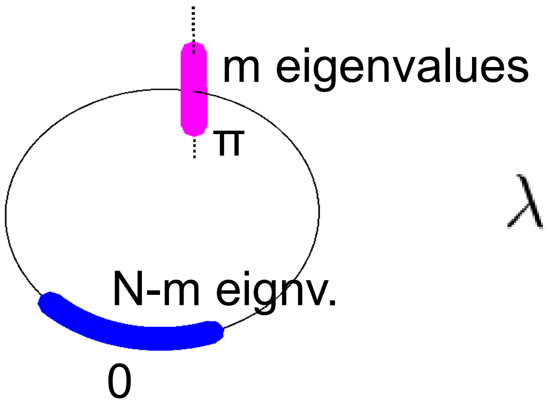
$$z^{n+1} = z^n - (\partial S(z)/\partial z_i \partial z_j)^{-1}(z^n)S(z^n)$$

with random choice of initial vector z0 on the complex plane.

(In order to improve convergence of iterations, we actually use second order Halley iterations, which are an improvement of Newton iterations)

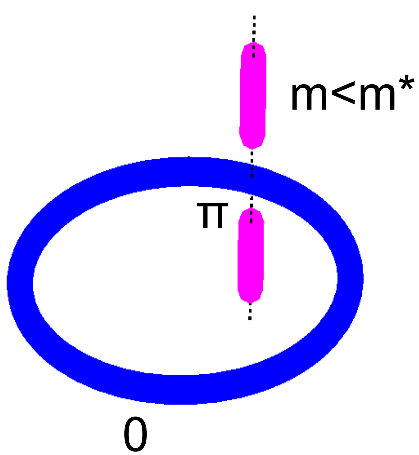
What have we found? **Making degrees of freedom complex, we promote unit circle to a cylinder and allow eigenvalues to move along it. Transversal direction of the cylinder represents imaginary part of eigenvalues.**

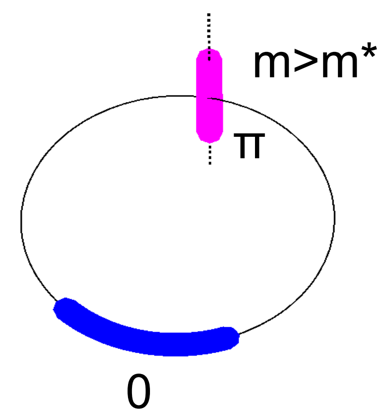
Situation changes dramatically: in both phases there are N saddle points and therefore many instantons.



$\lambda < 2$

In the strong coupling regime we have 2 distinct types of saddle points:



