Induced YM theory with auxiliary bosons

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Outline

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Motivation

- Persistent challenges for lattice QCD, e.g., sign problem for real chemical potential, critical slowing down, volume scaling...
  - → worthwhile to explore alternative discretizations of Yang-Mills theory
- Idea: induce gauge dynamics by auxiliary fields coupled linearly to the gauge field
  - → analytical methods / simulation algorithms used in strong-coupling approaches become applicable
- Most earlier approaches require infinite number of infinitely heavy auxiliary fields (or do not have the desired YM continuum limit)
- Approach of Budczies and Zirnbauer [arXiv:math-ph/0305058, 2003]: induced $\mathrm{U}(N_c)$ pure YM theory with $N_b \geq N_c$ auxiliary fields
- In the following: slightly modified version of BZ approach (curing a trivial sign problem) adapted to $\mathrm{SU}(N_c)$

Outline

Plaquette weight function

- Instead of familiar Wilson weight factor $\omega_W(U_p) = \exp\left[\frac{1}{g_W^2}\,\mathrm{Tr}\left(U_p + U_p^\dagger - 2\right)\right]$ for plaquette variables $U_p \equiv U_\mu(x)U_\nu(x+\mu)U_\mu^\dagger(x+\nu)U_\nu^\dagger(x) \in \mathrm{SU}(N_c)$, consider pure gauge plaquette weight factor

$$\omega(U_p) = \det{}^{-N_b}\left(1 - \frac{\alpha}{2}(U_p + U_p^\dagger)\right) \qquad 0 < \alpha \leq 1, \quad N_b > 0$$

$$Z = \int [dU_\mu] \prod_p \omega(U_p) \qquad p \equiv (x; \mu < \nu)$$

- For integer N_b, inverse determinants can be written as integrals over N_b complex auxiliary boson fields (in the fundamental representation) with $m \geq 2$, $\frac{2}{\alpha} = m^4 - 4m^2 + 2$

$$\prod_p \omega(U_p) = \int [d\phi]\, e^{-S_b[\bar{\phi},\phi,U]}$$

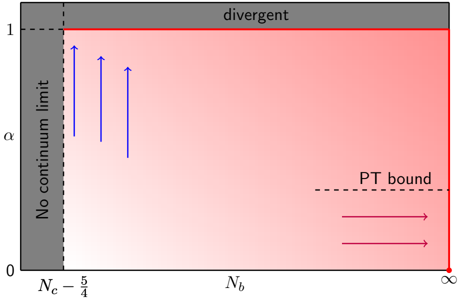
$$S_b[\bar{\phi},\phi,U] = \sum_{b=1}^{N_b} \sum_p \sum_{j=1}^{4} \Big[m\bar{\phi}_{b,p}(x_j^p)\phi_{b,p}(x_j^p) - \bar{\phi}_{b,p}(x_{j+1}^p)U(x_{j+1}^p, x_j^p)\phi_{b,p}(x_j^p) - \bar{\phi}_{b,p}(x_j^p)U(x_j^p, x_{j+1}^p)\phi_{b,p}(x_{j+1}^p) \Big]$$

Limits

- Trivial limit $N_b \to \infty$, $\alpha \to 0$ with $N_b\alpha$ fixed

$$S(U_p) = -\log\det^{-N_b}\left(1 - \frac{\alpha}{2}(U_p + U_p^\dagger)\right) = N_b \,\mathrm{Tr}\log(\ldots) \to -\frac{\alpha N_b}{2}\,\mathrm{Tr}\left(U_p + U_p^\dagger\right)$$

- Limit $\alpha \to 1$ at fixed N_b à la Budczies and Zirnbauer: existence and nature of the continuum limit depend on N_c and N_b (details for $\mathrm{SU}(N_c)$ later)
- Limit $N_b \to \infty$ at fixed α allows for systematic saddle-point analysis (perturbation theory)



Outline

Character expansion

- At fixed N_b, determine if $\omega(U) = \det^{-N_b}\left(1 - \frac{\alpha}{2}(U + U^\dagger)\right) \to \delta(U)$ as $\alpha \to 1$
- Peter-Weyl theorem: δ-function on group manifold as linear combination of characters χ_λ of all irreducible representations λ (with dim. d_λ)

$$\delta(U) \propto \sum_{\text{all irreps } \lambda} d_\lambda \chi_\lambda(U)$$

