# Complex Langevin simulation in condensed matter physics

Arata Yamamoto (University of Tokyo)

with Tomoya Hayata (RIKEN)

## Complex Langevin Method

Euclidean path integral:

$$Z = \int D\Phi \, e^{-S[\Phi]}$$

$$S[\Phi] \neq S^*[\Phi] \longrightarrow$$ "sign problem"

ex) chemical potential, electric field, theta term, real time, matrix model, non-relativistic system, spin system

# Complex Langevin Method

complex Langevin equation:

$$\Phi \in \mathbf{R} \rightarrow \tilde{\Phi} \in \mathbf{C}$$

$$\frac{\partial}{\partial \theta}\tilde{\Phi} = -\frac{\delta}{\delta\tilde{\Phi}}S[\tilde{\Phi}] + \eta$$

fictitious time ($\theta$) noise field ($\eta$)

expectation value:

$$\langle \hat{O}[\tilde{\Phi}] \rangle = \lim_{\theta \to \infty} \langle \hat{O}[\tilde{\Phi}(\theta)] \rangle_\eta$$

## Complex Langevin Method

complex Langevin simulations of

**Bose system & Fermi system**

in condensed matter physics

- ✓ for the ab-initio analysis of condensed matter systems
- ✓ for a test of the complex Langevin method

# Bose System

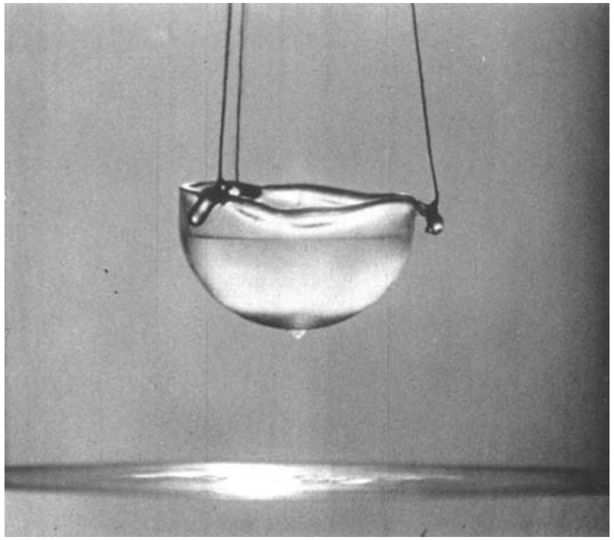
non-relativistic Bose gas:

$$S_0[\Phi_1, \Phi_2] = \int d\tau d^3x \left[ \Phi^*(x) \left( \frac{\partial}{\partial \tau} - \mu - \frac{1}{2m}\Delta \right) \Phi(x) + \frac{1}{4}\lambda |\Phi(x)|^4 \right]$$