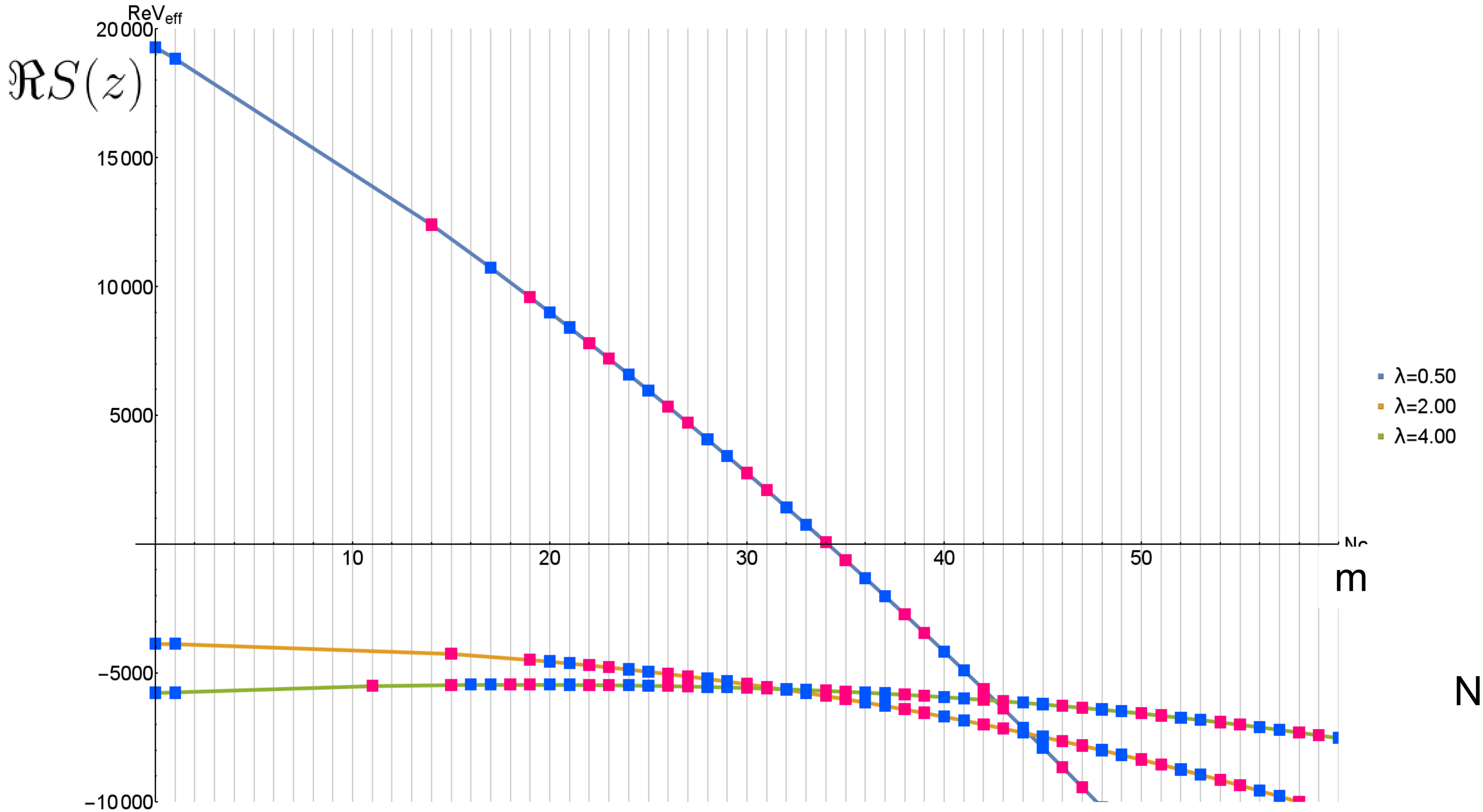


$\lambda > 2$

There is some "critical" number of complex eigenvalues m* which distinguishes between two different types of saddle points.

Weight of saddle points is always real:

$$\exp(S(z)) = \exp(\Re S(z) + i\pi n) \quad n = [m/2]$$

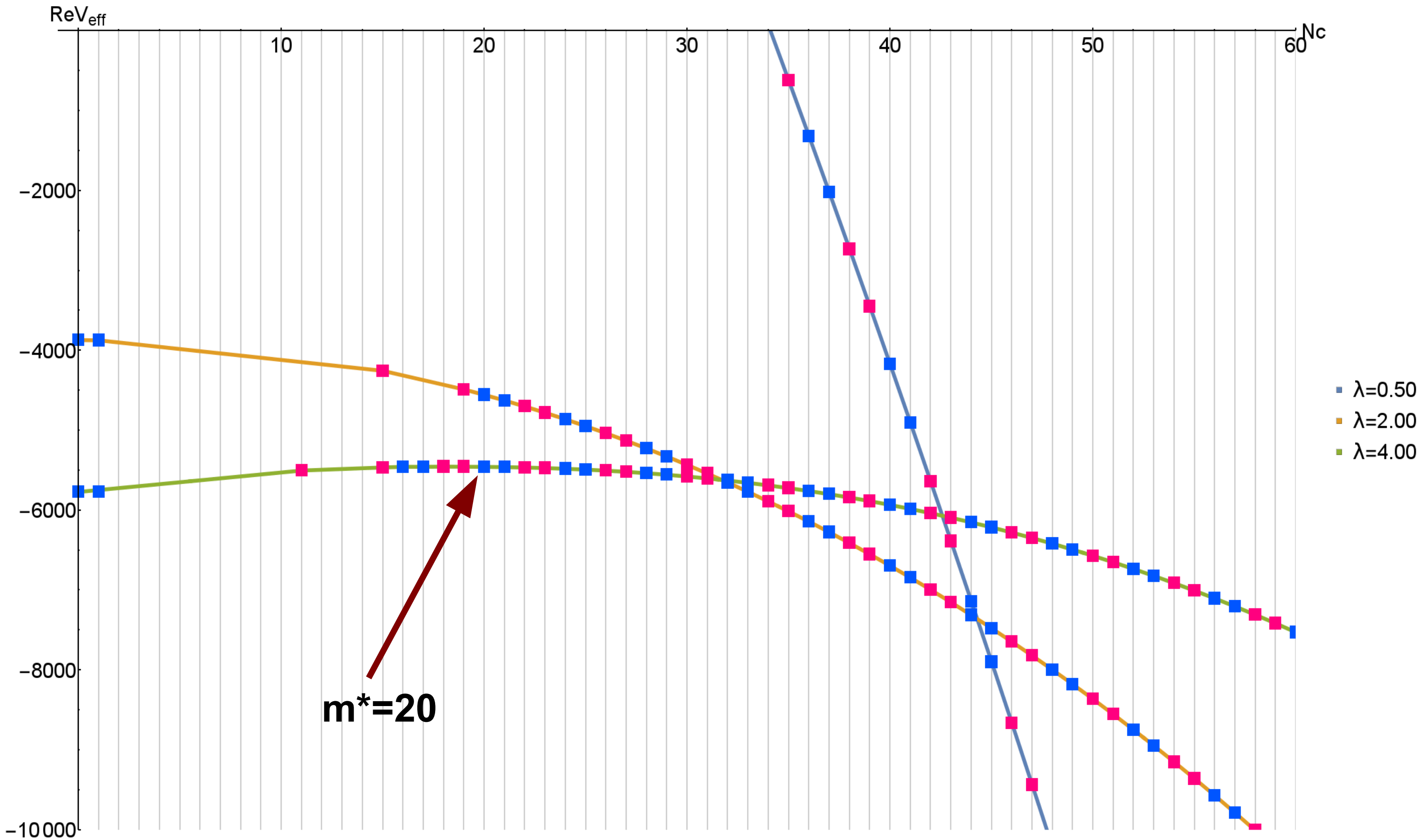


N=100

Blue dots → weight is +exp(ReS(z)) Red dots → weight is -exp(ReS(z))

Dilute instanton gas in the weak coupling regime. We observe indications for condensation of instantons at the transition point.

In the strong coupling regime configuration with maximal weight has "instanton" number m = m* > 1