- Expand weight in irreducible characters

$$\omega(U) = \sum_\lambda c_\lambda(\alpha)\chi_\lambda(U) \qquad c_\lambda(\alpha) = \int dU \omega(U)\chi_\lambda(U^{-1})$$

- Properly normalized weight function ($\lambda = 0$ corresponds to the trivial rep. $U = 1$)

$$\mathcal{Z} \equiv \int dU \omega(U) = c_0(\alpha)$$

$$\overline{\omega}(U) \equiv \frac{1}{\mathcal{Z}}\omega(U) = \sum_\lambda \frac{c_\lambda(\alpha)}{c_0(\alpha)}\chi_\lambda(U)$$

Expansion for small $1-\alpha$

To compute $\lim_{\alpha\to 1} c_\lambda / c_0$, parametrize $U = e^{i\sqrt{\frac{2}{\alpha}(1-\alpha)}H}$ with $H \in \mathrm{su}(N_c)$

$$dU = \sqrt{\det g(H)} \left(\frac{2}{\alpha}(1-\alpha)\right)^{\frac{N_c^2-1}{2}} dH$$

$$g(H) = \frac{1}{2} + \sum_{k=1}^{\infty} \frac{(-1)^k}{(2k+2)!} \left(\frac{2}{\alpha}(1-\alpha)\right)^k H_{\mathrm{adj}}^{2k}$$

and expand the integrand in $1-\alpha$

$$\det{}^{-N_b}\left(1 - \frac{\alpha}{2}\left(U + U^\dagger\right)\right) = \frac{\det{}^{-N_b}\left(1+H^2\right)}{(1-\alpha)^{N_b N_c}} \left(1 + N_b \frac{1-\alpha}{6\alpha} \operatorname{Tr} \frac{H^4}{1+H^2} + \ldots\right)$$

$$\chi_\lambda\left(e^{-i\sqrt{\frac{2}{\alpha}(1-\alpha)}H}\right) = d_\lambda \left(1 - \frac{C_2^{\mathrm{SU}(N_c)}(\lambda)}{N_c^2 - 1}(1-\alpha) \operatorname{Tr} H^2 + \ldots\right)$$

Interchange limit and integral

- As long as $\int_{\mathrm{su}(N_c)} dH \det^{-N_b}(1+H^2)$ exists, interchanging $\lim_{\alpha\to 1}$ and $\int dH$ gives

$$\lim_{\alpha\to 1}\frac{c_\lambda(\alpha)}{c_0(\alpha)} = d_\lambda \qquad \Rightarrow \qquad \lim_{\alpha\to 1}\overline{\omega}(U) = \sum_\lambda d_\lambda \chi_\lambda(U) \propto \delta(U)$$

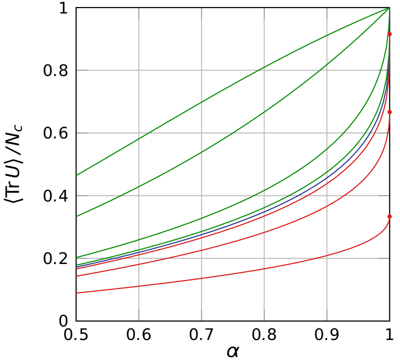
- In the eigenvalue parametrization

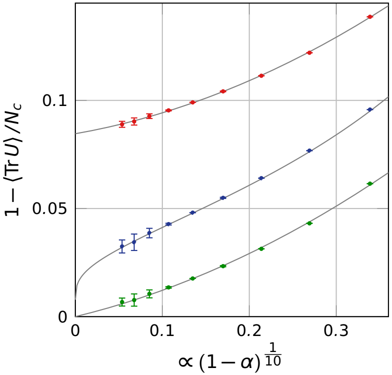
$$\int_{\mathrm{su}(N_c)} dH \det^{-N_b}(1+H^2) \propto \int_{-\infty}^{\infty}\prod_{j=1}^{N_c} dz_j\, \delta\left(\sum_{j=1}^{N_c} z_j\right)\left(\prod_{j<k}(z_j-z_k)^2\right)\prod_{j=1}^{N_c}\left(1+z_j^2\right)^{-N_b}$$

  we see that the integral exists for $N_b > N_c - \frac{5}{4}$
- $N_b > N_c - \frac{5}{4}$ is a sufficient but not necessary condition for the existence of the continuum limit
- Alternative, more direct approach (using the eigenvalue parametrization of U) shows that the continuum limit exists if and only if $N_b \geq N_c - \frac{5}{4}$
  (same method for $\mathrm{U}(N_c)$ leads to $N_b \geq N_c - \frac{1}{2}$ as condition for continuum limit)