anti-Hermitian

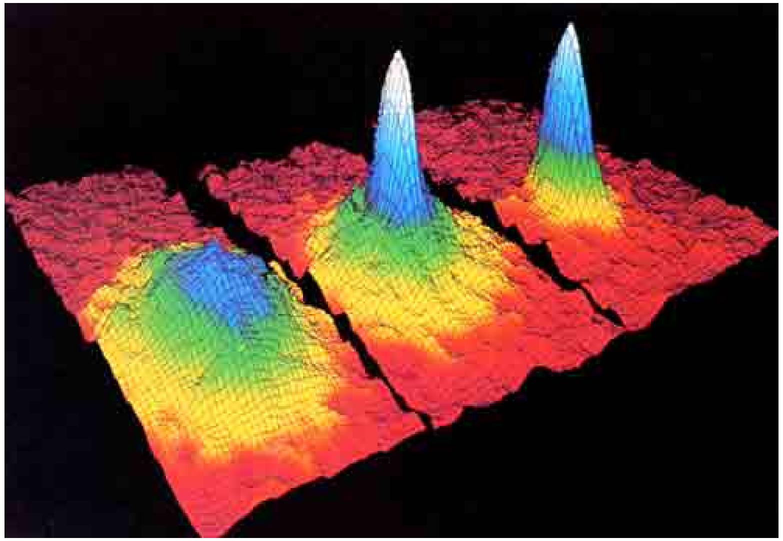
superfluid $^4$He

from wikipedia



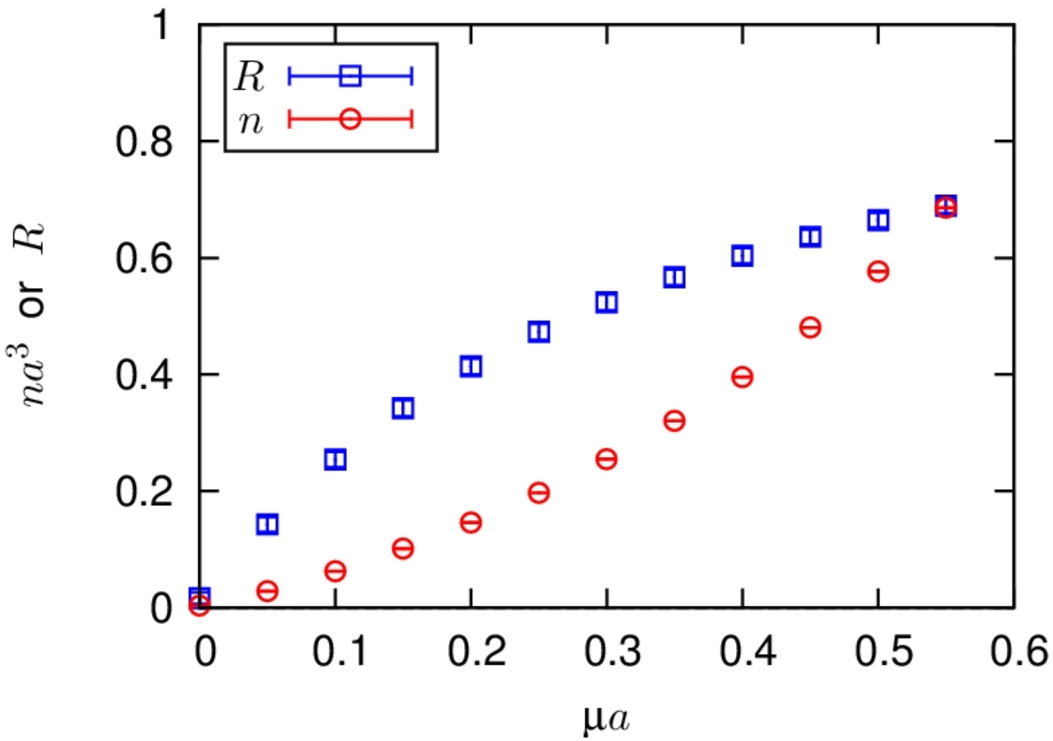
atomic BEC

JILA group experiment



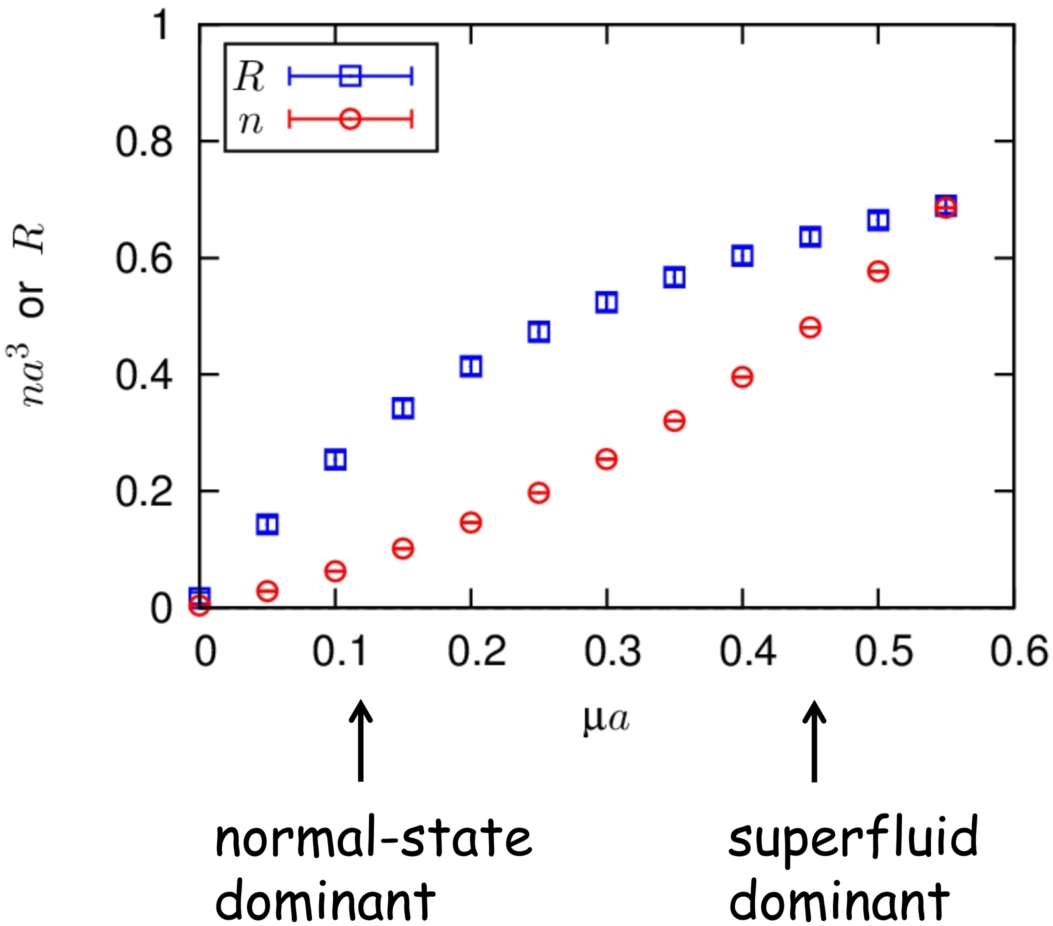
# Bose System

condensate fraction: $R = \lim_{|x-y|\to\infty} \frac{\langle \Phi^*(x)\Phi(y)\rangle}{\langle \Phi^*(x)\Phi(x)\rangle}$



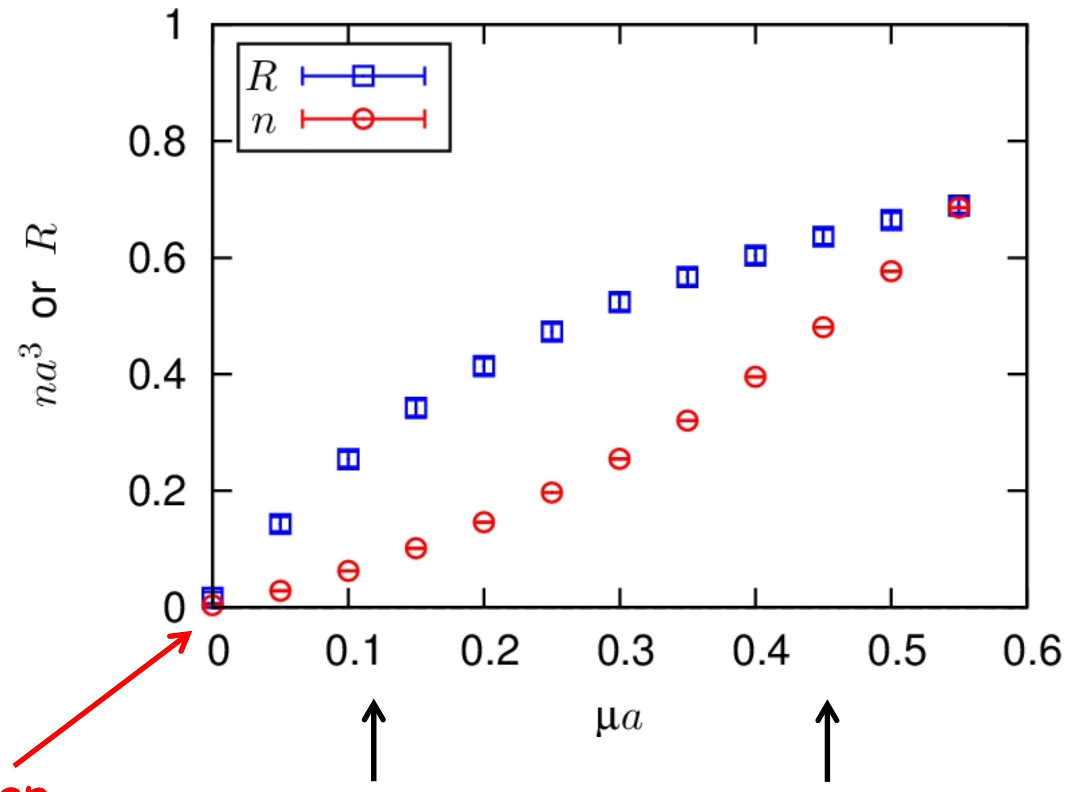
# Bose System

condensate fraction: $R = \lim_{|x-y|\to\infty} \frac{\langle \Phi^*(x)\Phi(y)\rangle}{\langle \Phi^*(x)\Phi(x)\rangle}$