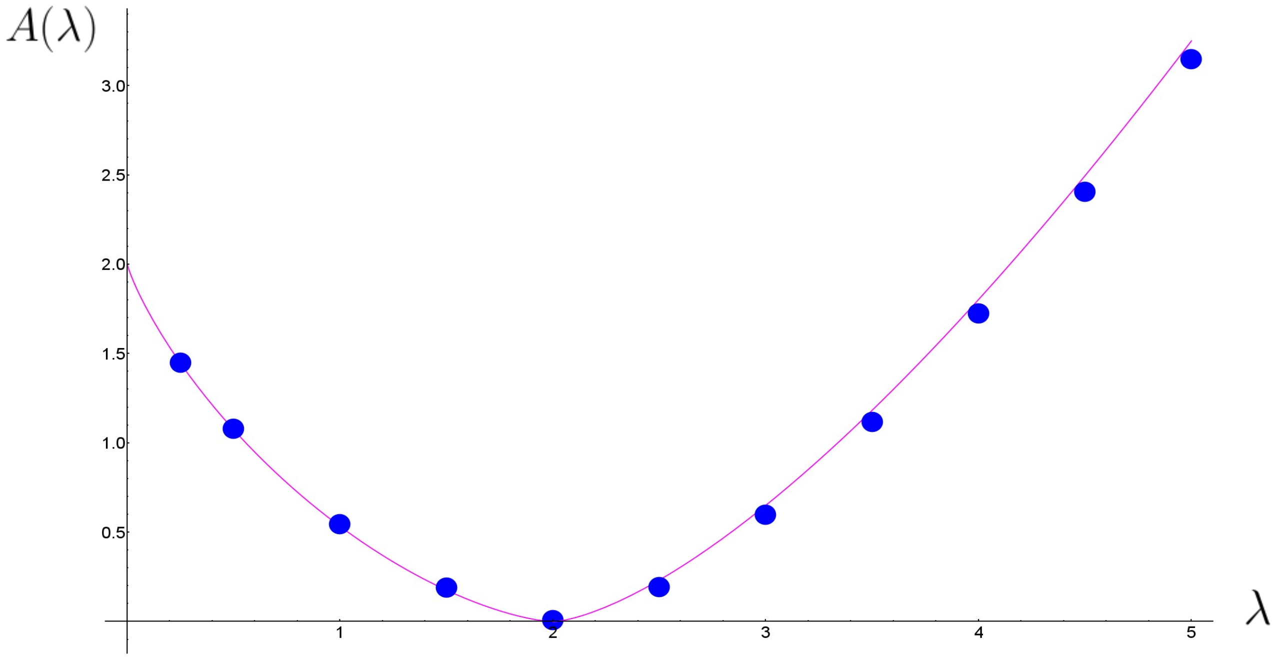
N=100

Next, we would like to address the question of instanton action.

Assuming that complex saddle points contribute to the path integral (intersection numbers $n_1 \neq 0, n_2 \neq 0$), we can calculate instanton action as

$$A(\lambda) = \frac{\lambda}{2N}(S(z_1) - S(z_2))$$

Then we compare our one-instanton action to analytical result obtained from purely algebraic consideration without any knowledge about what instantons are in the strong coupling regime:



Blue dots – numerical data, Solid line – Marino, JHEP, 0812:114, 2008

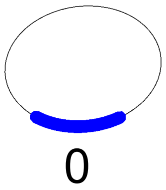
Based on this result, we suggest that we have found missing ingredient in previous studies of this model in the context of resurgence – complex instantons.

Finally, we have studied eigenvalues of Hessian matrix at saddle points:

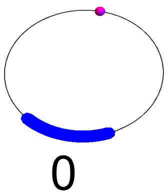
$$H_{ij} = \partial^2 S(z)/\partial z_i \partial z_j$$

In the weak coupling regime $\lambda < 2$ we have found what we expected:

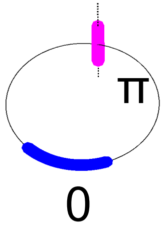
1. Real-valued configuration with m=0 delivers global maximum on the unit circle, no zero modes:



2. Configuration with one tunneled eigenvalue has 1 unstable direction associated with instanton:

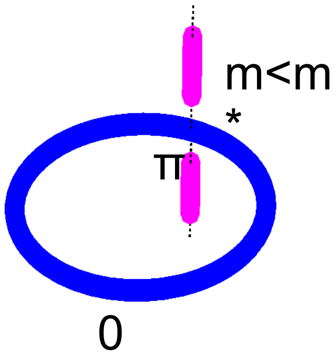


3. All complex saddle points are non-degenerate:



But in the strong coupling regime $\lambda > 2$ we came across to something very interesting:

All saddles of this type (including ground sate)

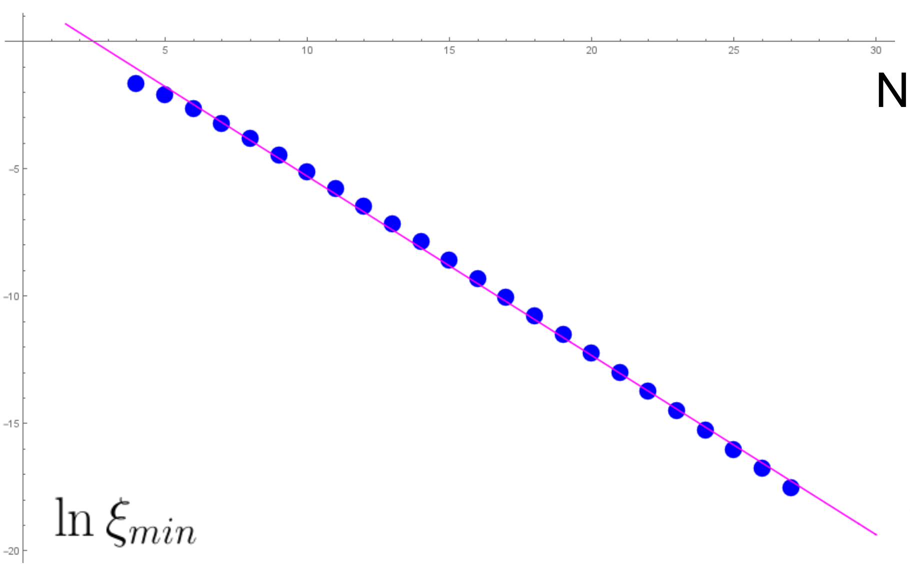


have precisely one zero eigenvalue of Hessian matrix in the limit of large N

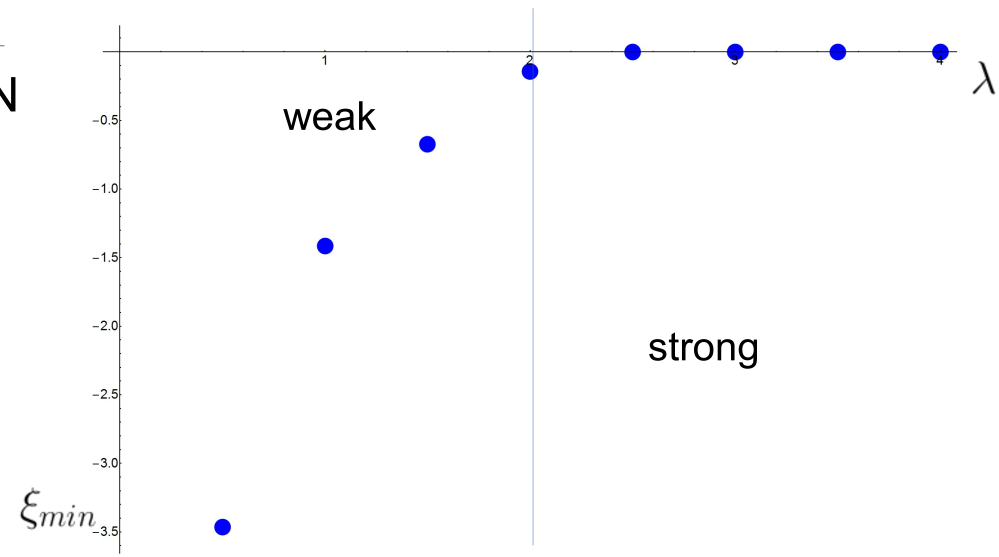
We'd like to address the issue of zero mode in more details.

