Background field method in the gradient flow

Hiroshi Suzuki

(Kyushu University)

2015/07/18 @ Lattice 2015

- H.S, arXiv:1507.02360 [hep-lat]

Yang–Mills gradient flow (Lüscher 2009–)

- Evolution along a fictitious time $t \in [0, \infty)$,

$$\partial_t B_\mu(t,x) = D_\nu G_{\nu\mu}(t,x) = -g_0^2 \frac{\delta S_{\mathrm{YM}}}{\delta B_\mu(t,x)}, \qquad B_\mu(0,x) = A_\mu(x).$$

where

$$D_\mu = \partial_\mu + [B_\mu, \cdot], \qquad G_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu + [B_\mu, B_\nu]$$

Yang–Mills gradient flow (Lüscher 2009–)

- Evolution along a fictitious time $t \in [0, \infty)$,

$$\partial_t B_\mu(t,x) = D_\nu G_{\nu\mu}(t,x) = -g_0^2 \frac{\delta S_{\text{YM}}}{\delta B_\mu(t,x)}, \qquad B_\mu(0,x) = A_\mu(x).$$

where

$$D_\mu = \partial_\mu + [B_\mu, \cdot], \qquad G_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu + [B_\mu, B_\nu]$$

- Smoothing gauge configuration

Yang–Mills gradient flow (Lüscher 2009–)

- Evolution along a fictitious time $t \in [0, \infty)$,

$$\partial_t B_\mu(t, x) = D_\nu G_{\nu\mu}(t, x) = -g_0^2 \frac{\delta S_{\text{YM}}}{\delta B_\mu(t, x)}, \qquad B_\mu(0, x) = A_\mu(x).$$

  where

$$D_\mu = \partial_\mu + [B_\mu, \cdot], \qquad G_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu + [B_\mu, B_\nu]$$

- Smoothing gauge configuration
- Operators are automatically renormalized (Lüscher–Weisz)

Yang–Mills gradient flow (Lüscher 2009–)

- Evolution along a fictitious time $t \in [0, \infty)$,

$$\partial_t B_\mu(t,x) = D_\nu G_{\nu\mu}(t,x) = -g_0^2 \frac{\delta S_{\rm YM}}{\delta B_\mu(t,x)}, \qquad B_\mu(0,x) = A_\mu(x).$$

  where

$$D_\mu = \partial_\mu + [B_\mu, \cdot], \qquad G_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu + [B_\mu, B_\nu]$$

- Smoothing gauge configuration
- Operators are automatically renormalized (Lüscher–Weisz)
- Topological charge, scale setting, renormalized coupling, chiral condensate, . . .

Yang–Mills gradient flow (Lüscher 2009–)

- Evolution along a fictitious time $t \in [0, \infty)$,

$$\partial_t B_\mu(t, x) = D_\nu G_{\nu\mu}(t, x) = -g_0^2 \frac{\delta S_{\mathrm{YM}}}{\delta B_\mu(t, x)}, \qquad B_\mu(0, x) = A_\mu(x).$$

where

$$D_\mu = \partial_\mu + [B_\mu, \cdot], \qquad G_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu + [B_\mu, B_\nu]$$

- Smoothing gauge configuration
- Operators are automatically renormalized (Lüscher–Weisz)
- Topological charge, scale setting, renormalized coupling, chiral condensate, . . .
- Lattice energy–momentum tensor

For perturbation theory…

- "Gauge fixing term" (Lüscher)

$$\partial_t B_\mu(t,x) = D_\nu G_{\nu\mu}(t,x) + \alpha_0 D_\mu \partial_\nu B_\nu(t,x)$$

For perturbation theory…

- "Gauge fixing term" (Lüscher)

$$\partial_t B_\mu(t,x) = D_\nu G_{\nu\mu}(t,x) + \alpha_0 D_\mu \partial_\nu B_\nu(t,x)$$

- Tree-level propagator

$$\langle B_\mu B_\nu \rangle_0 \sim \frac{1}{(p^2)^2} \left[(\delta_{\mu\nu} p^2 - p_\mu p_\nu) e^{-(t+s)p^2} + \frac{1}{\lambda_0} p_\mu p_\nu e^{-\alpha_0 (t+s) p^2} \right]$$

  and this ensures a good convergence property of momentum integrals

For perturbation theory...