## Bose System

condensate fraction: $R = \lim_{|x-y| \to \infty} \frac{\langle \Phi^*(x)\Phi(y) \rangle}{\langle \Phi^*(x)\Phi(x) \rangle}$




no gap
(no Silver Blaze)

normal-state dominant

superfluid dominant

## Rotation

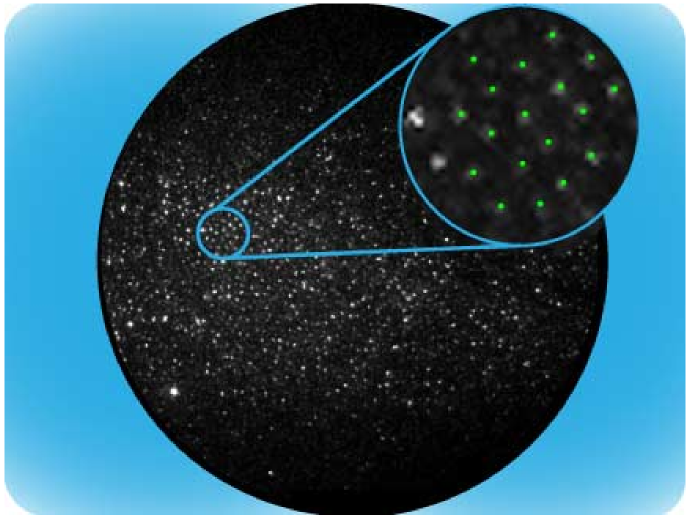
rotating Bose gas:

$$S_\Omega = S_0 - \Omega L_z$$

$$L_z = -i \int d\tau d^3x\, \Phi^*(x) \left( x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} \right) \Phi(x)$$