To do so, we have studied lowest eigenvalue ξ_{min} of Hessian matrix of suspicious configurations and found that it decays exponentially with N in the strongly coupled phase:

$$\xi_{min} \sim e^{-cN}$$



Lowest eigenvalue in a Log-scale versus N at $\lambda = 6$

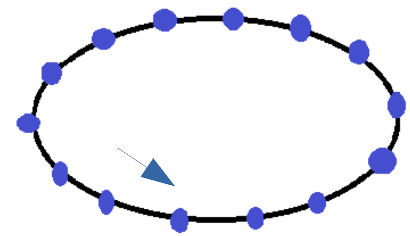


Lowest eigenvalue versus λ at N=400

To deal properly with this eigenvalue in the limit of large N, one can pick it from determinant and include to the action of corresponding saddle:

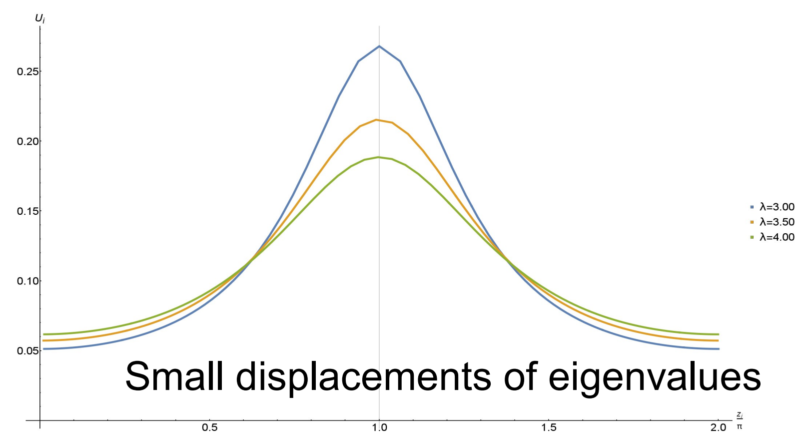
$$\int_{\mathcal{J}_0} dz \exp(S(z)) \sim \frac{\exp(S(z_0))}{\sqrt{-\det H_{ij}}} = \frac{\exp(S(z_0) + cN/2)}{\sqrt{-\det H_{ij}/\xi_{min}}}$$

The origin of this zero mode is clear: in the limit of large λ we can neglect external potential and eigenvalues tend to arrange themselves equidistantly along the unit circle:



Therefore, there is a global U(1) rotational symmetry which can be broken spontaneously.

Corresponding zero mode is nothing else but infinitesimally small displacements of eigenvalues from their original positions which do not change the action: just like a sound wave in a crystalline.



Surprisingly that this zero mode survives even at finite λ then rotational U(1) symmetry is broken explicitly by the potential.

Conclusions

1) We have found many saddle points in the complex plane and studied their structure.

2) We managed to reproduce known instanton action in both phases. This observation allows us to identify complex saddle points which most likely govern divergences in the 1/N expansion (this was not known).

3) Structure of saddles reveals picture of dilute instanton gas in the weak coupling regime and condensation (“melting”) of instantons at the transition point. In the strong coupling all instantons simply move to the complex plane.

4) In the strong coupling regime we observe emergence of zero modes, which potentially can have some impact on the 1/N expansion.

Outlook

1) All complex saddle points presumably can be found analytically as multi-cut solutions.

2) Knowing all saddle points and Lefcshetz thimbles it is possible to construct uniform approximation, which will connect the strong coupling and the weak coupling phases.

From this point of view, it is very important to study structure of these thimbles. However, there are many complications coming from branch cuts of Log-term in the action, which greatly influence on thimbles:

$$S(z) = \frac{2N}{\lambda}\sum_i \cos(z_i) + \sum_{i<j} \ln\sin^2\left(\frac{z_i - z_j}{2}\right)$$

It seems that simple Morse theory does not give a proper answer.