Check for $N_c = 2, 3$

- For SU(2): explicit calculation of coefficients in character expansion confirms bound
- For SU(3): numerically compute $\frac{1}{N_c}\langle \mathrm{Tr}\, U \rangle$ with single plaquette weight for $N_b = 1, 1.5, 1.7$, $N_b = 1.75$, $N_b = 1.8, 2, 3, 4$





Nature of continuum limit

- To determine the nature of the continuum limit, expand the coefficients c_λ/c_0 to linear order in $1-\alpha$
- Interchange again expansion in $(1-\alpha)$ and $\int dH$, as long as integrals are finite
- $\int_{\mathrm{su}(N_c)} dH \det^{-N_b}(1+H^2)\,\mathrm{Tr}\,H^2$ exists if $N_b > N_c - \frac{3}{4}$
- We get $N_b > N_c - \frac{3}{4}$ as sufficient but not necessary condition for

$$\frac{c_\lambda(\alpha)}{c_0(\alpha)} = d_\lambda \left(1-(1-\alpha)\frac{C_2^{\mathrm{SU}(N_c)}(\lambda)}{N_c^2-1}\frac{\int_{\mathrm{su}(N_c)} dH \det^{-N_b}(1+H^2)\,\mathrm{Tr}\,H^2}{\int_{\mathrm{su}(N_c)} dH \det^{-N_b}(1+H^2)} + \dots\right)$$

- In this case, $\bar{\omega}(U)$ reduces to the heat-kernel weight factor for small $1-\alpha$

$$\frac{c_\lambda(\alpha)}{c_0(\alpha)} = d_\lambda e^{-t\,C_2^{\mathrm{SU}(N_c)}(\lambda)}$$

  analogous to Wilson's weight factor for small g^2
- Continuum limit in two dimensions is in the universality class of SU(N_c) Yang-Mills theory for $N_b > N_c - \frac{3}{4}$ (condition for U(N_c): $N_b > N_c + \frac{1}{2}$) (Continuum limit in $2d$ is trivial with Migdal's recursion: heat-kernel weight is invariant under subdivision of lattice cells due to character orthogonality)
- Conjecture for $3d$ and $4d$: equivalence with YM persists (collective nature of fields gets enhanced (compared to 2d); works in favor of universality)

Outline

Setup

- Aim: determine relation between $1-\alpha$ and Wilson's coupling g_W for fixed N_b
- Problem I: expansion of the log

$$S = N_b \sum_p \mathrm{Tr} \log\left(1 - \frac{\alpha}{2}\left(U_p + U_p^\dagger\right)\right) \widehat{=}\; N_b \sum_p \mathrm{Tr} \log\left(1 - \frac{\alpha}{2(1-\alpha)}\left(U_p + U_p^\dagger - 2\right)\right)$$

$$\left|\frac{\alpha}{1-\alpha}(\cos\varphi - 1)\right| \leq 1 \qquad \forall\varphi \qquad \Rightarrow \qquad \alpha \leq \frac{1}{3}$$

- Problem II: saddle-point analysis (after expansion of the log) is not possible

$$S = -N_b \sum_p \sum_{n=1}^{\infty} \frac{1}{n}\left(\frac{\alpha}{2(1-\alpha)}\right)^n \mathrm{Tr}\left(\left(U_p + U_p^\dagger - 2\right)^n\right)$$

  higher orders in $U + U^\dagger - 2$ are not suppressed (parametr. $U_p = e^{i\sqrt{1-\alpha}H}$)
  $\rightarrow$ non-Gaussian integrals
- Workaround: first keep $\alpha \leq \frac{1}{3}$ fixed, take $N_b \to \infty$; then analytically continue $g_W(\alpha, N_b)$ to small $1-\alpha$
- $n = 1$-term corresponds to Wilson's gauge action, with coupling constant

$$\frac{1}{g_l^2} \equiv N_b \frac{\alpha}{2(1-\alpha)}$$

Background field method

- For fixed α, compute

$$\frac{1}{g_W^2} = \frac{1}{g_I^2}\left(1 + c_1(\alpha)g_I^2 + c_2(\alpha)g_I^4 + \dots\right) \qquad \frac{1}{g_I^2} = N_b \frac{\alpha}{2(1-\alpha)}$$

