

On calculating **disconnected-type**  
**hadronic light-by-light scattering diagrams**  
from lattice QCD

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## Background and motivation

- We focus on the **anomalous magnetic dipole moment** of the muon,  $a_\mu$ .

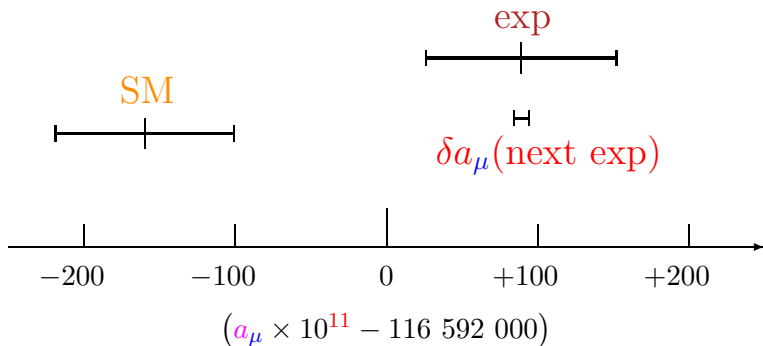


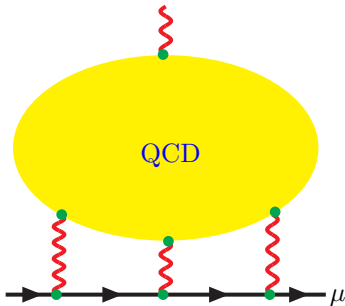
Figure: Theoretical prediction (SM) to  $a_\mu$  and its measured value (exp).

## Hadronic light-by-light scattering contribution

- There are *two* kinds of QCD contributions important to the most precise comparison between  $a_\mu(\text{SM})$  and  $a_\mu(\text{exp})$ :
  - **Hadronic vacuum polarization contribution (HVP)** induced by



- **Hadronic light-by-light scattering contribution (HLBL)**,  $a_\mu(\text{had-lbyl})$



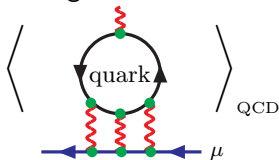
## Significance of lattice study of HLBL

$10^{11} \times a_\mu(\text{had-lbyl}) =$	116 (40)
$10^{11} \times \{a_\mu(\text{exp}) - a_\mu(\text{SM})\} =$	249 (87)
$10^{11} \times \delta a_\mu(\text{next exp}) =$	$O(1)$

- Despite its importance,  $a_\mu(\text{had-lbyl})$  has been *only estimated* by the low energy effective theory and the models with several hadrons as dynamical variables.
- Imprecise knowledge on chiral dynamics and *ignorance on the other hadronic intermediate state contributions hinder quantification of uncertainty.*
- **Computation of  $a_\mu(\text{had-lbyl})$  by means of the lattice QCD is indispensable** to make a more reliable prediction of  $a_\mu$ .

## Connected-type diagrams in HLBL

- Check of feasibility of lattice QCD study on HLBL :
  - T. Blum, S. Chowdhury, M. H and T. Izubuchi, Phys. Rev. Lett. **114**, 012001 (2015) [arXiv:1407.2923 [hep-lat]]
  - Luchang's talk on new methods and extensive calculation
- For the hadronic light-by-light scattering amplitude, J. Green. O. Gryniuk, G. Hippel, H. B. Meyer and V. Pascalutsa, arXiv:1507.01577.
- Lattice QCD simulation has been carried out for the **connected-type** diagrams :

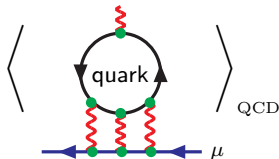


+ (permutations of QED vertices(●)  
on muon side),

where each **line in the quark part** denotes  $D_q[U]^{-1}$  for a given gluon field  $U$ .

## Disconnected-type diagrams in HLBL

- The **connected-type** diagrams are the ones in which four **QED interaction vertices** in the QCD part lie on **a single quark loop**.