- "Gauge fixing term" (Lüscher)

$$\partial_t B_\mu(t,x) = D_\nu G_{\nu\mu}(t,x) + \alpha_0 D_\mu \partial_\nu B_\nu(t,x)$$

- Tree-level propagator

$$\langle B_\mu B_\nu \rangle_0 \sim \frac{1}{(p^2)^2}\left[(\delta_{\mu\nu}p^2 - p_\mu p_\nu)e^{-(t+s)p^2} + \frac{1}{\lambda_0}p_\mu p_\nu e^{-\alpha_0(t+s)p^2}\right]$$

  and this ensures a good convergence property of momentum integrals
- This term breaks the gauge covariance, but any gauge invariant quantity (that does not contain $\partial/\partial t$) is independent of α_0

Here, we propose...

- Background gauge covariant gauge fixing term

$$\partial_t B_\mu(t,x) = D_\nu G_{\nu\mu}(t,x) + \alpha_0 D_\mu \hat{D}_\nu b_\nu(t,x),$$

where we split fields into

$$B_\mu(t,x) = \underbrace{\hat{B}_\mu(t,x)}_{\text{background}} + \underbrace{b_\mu(t,x)}_{\text{quantum}},$$

and the background covariant derivative

$$\hat{D}_\mu = \partial_\mu + [\hat{B}_\mu, \cdot]$$

Here, we propose...

- Background gauge covariant gauge fixing term

$$\partial_t B_\mu(t,x) = D_\nu G_{\nu\mu}(t,x) + \alpha_0 D_\mu \hat{D}_\nu b_\nu(t,x),$$

where we split fields into

$$B_\mu(t,x) = \underbrace{\hat{B}_\mu(t,x)}_{\text{background}} + \underbrace{b_\mu(t,x)}_{\text{quantum}},$$

and the background covariant derivative

$$\hat{D}_\mu = \partial_\mu + [\hat{B}_\mu, \cdot]$$

- This term breaks the full gauge covariance, but preserves covariance under the background gauge transformation

$$\hat{B}_\mu \to \hat{B}_\mu + \hat{D}_\mu \omega(x), \qquad b_\mu \to b_\mu + [b_\mu, \omega(x)].$$

Background field method ('t Hooft, DeWitt, Boulware, Abbott, Omote–Ichinose, . . .)

- Background–quantum splitting

$$A_\mu(x) = \underbrace{\hat{A}_\mu(x)}_{\text{background}} + \underbrace{a_\mu(x)}_{\text{quantum}},$$

Background field method ('t Hooft, DeWitt, Boulware, Abbott, Omote–Ichinose, ...)

- Background–quantum splitting

$$A_\mu(x) = \underbrace{\hat{A}_\mu(x)}_{\text{background}} + \underbrace{a_\mu(x)}_{\text{quantum}},$$

- Background covariant gauge fixing term

$$S_{\text{gauge fixing}} = \frac{\lambda_0}{2g_0^2} \int \mathrm{d}^4 x\, \hat{D}_\mu a_\mu^a(x) \hat{D}_\nu a_\nu^a(x),$$

which preserves covariance under the background gauge transformation

$$\hat{A}_\mu(x) \to \hat{A}_\mu(x) + \hat{D}_\mu \omega(x), \qquad a_\mu(x) \to a_\mu(x) + [a_\mu(x), \omega(x)].$$

Background field method ('t Hooft, DeWitt, Boulware, Abbott, Omote–Ichinose, . . .)

- Background–quantum splitting

$$A_\mu(x) = \underbrace{\hat{A}_\mu(x)}_{\text{background}} + \underbrace{a_\mu(x)}_{\text{quantum}},$$

- Background covariant gauge fixing term

$$S_{\text{gauge fixing}} = \frac{\lambda_0}{2g_0^2} \int \mathrm{d}^4x\, \hat{D}_\mu a_\mu^a(x) \hat{D}_\nu a_\nu^a(x),$$

  which preserves covariance under the background gauge transformation

$$\hat{A}_\mu(x) \to \hat{A}_\mu(x) + \hat{D}_\mu\omega(x), \qquad a_\mu(x) \to a_\mu(x) + [a_\mu(x), \omega(x)].$$

- Greatly simplifies the consideration of counterterms, for example. . .

With our flow equation

- Any gauge invariant quantity (that does not contain $\partial/\partial t$) is independent of α_0

With our flow equation

- Any gauge invariant quantity (that does not contain $\partial/\partial t$) is independent of α_0
- Manifestly background gauge covariant expressions. . .