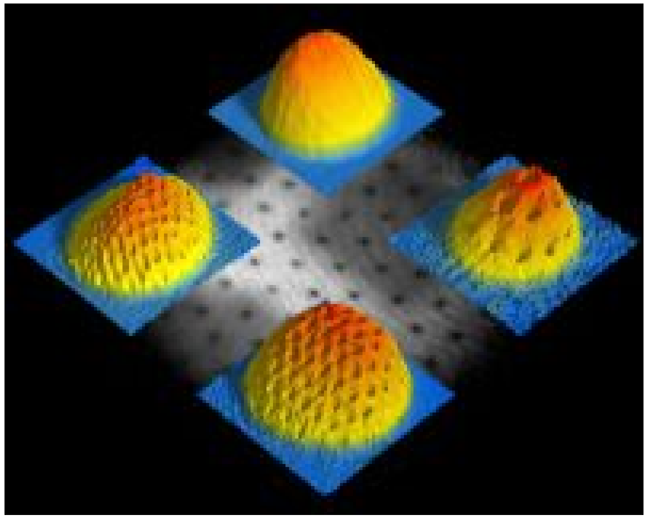
vortex in $^4$He

Maryland group experiment

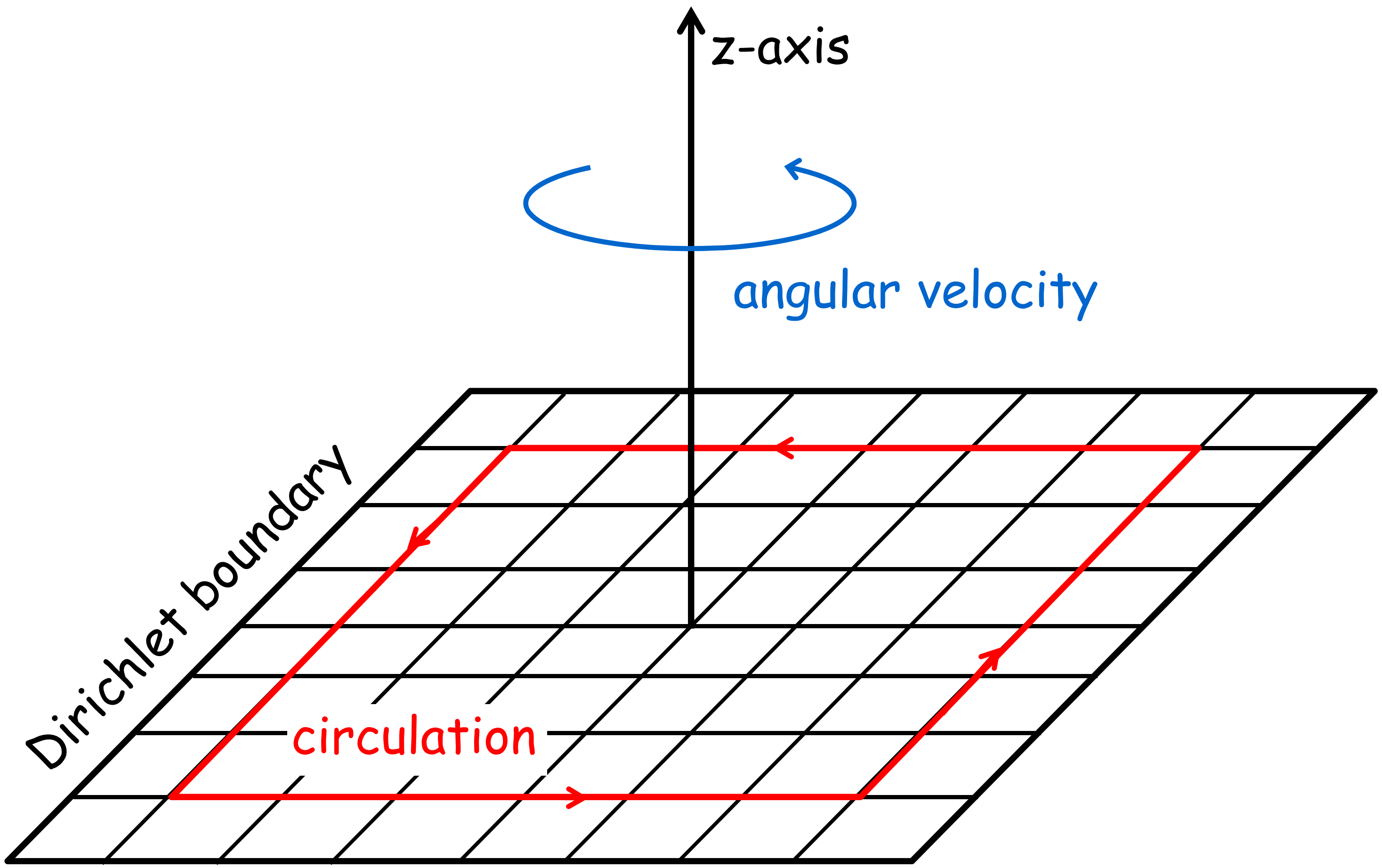


vortex in atomic BEC

MIT group experiment



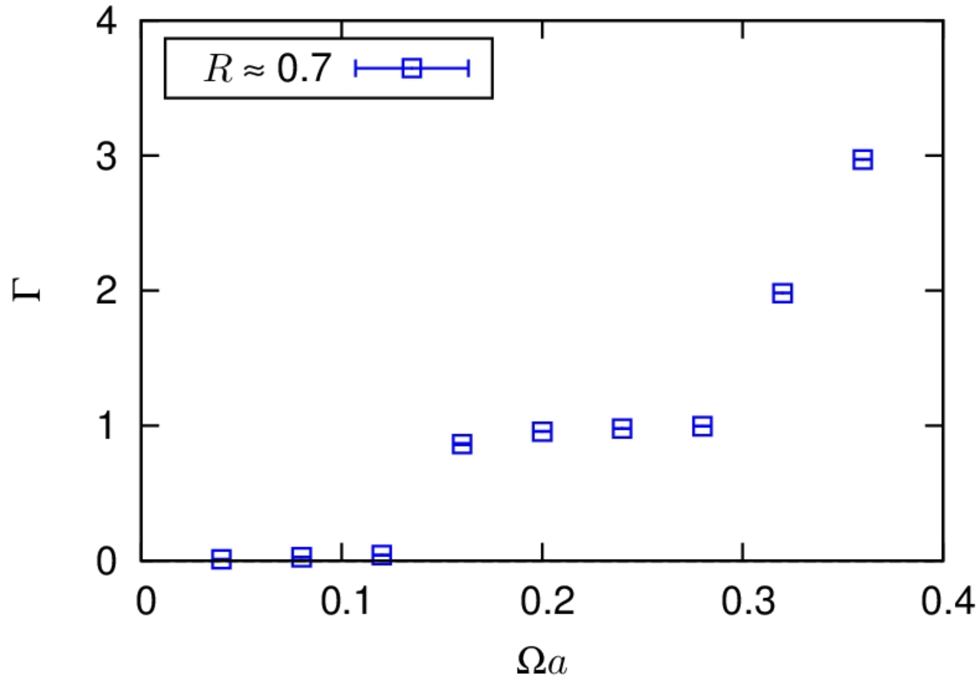
## Rotation




circulation : $\hat{\Gamma} = \frac{1}{2\pi} \oint \delta\Theta = N_{\rm vortex}$

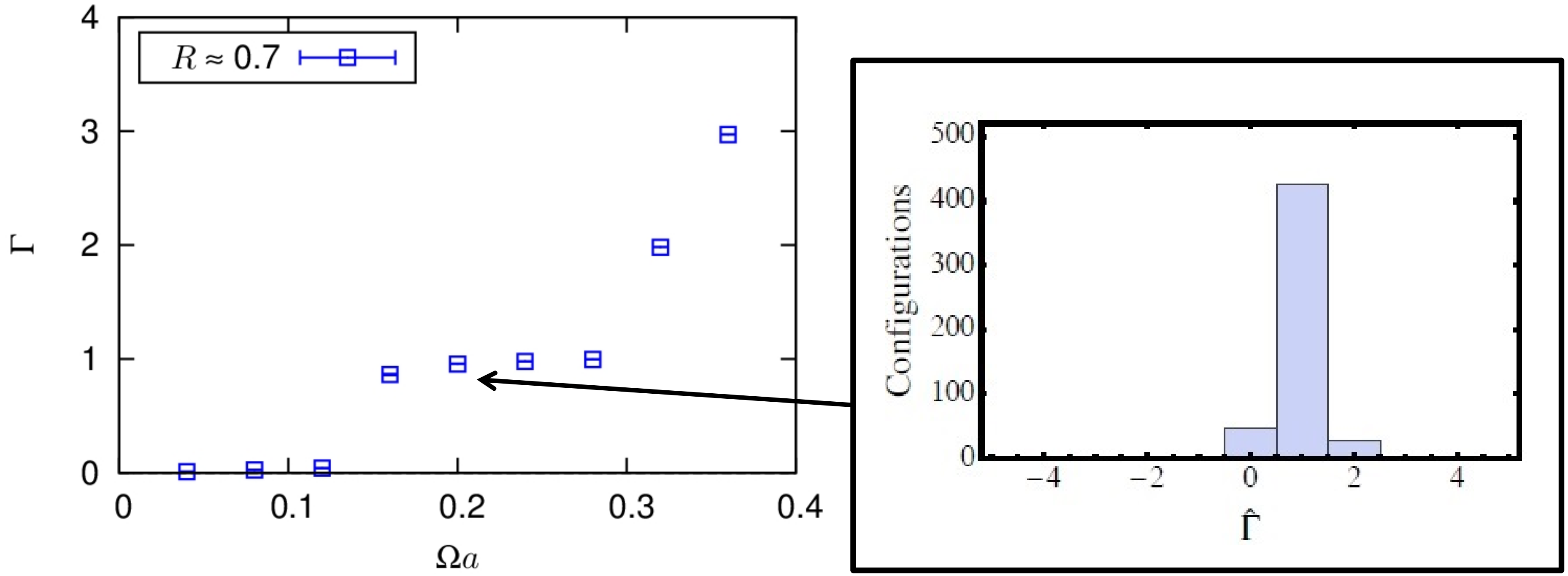
# Rotation

circulation (superfluid dominant)