- Parametrize link variables through quantum field q_μ and background field A_μ

$$U_\mu(x) = e^{iagq_\mu(x)} U_\mu^{(0)}(x), \qquad U_\mu^{(0)}(x) = e^{iaA_\mu(x)}$$

- Compute effective action $\Gamma_{I/W}[A]$ to quadratic order in A

$$e^{-\Gamma_{I/W}[A]} \propto \int_{\text{1-PI}} [Dq] e^{-S_{I/W}[A,q]}$$

- Relation between g_I and g_W from $\Gamma_I[A] = \Gamma_W[A]$ in the continuum limit

Expansion of the action

- Gauge-fixing in background-field Feynman gauge, $S_{\mathrm{gf}} = a^4 \sum_x \mathrm{Tr}\left(\sum_\mu \bar{D}^{(0)}_\mu q_\mu\right)^2$ with lattice covariant derivative (involving only the background field) $\bar{D}^{(0)}_\mu g(x) = \frac{1}{a}\left(U^{(0)\dagger}_\mu(x-\mu) g(x-\mu) U^{(0)}_\mu(x-\mu) - g(x)\right)$
  $\rightarrow \Gamma[A]$ is invariant under gauge transformations
- Terms of order A^k ($k = 0, 1, 2$) from $\mathrm{Tr}(U_p + U_p^\dagger - 2)^n$ in the induced action: $S_I^{(n,k)}$

$$S_I = S_W|_{g_W = g_I} + \sum_{n=2}^{\infty}\left(S_I^{(n,0)} + S_I^{(n,1)} + S_I^{(n,2)} + \mathcal{O}(A^3)\right)$$

$$S_I^{(n,0)} = (-1)^{n+1} \frac{a^{2n} g^{2n-2}}{(2/\alpha - 2)^{n-1}} \sum_{x,\mu,\nu} \frac{1}{2n} \mathrm{Tr}\left[q_{\mu\nu}(x)^{2n} + \mathcal{O}\left(g q^{2n+1}\right)\right]$$

$$S_I^{(n,1)} = (-1)^{n+1} \frac{a^{2n} g^{2n-3}}{(2/\alpha - 2)^{n-1}} \sum_{x,\mu,\nu} \mathrm{Tr}\left[A_{\mu\nu}(x) q_{\mu\nu}(x)^{2n-1} + \mathcal{O}\left(g A q^{2n}\right)\right]$$

$$S_I^{(n,2)} = (-1)^{n+1} \frac{a^{2n} g^{2n-4}}{(2/\alpha - 2)^{n-1}} \sum_{x,\mu,\nu} \mathrm{Tr}\Bigg[\sum_{m=0}^{n-2} A_{\mu\nu}(x) q_{\mu\nu}(x)^m A_{\mu\nu}(x) q_{\mu\nu}(x)^{2n-m-2} + \frac{1}{2}\left(A_{\mu\nu}(x) q_{\mu\nu}(x)^{n-1}\right)^2 + \mathcal{O}\left(A^2 g q^{2n-1}\right)\Bigg]$$

$$q_{\mu\nu}(x) \equiv q_\mu(x) + q_\nu(x+\mu) - q_\mu(x+\nu) - q_\nu(x)$$
$$A_{\mu\nu}(x) \equiv A_\mu(x) + A_\nu(x+\mu) - A_\mu(x+\nu) - A_\nu(x)$$

Expectation values

- Split action in free action $S_f[q]$ for quantum field q (terms of order $q^2 A^0$), classical piece $S_{\rm cl}[A]$ (terms independent of q) and 'interaction' terms

$$e^{-\Gamma[A]} \propto e^{-S_{\rm cl}[A]} \int_{\text{1-PI}} [Dq] e^{-S_f[q]} \sum_k \frac{1}{k!} \left(-S_{\rm int}[A,q]\right)^k \propto e^{-S_{\rm cl}[A]} \sum_k \frac{1}{k!} \left\langle \left(-S_{\rm int}[A,q]\right)^k \right\rangle_{\text{1-PI}}$$