- There are **disconnected-type** diagrams in which four **QED vertices** in the QCD part are distributed over **more than one quark loop**.

## An example of disconnected-type diagram; ( $1_E, 1, 1, 1$ )-type

- If four QED interaction vertices are distributed over *four separate quark loops*,

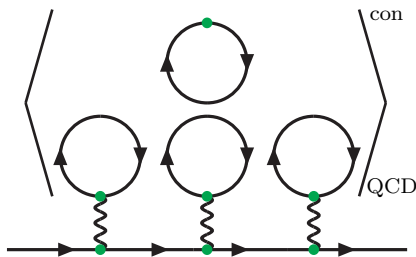


Figure: ( $1_E, 1, 1, 1$ )-type diagrams

Quark loops are connected with one another by the gluons to be supplied by the average over QCD configurations.

## All disconnected-type diagrams

- Disconnected-type diagrams can be classified *according to how* four QED interaction vertices are distributed over more than one quark loop :

$$(3_E, 1)$$

$$(2_E, 2)$$

$$(1_E, 3)$$

$$(2_E, 1, 1)$$

$$(1_E, 1, 2)$$

$$(1_E, 1, 1, 1)$$

- Each number denotes the total number of QED vertices contained in a quark loop.
- In particular, the number with the subscript  $E$  denotes the total number of QED vertices contained in the quark loop with the external QED vertex.



## Diagrams of $(3_E, 1)$ -type

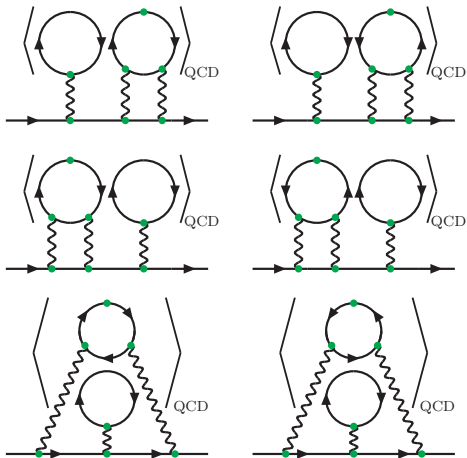


FIG. 1:  $(3_E, 1)$ -type diagrams. The diagrams with  $O(a)$  local QED vertices are not shown.