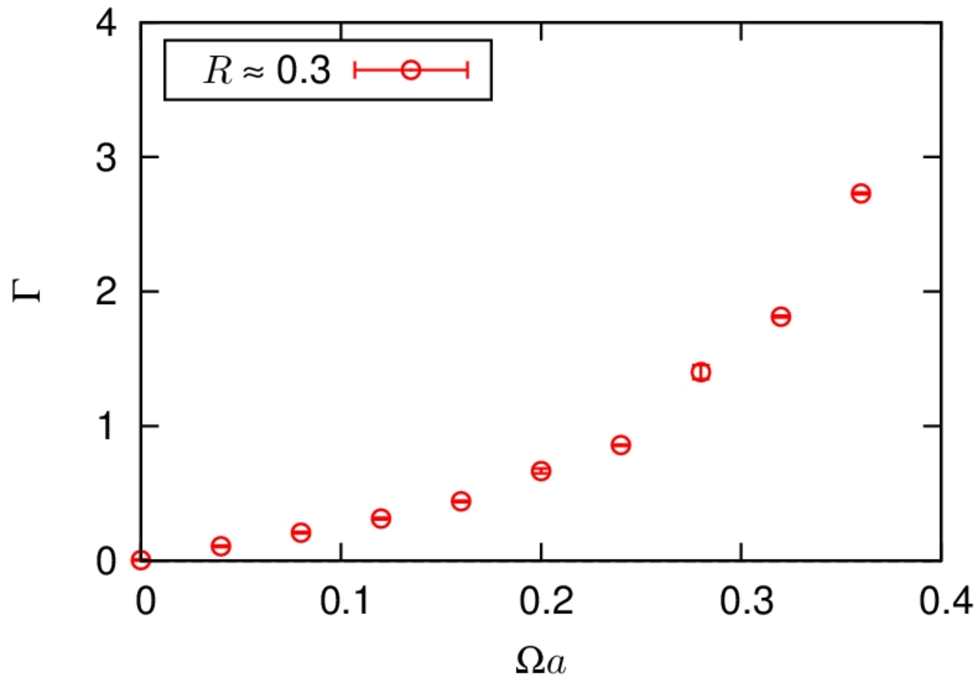
# Rotation

circulation (superfluid dominant)



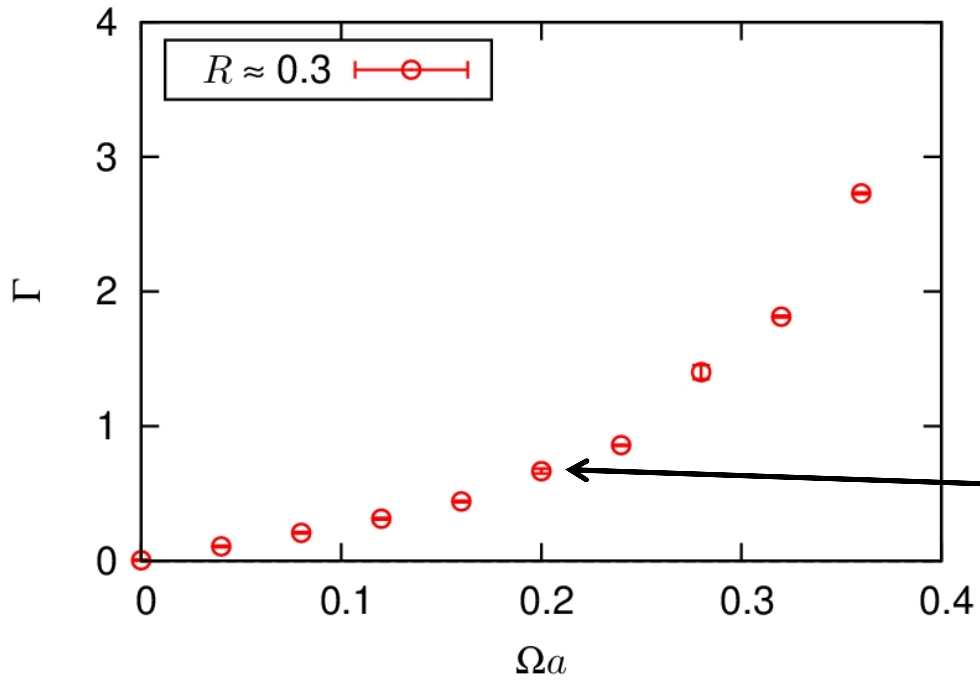
# Rotation

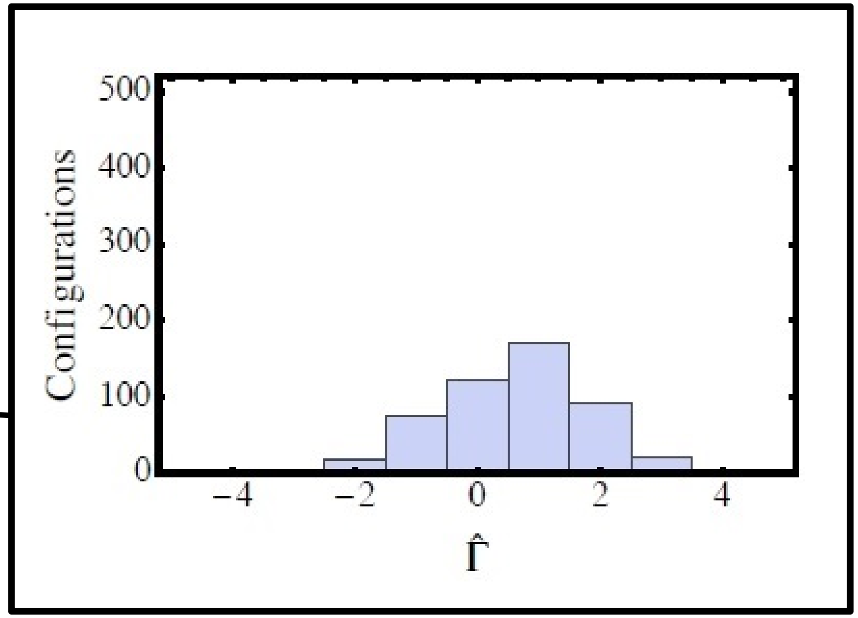
circulation (normal-state dominant)



# Rotation

circulation (normal-state dominant)



# Fermi System

Hubbard model:

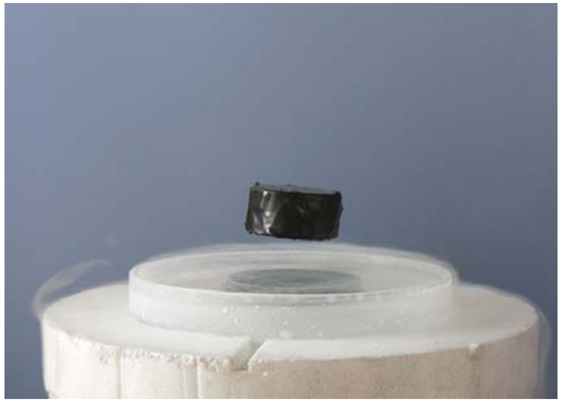
$$S[\Psi_\uparrow^*,\Psi_\uparrow,\Psi_\downarrow^*,\Psi_\downarrow] = \int d\tau \sum_x \Bigg[ \sum_{i=\uparrow,\downarrow} \Psi_i^*(x) \left( \frac{\partial}{\partial\tau} - \mu_i \right) \Psi_i(x)$$

$$- \sum_{i=\uparrow,\downarrow} \sum_j t_i \left( \Psi_i^*(x)\Psi_i(x+\hat{j}) + \Psi_i^*(x+\hat{j})\Psi_i(x) \right)$$

$$+ U \Psi_\uparrow^*(x)\Psi_\uparrow(x)\Psi_\downarrow^*(x)\Psi_\downarrow(x) \Bigg]$$

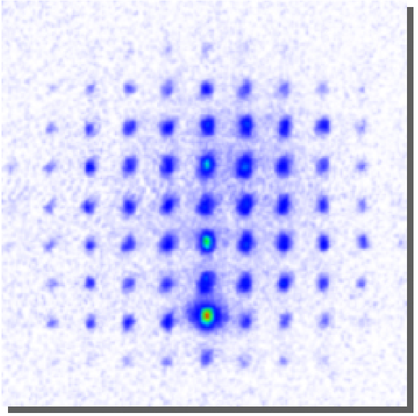
high-Tc superconductor

from wikipedia



optical lattice of atom gas

LMU-MPQ group experiment



## Fermi System

Hubbard model:

$$Z = \int D\Phi \, \det K_{\uparrow}[\Phi] \det K_{\downarrow}[\Phi] e^{-S_A[\Phi]}$$

$$K_i[\Phi] = \frac{\partial}{\partial \tau} - \mu_i + \Phi(x) - \sum_j t_i (T_{+j} + T_{-j})$$

$$S_A[\Phi] = \int d\tau \sum_x \frac{1}{2|U|} \Phi^2(x)$$

## Fermi System

$$\det K[t, \mu] \in \mathbf{R}$$

balanced:

$$\det K[t, \mu] \det K[t, \mu] \geq 0$$

no problem

imbalanced:

$$\det K[t_\uparrow, \mu_\uparrow] \det K[t_\downarrow, \mu_\downarrow] \gtrless 0$$

sign problem

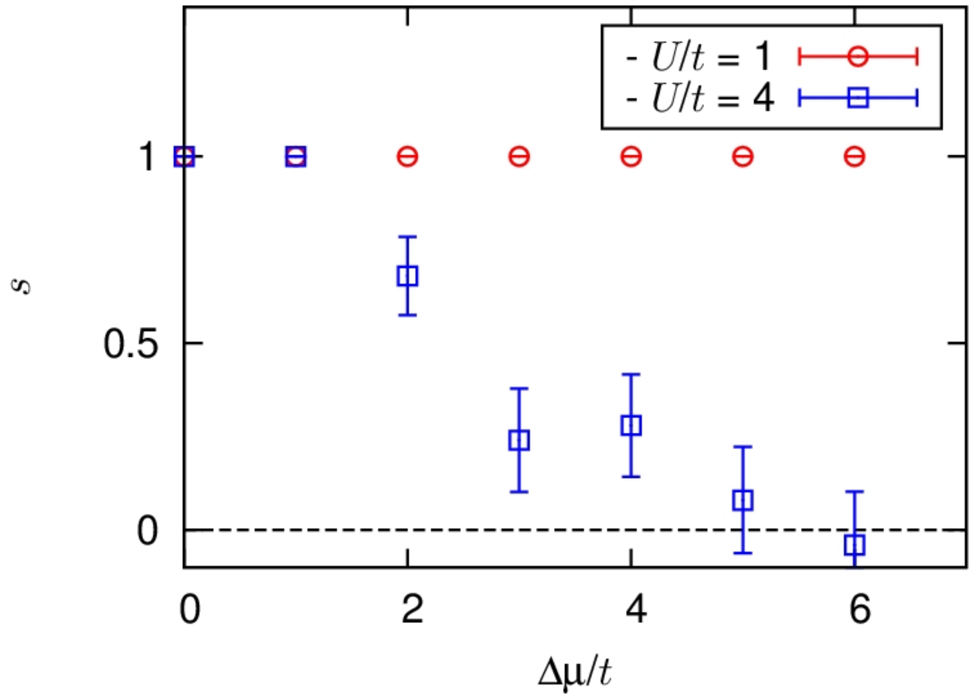
cf) QCD

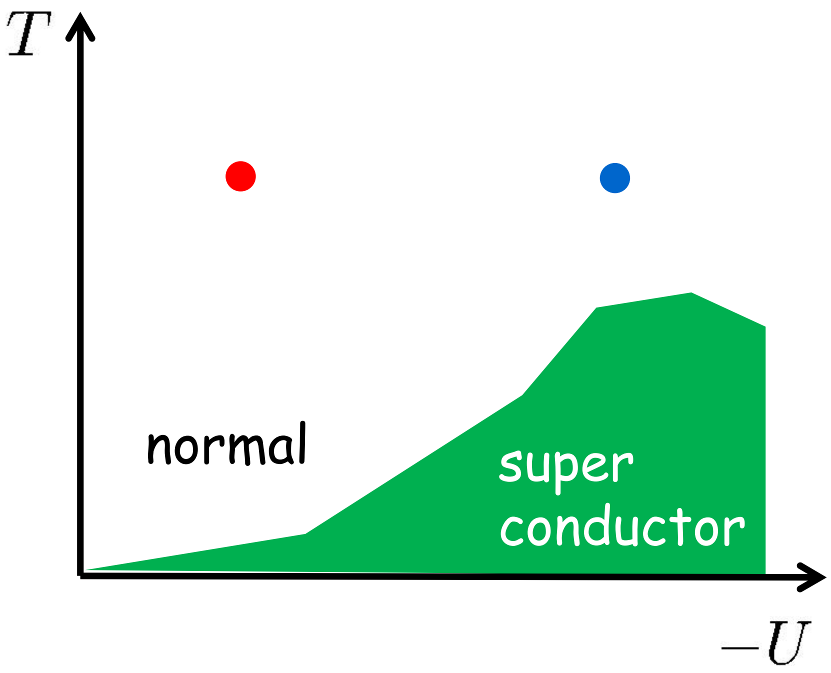
$$\det D[m, \mu] \det D[m, \mu] \in \mathbf{C}$$