With our flow equation

- Any gauge invariant quantity (that does not contain $\partial/\partial t$) is independent of α_0
- Manifestly background gauge covariant expressions...
- "Tree-level" propagator (in the "Feynman gauge", $\lambda_0 = \alpha_0 = 1$)

$$\left\langle b_\mu^a(t,x) b_\nu^b(s,y) \right\rangle_0 = g_0^2 \left(\mathrm{e}^{(t+s)[\hat{\mathcal{D}}_x^2 + 2\hat{\mathcal{F}}(x)]} \frac{-1}{\hat{\mathcal{D}}_x^2 + 2\hat{\mathcal{F}}(x)} \right)_{\mu\nu}^{ab} \delta(x-y),$$

where

$$\hat{\mathcal{D}}_\mu^{ab} \equiv \delta^{ab}\partial_\mu + \hat{A}_\mu^c f^{acb}, \qquad \hat{\mathcal{F}}_{\mu\nu}^{ab} \equiv \hat{F}_{\mu\nu}^c f^{acb}$$

With our flow equation

- Any gauge invariant quantity (that does not contain $\partial/\partial t$) is independent of α_0
- Manifestly background gauge covariant expressions...
- "Tree-level" propagator (in the "Feynman gauge", $\lambda_0 = \alpha_0 = 1$)

$$\left\langle b_\mu^a(t,x) b_\nu^b(s,y) \right\rangle_0 = g_0^2 \left(\mathrm{e}^{(t+s)[\hat{\mathcal{D}}_x^2 + 2\hat{\mathcal{F}}(x)]} \frac{-1}{\hat{\mathcal{D}}_x^2 + 2\hat{\mathcal{F}}(x)} \right)_{\mu\nu}^{ab} \delta(x-y),$$

where

$$\hat{\mathcal{D}}_\mu^{ab} \equiv \delta^{ab}\partial_\mu + \hat{A}_\mu^c f^{acb}, \qquad \hat{\mathcal{F}}_{\mu\nu}^{ab} \equiv \hat{F}_{\mu\nu}^c f^{acb}$$

- We have assumed that the background field is evolved by its own flow equation

$$\partial_t \hat{B}_\mu(t,x) = \hat{D}_\nu \hat{G}_{\nu\mu}(t,x), \qquad \hat{B}_\mu(0,x) = \hat{A}_\mu(x),$$

and, moreover $\hat{D}_\nu \hat{F}_{\nu\mu}(x) = 0$ for simplicity.

Small flow time expansion relevant to EMT

- Small flow time expansion (Lüscher–Weisz)

$$
\begin{aligned}
&G^a_{\mu\rho}(t,x)G^a_{\nu\rho}(t,x)\\
&\stackrel{t\to0}{\sim}\left\langle G^a_{\mu\rho}(t,x)G^a_{\nu\rho}(t,x)\right\rangle\\
&\qquad+\zeta_{11}(t)F^a_{\mu\rho}(x)F^a_{\nu\rho}(x)+\zeta_{12}(t)\delta_{\mu\nu}F^a_{\rho\sigma}(x)F^a_{\rho\sigma}(x)+O(t),
\end{aligned}
$$

Small flow time expansion relevant to EMT

- Small flow time expansion (Lüscher–Weisz)

$$G_{\mu\rho}^a(t,x)G_{\nu\rho}^a(t,x)$$
$$\overset{t\to 0}{\sim} \left\langle G_{\mu\rho}^a(t,x)G_{\nu\rho}^a(t,x)\right\rangle$$
$$+ \zeta_{11}(t)F_{\mu\rho}^a(x)F_{\nu\rho}^a(x) + \zeta_{12}(t)\delta_{\mu\nu}F_{\rho\sigma}^a(x)F_{\rho\sigma}^a(x) + O(t),$$

- EMT can then be expressed as (H.S., Del Debbio–Patella–Rago)

$$\{T_{\mu\nu}\}_R = \lim_{t\to 0}\left\{c_1(t)\left[G_{\mu\rho}^a G_{\nu\rho}^a - \frac{1}{4}\delta_{\mu\nu}G_{\rho\sigma}^a G_{\rho\sigma}^a\right]\right.$$
$$\left. + c_2(t)\left[\delta_{\mu\nu}G_{\rho\sigma}^a G_{\rho\sigma}^a - \left\langle\delta_{\mu\nu}G_{\rho\sigma}^a G_{\rho\sigma}^a\right\rangle\right]\right\},$$

Small flow time expansion relevant to EMT

- Small flow time expansion (Lüscher–Weisz)

$$
\begin{aligned}
&G^a_{\mu\rho}(t,x)G^a_{\nu\rho}(t,x) \\
&\overset{t\to 0}{\sim} \left\langle G^a_{\mu\rho}(t,x)G^a_{\nu\rho}(t,x)\right\rangle \\
&\qquad + \zeta_{11}(t)F^a_{\mu\rho}(x)F^a_{\nu\rho}(x) + \zeta_{12}(t)\delta_{\mu\nu}F^a_{\rho\sigma}(x)F^a_{\rho\sigma}(x) + O(t),
\end{aligned}
$$