## Diagrams of $(2_E, 2)$ -type

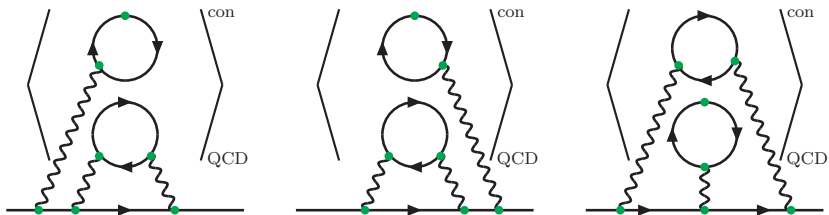


FIG. 1:  $(2_E, 2)$ -type diagrams

## Diagrams of $(1_E, 3)$ -type

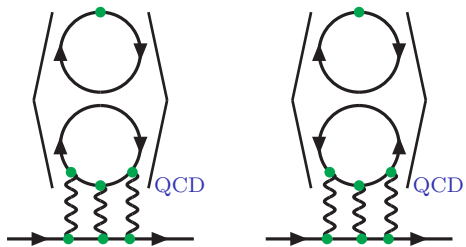


Figure:  $(1_E, 3)$ -type diagrams

## Diagrams of $(2_E, 1, 1)$ -type

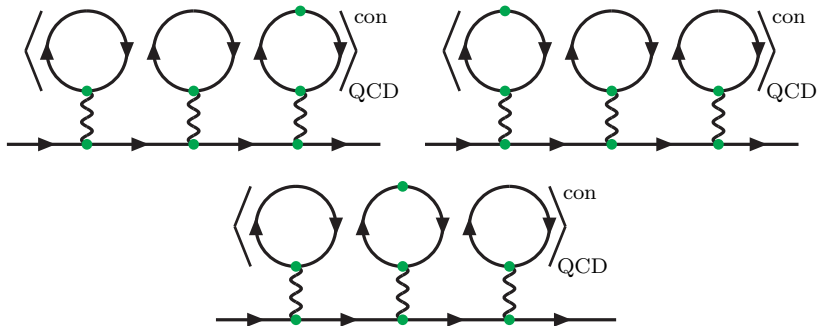


Figure:  $(2_E, 1, 1)$ -type diagrams

## Diagrams of $(1_E, 1, 2)$ -type

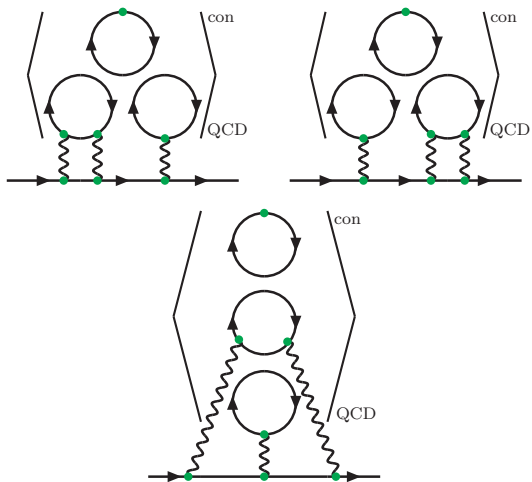


FIG. 1:  $(1_E, 1, 2)$ -type diagrams

## Diagrams of $(1_E, 1, 1, 1)$ -type

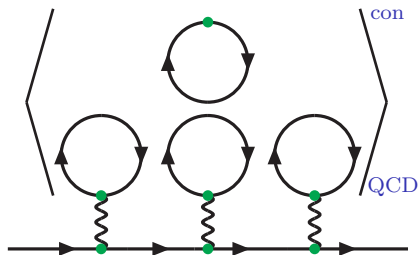


Figure:  $(1_E, 1, 1, 1)$ -type diagrams

# Method to calculate disconnected-type diagrams

- One method to calculate all HLBL including disconnected-type diagrams was proposed in  
T. Blum, M. H. and T. Izubuchi, PoS LATTICE 2012, 022 (2012)  
[arXiv:1301.2607 [hep-lat]]  
which will be reviewed later on.
- It needs modification because **one point was overlooked**.
- The succeeding content of this talk is as follows;
  - I explain the overlooked point, which is actually **a general remark that must be taken into account** in any method to calculate the disconnected-type HLBL diagrams.
  - I illustrate **how to reflect such a remark in the calculation** by modifying the **nonperturbative QED method** in PoS LATTICE 2012, 022 (2012).

## Field-theoretically disconnected components

- Lattice QCD simulation may enable to compute the VEV of four **hadronic electromagnetic (EM) currents**  $j_\mu(x)$

$$\begin{aligned} & \left\langle j_{\mu(1)}(x(1)) j_{\mu(2)}(x(2)) j_{\mu(3)}(x(3)) j_{\mu(4)}(x(4)) \right\rangle_{\text{QCD}} \\ &= \frac{1}{Z_{\text{QCD}}} \int dU \int dq d\bar{q} e^{-S_{\text{QCD}}[U, q, \bar{q}]} \\ & \quad j_{\mu(1)}(x(1)) j_{\mu(2)}(x(2)) j_{\mu(3)}(x(3)) j_{\mu(4)}(x(4)). \end{aligned}$$

- It *inevitably contains* the contribution caused by *field-theoretically disconnected diagrams*.
- A **field-theoretically disconnected diagram** is a *Feynman diagram consisting of more than one connected component*.