sign problem

# Fermi System

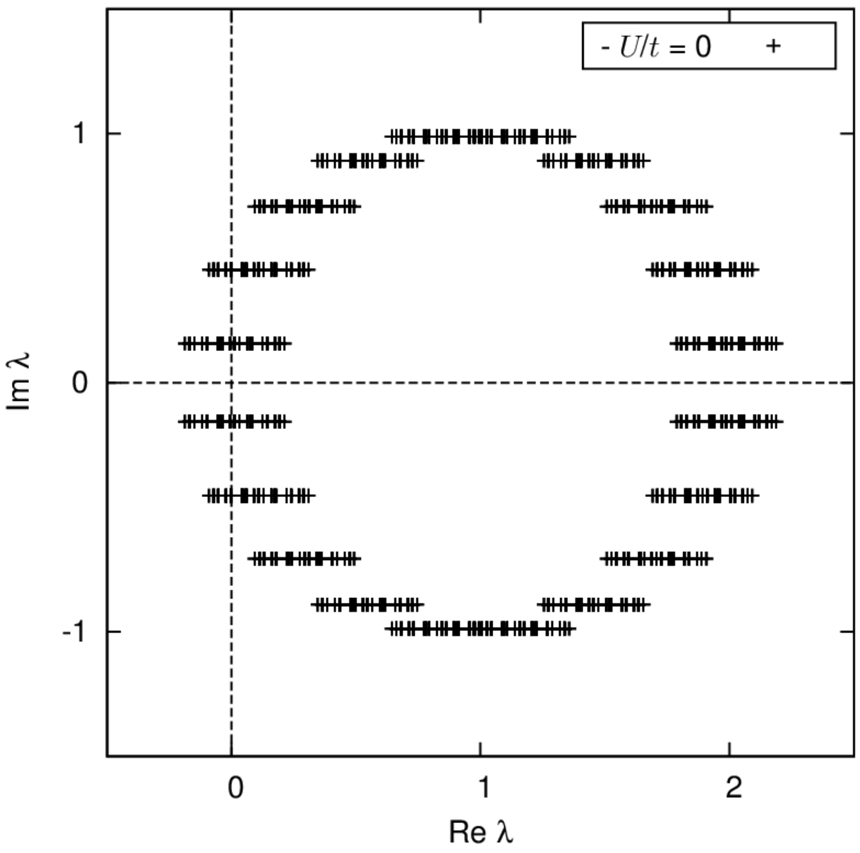
average sign in "sign-quenched" Monte Carlo



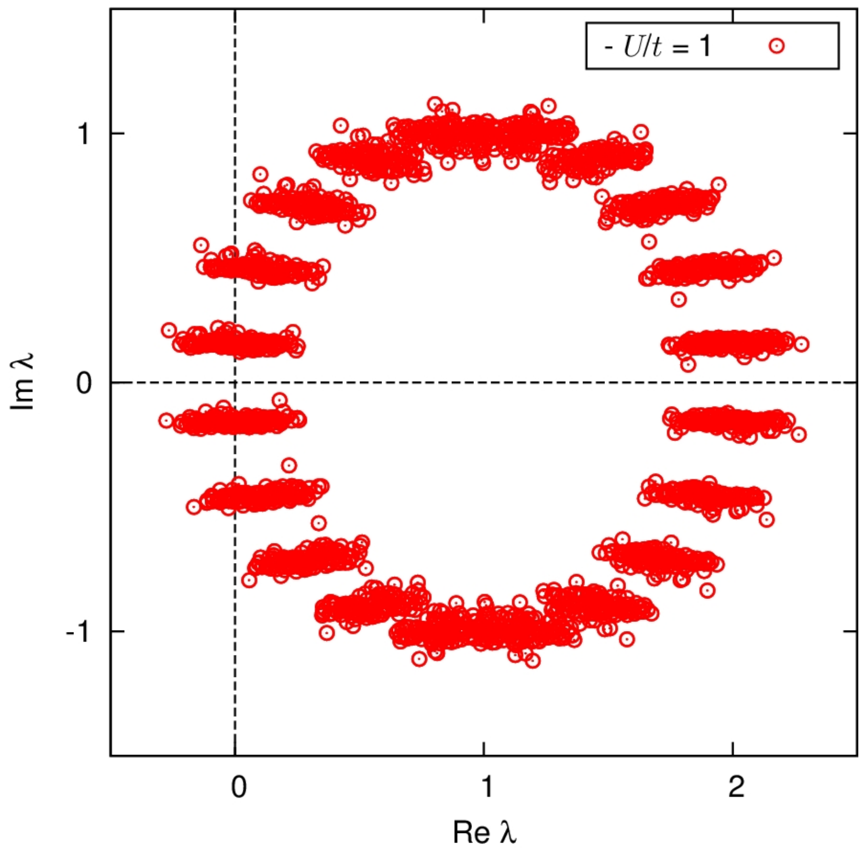
# Fermi System

eigenvalues (free)