- Omitted: integrals over ghost fields (don't contribute to LO and relevant NLO terms)
- Expectation values w.r.t. free action

$$S_f = \frac{a^4}{2} \sum_{x,\mu,b} q_\mu^b(x) \Box q_\mu^b(x)$$

$$\left\langle q_\mu^a(x) q_\nu^b(y) \right\rangle = \delta_{ab} \delta_{\mu\nu} D(x-y)$$

with standard lattice propagator for a massless scalar field

$$D(x) = \int_{-\pi/a}^{\pi/a} \frac{d^d p}{(2\pi)^d} e^{ipx} \frac{a^{d-2}}{\sum_\mu 2\left(1 - \cos\left(a p_\mu\right)\right)}$$

One-loop result

- S_I includes S_W ($n=1$-term), corresponding terms cancel in $\Gamma_I[A]-\Gamma_W[A]$
- coefficient $c_1(\alpha)$ (in $g_W^{-2}=g_I^{-2}+c_1(\alpha)+\ldots$) is determined from

$$\left\langle S_I^{(2,2)}\right\rangle = -\frac{a^4\alpha}{2(1-\alpha)}\sum_x\sum_{\mu,\nu}A^a_{\mu\nu}(x)A^b_{\mu\nu}(x)\left\langle q^c_{\mu\nu}(x)q^d_{\mu\nu}(x)\right\rangle \mathrm{Tr}\left[t_at_bt_ct_d+\frac{1}{2}t_at_ct_bt_d\right]$$

$$\to -\frac{4}{d}\left(\frac{2N_c^2-3}{8N_c}\right)\frac{\alpha}{2(1-\alpha)}a^{4-d}\int d^dx\sum_{\mu,\nu}\mathrm{Tr}\,F_{\mu\nu}(x)^2+\ldots$$

- Comparison with $S_{\mathrm{cl}}[A]=\frac{1}{2g_I^2}a^{4-d}\int d^dx\sum_{\mu,\nu}\mathrm{Tr}\,F_{\mu\nu}(x)^2+\ldots$ leads to (in d dimensions)

$$c_1(\alpha)=-\left(\frac{2N_c^2-3}{N_cd}\right)\frac{\alpha}{2(1-\alpha)}$$

Relevant two-loop result

- At order g_I^2, Γ_I contains terms of order $(1-\alpha)^{-2}$ and $(1-\alpha)^{-1}$

$$c_2(\alpha) = c_{2,-2}\left(\frac{\alpha}{2(1-\alpha)}\right)^2 + c_{2,-1}\left(\frac{\alpha}{2(1-\alpha)}\right)$$

- We are only interested in $c_{2,-2}$, obtained from

$$\left\langle S_I^{(3,2)} - S_I^{(2,0)} S_I^{(2,2)} - \frac{1}{2} S_I^{(2,1)} S_I^{(2,1)} \right\rangle$$

- Result:

$$c_{2,-2} = \frac{N_c^4 - 3N_c^2 + 6}{d^2 N_c^2} - \frac{N_c^4 - 6N_c^2 + 18}{2N_c^2(d-1)}\left(\frac{3}{d^2} - 4(4-d)J_d\right)$$

$$J_d \equiv \int_{-\pi}^{\pi} \frac{d^d k}{(2\pi)^d} \frac{d^d q}{(2\pi)^d} \frac{(\sin(k_1) + \sin(q_1) - \sin(k_1+q_1))^2}{\left(2\sum_\gamma(1-\cos(k_\gamma))\right)\left(2\sum_\mu(1-\cos(q_\mu))\right)\left(2\sum_\rho(1-\cos(k_\rho+q_\rho))\right)}$$

$$J_2 = \frac{1}{32}, \qquad J_3 \approx 0.0085535415$$

Reinterpretation of the result

In terms of α and N_b, we have (from $N_b \to \infty$ at fixed $\alpha \leq \frac{1}{3}$)