## Field-theoretically disconnected components

- The contribution from all field-theoretically connected diagrams to  $\mathcal{A}$  is denoted by  $\langle \mathcal{A} \rangle_{\text{QCD}}^{\text{con}}$ .
- The quantity necessary for HLBL is  $\langle j_{\mu(1)}(x(1)) j_{\mu(2)}(x(2)) j_{\mu(3)}(x(3)) j_{\mu(4)}(x(4)) \rangle_{\text{QCD}}^{\text{con}}$ .
- For instance, (In momentum space, disconnected components appear only for special kinematics.)

The diagrammatic equation illustrates the decomposition of a two-loop disconnected correlator into connected and disconnected parts. On the left, a large bracketed expression contains two separate one-loop diagrams, each with two external vertices (green dots) and a loop with two arrows. This is equal to the product of two one-loop diagrams (connected part) plus a term with two one-loop diagrams (disconnected part) labeled with a 'con' superscript. The 'con' label is positioned above the second diagram in the sum.

$$\langle \text{two loops} \rangle_{\text{QCD}} = \langle \text{one loop} \rangle_{\text{QCD}} \times \langle \text{one loop} \rangle_{\text{QCD}} + \langle \text{two loops} \rangle_{\text{QCD}}^{\text{con}}$$

## Field-theoretically disconnected components

- In lattice QCD simulation, it is **difficult to isolate**  $\langle \mathcal{A} \rangle_{\text{QCD}}^{\text{con}}$  *directly*.
- **Every method** to calculate the HLBL diagrams with more than one quark loop with QED vertices (*i.e.* disconnected-type ones in terms of lattice field theory,) **inevitably picks up**  $O(\alpha^3)$  HVP, called *unwanted contribution* hereafter.
- If we adopt the value for HVP derived using experimental data, we **must subtract such an unwanted contribution in the Green function to avoid overcounting**.
- The **feasibility of the method** is affected by the possibility of subtraction of the unwanted contribution, and by the efficiency of subtraction scheme even if possible.

# Nonperturbative QED method

- To see how subtraction of disconnected components is realized, we try to modify the method proposed in  
T. Blum, M. H. and T. Izubuchi, PoS LATTICE 2012, 022 (2012)  
[arXiv:1301.2607 [hep-lat]]
- The term  $(-\mathcal{K}_D)$  is introduced to remove  $O(\alpha^3)$  HVP contribution

$$\frac{1}{3} \{ (\mathcal{M}_C - \mathcal{S}_C) + (\mathcal{M}_{C'} - \mathcal{S}_{C'}) + (\mathcal{M}_D - \mathcal{S}_D) - \mathcal{K}_D \}$$

# Nonperturbative QED method

$$\frac{1}{3} \{ (\mathcal{M}_C - \mathcal{S}_C) + (\mathcal{M}_{C'} - \mathcal{S}_{C'}) + (\mathcal{M}_D - \mathcal{S}_D) - \mathcal{K}_D \}$$

$$\mathcal{M}_C = \left\langle \begin{array}{c} \text{Diagram: A fermion loop with a photon (wavy line) and a gluon (curly line) attached to the loop. The loop is connected to an external fermion line.} \\ \text{QCD + QED} \end{array} \right\rangle$$

$$\mathcal{S}_C = \left\langle \begin{array}{c} \text{Diagram: Same as } \mathcal{M}_C \text{, but with a ghost loop (curly line) instead of a gluon loop.} \\ \text{QCD + QED} \end{array} \right\rangle$$

$$\mathcal{M}_{C'} = \left\langle \begin{array}{c} \text{Diagram: Same as } \mathcal{M}_C \text{, but with a ghost loop (curly line) instead of a gluon loop.} \\ \text{QCD + QED} \end{array} \right\rangle$$

$$\mathcal{S}_{C'} = \left\langle \begin{array}{c} \text{Diagram: Same as } \mathcal{S}_C \text{, but with a gluon loop (curly line) instead of a ghost loop.} \\ \text{QCD + QED} \end{array} \right\rangle$$