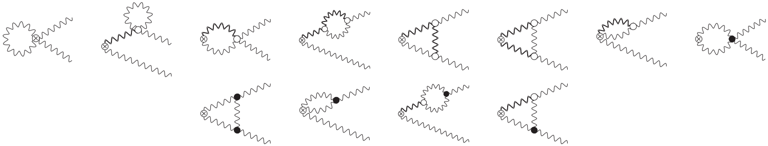
- EMT can then be expressed as (H.S., Del Debbio–Patella–Rago)

$$
\begin{aligned}
\{T_{\mu\nu}\}_R = \lim_{t\to 0}\Bigg\{ & c_1(t)\left[G^a_{\mu\rho}G^a_{\nu\rho} - \frac{1}{4}\delta_{\mu\nu}G^a_{\rho\sigma}G^a_{\rho\sigma}\right] \\
& + c_2(t)\left[\delta_{\mu\nu}G^a_{\rho\sigma}G^a_{\rho\sigma} - \left\langle\delta_{\mu\nu}G^a_{\rho\sigma}G^a_{\rho\sigma}\right\rangle\right]\Bigg\},
\end{aligned}
$$

- Tested for the bulk thermodynamics of quenched QCD (Asakawa–Hatsuda–Itou–Kitazawa–H.S.)

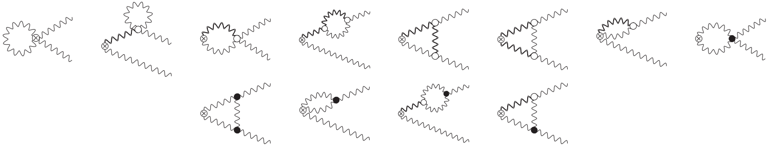
Background covariant expression

- Conventional approach requires cumbersome calculation of 12 diagrams



Background covariant expression

- Conventional approach requires cumbersome calculation of 12 diagrams



- Here, all the information (propagators, vertices) are summarized in ($\hat{\Delta} \equiv \hat{\mathcal{D}}^2 + 2\hat{\mathcal{F}}$):

$$\left\langle G^a_{\mu\rho}(t,x)G^a_{\nu\rho}(t,x)\Big|_{O(b^2)} - F^a_{\mu\rho}(x)F^a_{\nu\rho}(x)\Big|_{O(a^2)}\right\rangle_{1\text{PI}}$$

$$= 2g_0^2\int_0^t \mathrm{d}\xi \Bigg[(\delta_{\mu\alpha}\delta_{\nu\delta}\delta_{\beta\gamma} - \delta_{\mu\alpha}\delta_{\nu\gamma}\delta_{\beta\delta} - \delta_{\mu\beta}\delta_{\nu\delta}\delta_{\alpha\gamma} + \delta_{\mu\beta}\delta_{\nu\gamma}\delta_{\alpha\delta})$$

$$\times \hat{\mathcal{D}}^{ab}_\alpha \left(\mathrm{e}^{2\xi\hat{\Delta}}\right)^{bc}_{\beta\gamma} \hat{\mathcal{D}}^{ca}_\delta$$

$$+ \hat{\mathcal{F}}^{ab}_{\mu\rho}(x)\left(\mathrm{e}^{2\xi\hat{\Delta}}\right)^{ba}_{\rho\nu} + \hat{\mathcal{F}}^{ab}_{\nu\rho}(x)\left(\mathrm{e}^{2\xi\hat{\Delta}}\right)^{ba}_{\rho\mu}\Bigg]\,\delta(x-y)\big|_{y=x}\,.$$

Background covariant expression

- … and the expansion for $t \to 0$ is straightforward and quick (labor $\sim 1/10$)

$$
\begin{aligned}
&\left\langle G^a_{\mu\rho}(t,x)G^a_{\nu\rho}(t,x)\Big|_{O(b^2)} - F^a_{\mu\rho}(x)F^a_{\nu\rho}(x)\Big|_{O(a^2)}\right\rangle_{1\mathrm{PI}} \\
&\overset{t\to 0}{\sim} \frac{g_0^2}{(4\pi)^2}\left[\frac{11}{3}\epsilon(t)^{-1}+\frac{7}{3}\right]\mathrm{tr}\left[\hat{\mathcal{F}}(x)^2\right]_{\mu\nu} \\
&\qquad + \frac{g_0^2}{(4\pi)^2}\left[-\frac{11}{12}\epsilon(t)^{-1}-\frac{1}{6}\right]\delta_{\mu\nu}\,\mathrm{tr}\left[\hat{\mathcal{F}}(x)^2\right]_{\rho\rho} + O(t),
\end{aligned}
$$