$$\frac{1}{g_W^2} = \frac{1}{g_I^2}\left(1 + c_1(\alpha) g_I^2 + c_2(\alpha) g_I^4 + \dots\right) \qquad \frac{1}{g_I^2} = N_b \frac{\alpha}{2(1-\alpha)}$$

$$c_1(\alpha) = c_{1,-1}\left(\frac{\alpha}{2(1-\alpha)}\right)$$

$$c_2(\alpha) = c_{2,-2}\left(\frac{\alpha}{2(1-\alpha)}\right)^2 + c_{2,-1}\left(\frac{\alpha}{2(1-\alpha)}\right)$$

$$\frac{1}{g_W^2} = \frac{\alpha}{2(1-\alpha)} \underbrace{\left[N_b + c_{1,-1} + c_{2,-2}/N_b + \mathcal{O}\left(N_b^{-2}\right)\right]}_{\equiv d_0(N_b)} + \mathcal{O}((1-\alpha)^0)$$

For the limit $\alpha \to 1$ at fixed N_b, natural definition of coupling constant $\tilde{g}_I$:

$$\frac{1}{\tilde{g}_I^2} \equiv d_0(N_b) \frac{\alpha}{2(1-\alpha)}$$

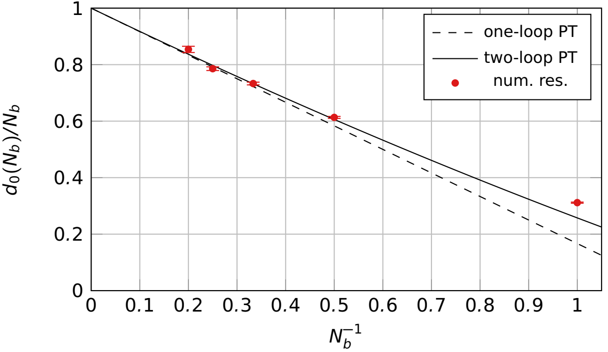
$$\frac{1}{g_W^2} = \frac{1}{\tilde{g}_I^2}\left(1 + d_1(N_b)\tilde{g}_I^2 + \dots\right)$$

Outline

Numerical results

- For $N_c = 2$ in $d = 3$ dimensions, simulate both Wilson and induced action
- Use Sommer parameter to set scale and match g_W to $1 - \alpha$ (at fixed N_b)
- Results for other observables agree well after matching
  [Brandt & Wettig, PoS(LATTICE2014)307]
- Compare coefficient of $\frac{1}{2(1-\alpha)}$ in $\frac{1}{g_W^2}$ to perturbative result d_0

$$\frac{d_0(N_b; N_c = 2, d = 3)}{N_b} = 1 - \frac{5}{6N_b} + \frac{0.0908283}{N_b^2} + \mathcal{O}(N_b^{-3})$$



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Summary and perspectives

- Induced $\mathrm{SU}(N_c)$ action exhibits continuum limit as $\alpha \to 1$ for fixed $N_b \geq N_c - \frac{5}{4}$ ($N_b \geq N_c - \frac{1}{2}$ for $\mathrm{U}(N_c)$)
- $N_b > N_c - \frac{3}{4}$ is sufficient but not necessary for the continuum limit in 2d to be in the universality class of YM theory ($N_b > N_c + \frac{1}{2}$ for $\mathrm{U}(N_c)$)
- Perturbation theory for $\alpha \to 1$ is problematic
- Relation between coupling constants g_I and g_W determined by first taking $N_b \to \infty$ at fixed $\alpha \leq \frac{1}{3}$ and analytic continuation to small $1-\alpha$
- Good agreement with numerical results for $\mathrm{SU}(2)$ in 3d

Future directions:

- Numerical simulations of $\mathrm{SU}(3)$ in 4d
- Make use of bosonized version of gauge action (for full QCD)
- Integration over link variables

$$\int_{\mathrm{SU}(N_c)} dU e^{\frac{1}{2}(UA+U^\dagger A^\dagger)} \propto \frac{1}{\Delta(\lambda^2)} \sum_{\nu=0}^{\infty} (2-\delta_{\nu 0}) \cos(\nu\phi) \det\left[\lambda_i^{j-1} I_{\nu+j-1}(\lambda_i)\right]$$

- Duality transformation (variant of color-flavor transformation)