# Nonperturbative QED method

$$\frac{1}{3} \{ (\mathcal{M}_C - \mathcal{S}_C) + (\mathcal{M}_{C'} - \mathcal{S}_{C'}) + (\mathcal{M}_D - \mathcal{S}_D) - \mathcal{K}_D \}$$

$$\begin{aligned}
 \mathcal{M}_D &= \left\langle \begin{array}{c} \text{QCD} \\ \text{QED} \end{array} \right\rangle_{\text{QCD} + \text{QED}} \\
 \mathcal{S}_D &= \left\langle \begin{array}{c} \text{QCD} \\ \text{QED} \end{array} \right\rangle_{\text{QCD} + \text{QED}} \\
 \mathcal{K}_D &= \left\langle \begin{array}{c} \text{D} [U_{(1)} e^{-i Q_q e A_{(1)}}]^{-1} \\ \text{D} [U_{(2)} e^{-i Q_{q'} e A_{(2)}}]^{-1} \\ \text{D} [e^{-i Q_\mu e A_{(1)}} e^{-i Q_\mu e A_{(2)}}]^{-1} \end{array} \right\rangle_{(U_{(1)}, A_{(1)}), (U_{(2)}, A_{(2)})}
 \end{aligned}$$

# Nonperturbative QED method

Every HLBL diagram is generated with *triplicate redundancy*:

**Table:** The multiplication factor of each diagram provided by the individual terms in nonperturbative QED method.  $C$ , say, denotes  $\mathcal{M}_C - \mathcal{S}_C$ .

	$C + C'$	$D$
$4_E$	3	0
$(1_E, 3)$	0	3
$(2_E, 2)$	1	2
$(3_E, 1)$	2	1
$(1_E, 1, 2)$	0	3
$(2_E, 1, 1)$	1	2
$(1_E, 1, 1, 1)$	0	3

## Nonperturbative QED method

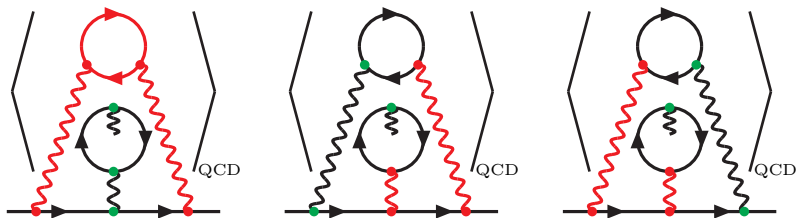
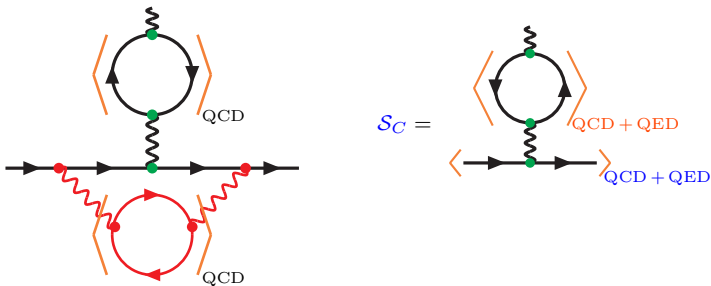


Figure: An identical diagram of  $(2_E, 2)$ -type is generated in three ways from  $\mathcal{M}_C$  (left) and  $\mathcal{M}_D$  (middle, right). The red stuffs are generated by the ensemble average of  $(\text{QCD} + \text{QED})$ .