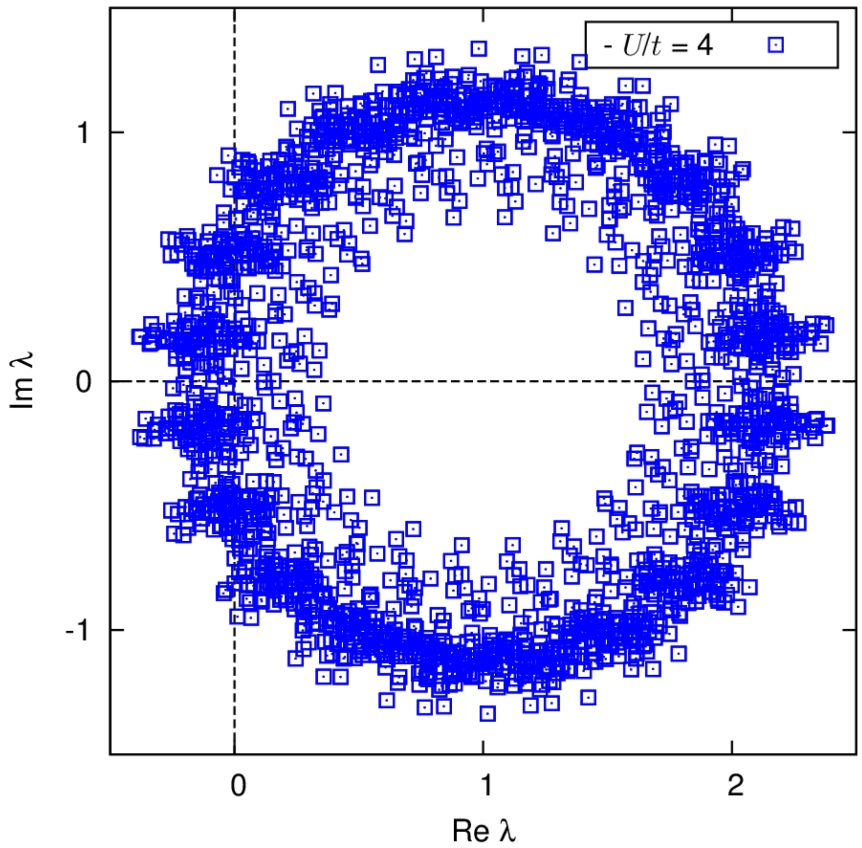
# Fermi System

eigenvalues (weak coupling)



# Fermi System

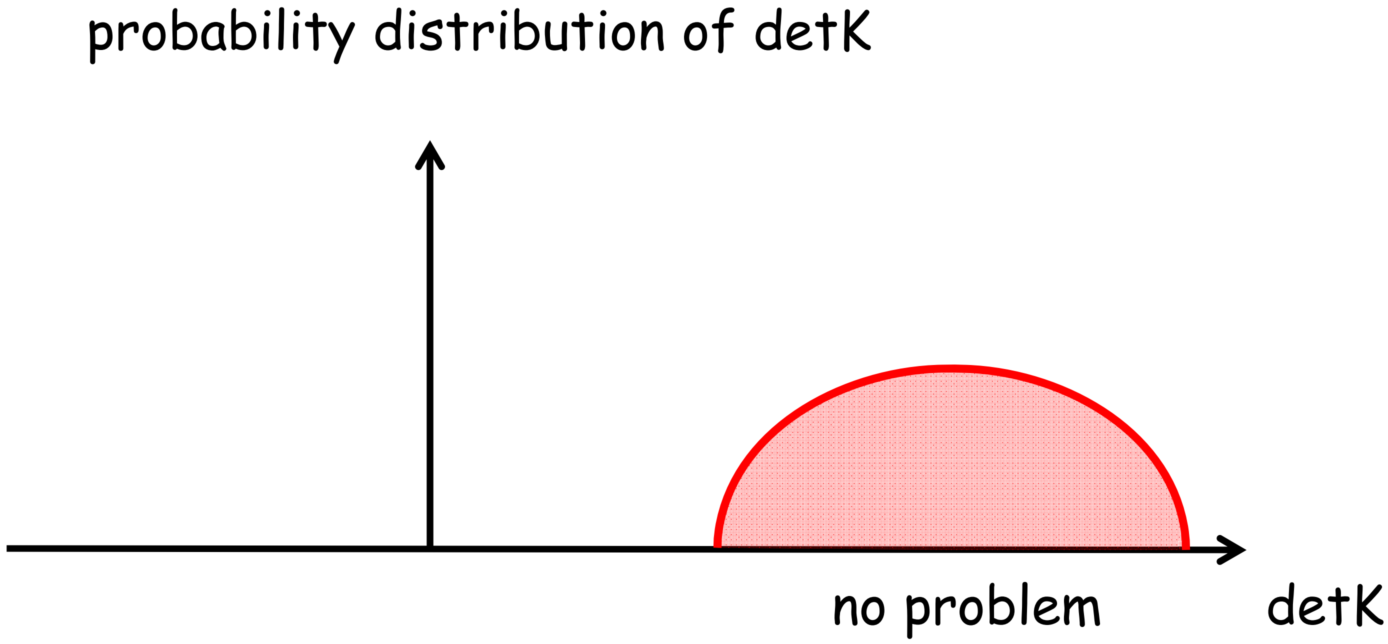
eigenvalues (strong coupling)




logarithmic singularity at zero eigenvalues

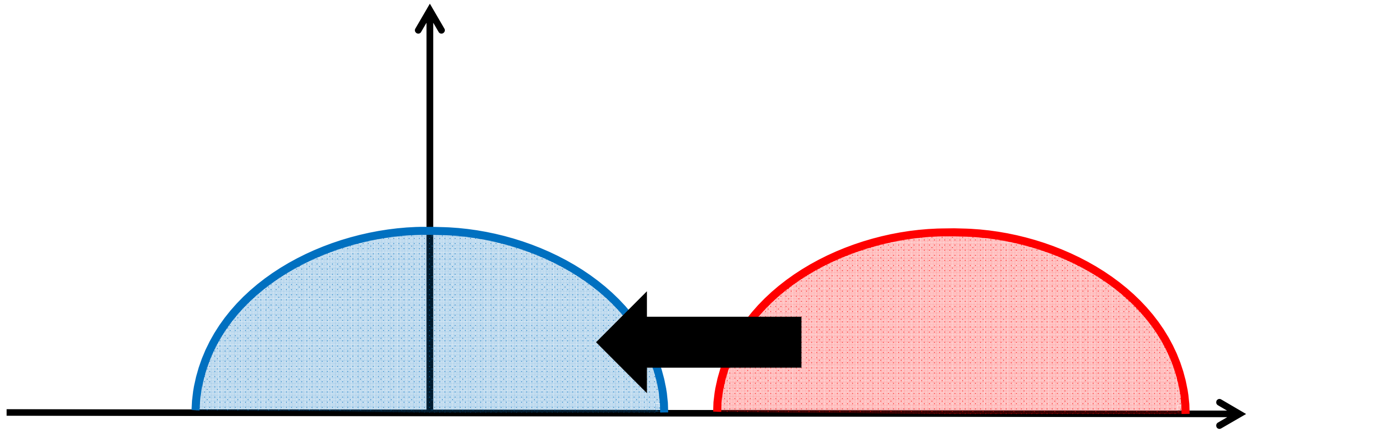
Mollgaard Splittorff '13, Greensite '14, Nishimura Shimasaki '15

## Fermi System



## Fermi System

probability distribution of detK




sign problem
&
singularity problem

no problem

## Summary

Boson is good.

Fermion is bad.