where

$$
\epsilon(t)^{-1} \equiv \frac{1}{\epsilon} + \ln(8\pi t).
$$

Background covariant expression

- … and the expansion for $t \to 0$ is straightforward and quick (labor $\sim 1/10$)

$$\left\langle G^a_{\mu\rho}(t,x)G^a_{\nu\rho}(t,x)\Big|_{O(b^2)} - F^a_{\mu\rho}(x)F^a_{\nu\rho}(x)\Big|_{O(a^2)}\right\rangle_{1\text{PI}}$$
$$\overset{t\to 0}{\sim} \frac{g_0^2}{(4\pi)^2}\left[\frac{11}{3}\epsilon(t)^{-1}+\frac{7}{3}\right]\operatorname{tr}\left[\hat{\mathcal{F}}(x)^2\right]_{\mu\nu}$$
$$+\frac{g_0^2}{(4\pi)^2}\left[-\frac{11}{12}\epsilon(t)^{-1}-\frac{1}{6}\right]\delta_{\mu\nu}\operatorname{tr}\left[\hat{\mathcal{F}}(x)^2\right]_{\rho\rho}+O(t),$$

where

$$\epsilon(t)^{-1} \equiv \frac{1}{\epsilon} + \ln(8\pi t).$$

- Actually, the present calculational scheme revealed that I made mistakes in the past diagrammatic calculation…(already identified and fixed)

Summary

- Background gauge covariant gauge fixing in the gradient flow

Summary

- Background gauge covariant gauge fixing in the gradient flow
- Calculational scheme with manifest background gauge covariance

Summary

- Background gauge covariant gauge fixing in the gradient flow
- Calculational scheme with manifest background gauge covariance
- Quick one-loop calculation of the small flow time expansion relevant to EMT

Summary

- Background gauge covariant gauge fixing in the gradient flow
- Calculational scheme with manifest background gauge covariance
- Quick one-loop calculation of the small flow time expansion relevant to EMT
- Also for the topological charge density,

$$\epsilon_{\mu\nu\rho\sigma} G_{\mu\nu}^a(t,x) G_{\rho\sigma}^a(t,x) \overset{t\to 0}{\sim} \left(1 + 0 \cdot g_0^2\right) \epsilon_{\mu\nu\rho\sigma} F_{\mu\nu}^a(x) F_{\rho\sigma}^a(x) + O(t).$$

Summary

- Background gauge covariant gauge fixing in the gradient flow
- Calculational scheme with manifest background gauge covariance
- Quick one-loop calculation of the small flow time expansion relevant to EMT
- Also for the topological charge density,

$$\epsilon_{\mu\nu\rho\sigma} G^a_{\mu\nu}(t,x) G^a_{\rho\sigma}(t,x) \overset{t\to 0}{\sim} \left(1 + 0 \cdot g_0^2\right) \epsilon_{\mu\nu\rho\sigma} F^a_{\mu\nu}(x) F^a_{\rho\sigma}(x) + O(t).$$

- Fermion flow can be included

Summary

- Background gauge covariant gauge fixing in the gradient flow
- Calculational scheme with manifest background gauge covariance
- Quick one-loop calculation of the small flow time expansion relevant to EMT
- Also for the topological charge density,

$$\epsilon_{\mu\nu\rho\sigma} G^a_{\mu\nu}(t,x) G^a_{\rho\sigma}(t,x) \overset{t\to 0}{\sim} \left(1 + 0 \cdot g_0^2\right) \epsilon_{\mu\nu\rho\sigma} F^a_{\mu\nu}(x) F^a_{\rho\sigma}(x) + O(t).$$

- Fermion flow can be included
- Simpler proof of the renormalizability (than Lüscher–Weisz)?

Summary

- Background gauge covariant gauge fixing in the gradient flow
- Calculational scheme with manifest background gauge covariance
- Quick one-loop calculation of the small flow time expansion relevant to EMT
- Also for the topological charge density,

$$\epsilon_{\mu\nu\rho\sigma} G^a_{\mu\nu}(t,x) G^a_{\rho\sigma}(t,x) \overset{t\to 0}{\sim} \left(1 + 0 \cdot g_0^2\right) \epsilon_{\mu\nu\rho\sigma} F^a_{\mu\nu}(x) F^a_{\rho\sigma}(x) + O(t).$$

- Fermion flow can be included
- Simpler proof of the renormalizability (than Lüscher–Weisz)?
- Two-loop calculation of the small flow time expansion?