## Structure of unwanted HVP

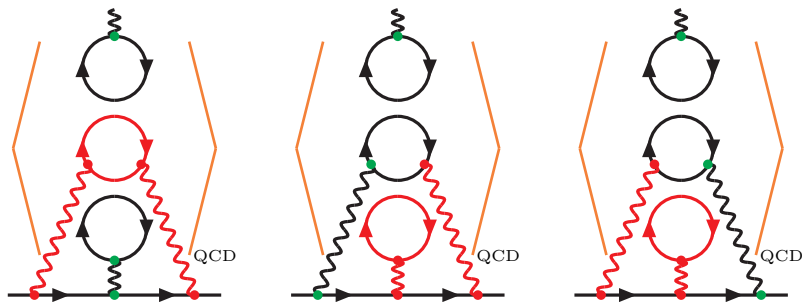


**Figure:** The left diagram is the disconnected component involved in a diagram of type  $(2_E, 2)$  induced from  $\mathcal{M}_C$ . It is canceled by  $(-\mathcal{S}_C)$ .

- The disconnected contribution **with a HVP function generated entirely by ensemble average** is canceled by  $(-\mathcal{S}_C)$ ,  $(-\mathcal{S}_{C'})$  or  $(-\mathcal{S}_D)$ . This cancellation takes place with **use of the photon field  $A$  generated by  $(QCD + QED)$  for  $D[A]^{-1}$  in the muon part.**



## Structure of unwanted HVP



**Figure:** An identical diagram of  $(1_E, 1, 2)$ -type is generated from  $\mathcal{M}_D$  in three ways. The disconnected component of the left diagram is canceled by  $(-\mathcal{S}_D)$ . However, *the other two disconnected components survive without being subtracted.*

## Structure of unwanted HPV

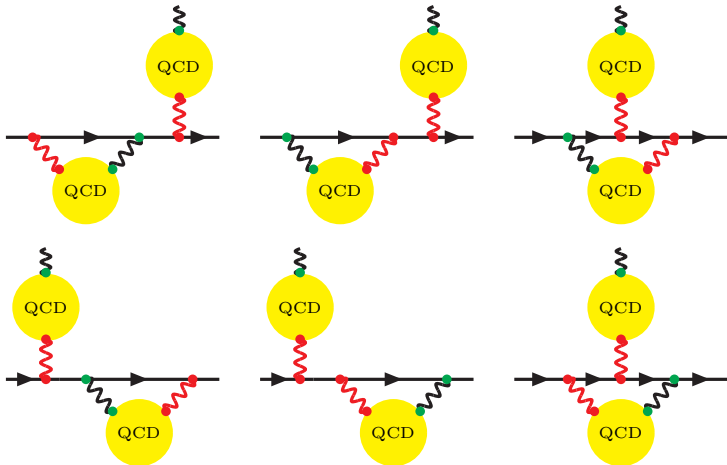


Figure: Summary of unwanted diagrams. Every diagram appears with *duplicate redundancy*.

## Structure of unwanted HPV

Here, HVP function consists of connected-type contribution and disconnected-type contribution in terms of lattice field theory (diagrams with  $O(a)$  QED vertices are not shown here)

The diagram shows the decomposition of the HVP function into connected and disconnected contributions. On the left, a wavy line with a green dot is labeled 'QCD' and is enclosed in a yellow circle. This is equal to the sum of two terms. The first term is a wavy line with a green dot connected to a loop of black lines, with a red wavy line attached to the loop and labeled 'QCD'. The second term is a wavy line with a green dot connected to a loop of black lines, with a red loop attached to the loop and labeled 'QCD'.

$$\text{QCD} = \text{QCD} + \text{QCD}$$

where the red stuffs are generated by the ensemble average of (QCD + QED).

## On $\mathcal{K}_D$

$$\mathcal{K}_D = \left\langle \begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \\ \text{Diagram 3} \end{array} \right\rangle (U_{(1)}, A_{(1)}) \cdot (U_{(2)}, A_{(2)})$$

The diagram shows three Feynman diagrams enclosed in large orange brackets. The top diagram is a quark loop with a photon (wavy line) attached to the top vertex, labeled  $D [U_{(1)} e^{-i Q_q e A_{(1)}}]^{-1}$ . The middle diagram is a quark loop with a photon attached to the bottom vertex, labeled  $D [U_{(2)} e^{-i Q_{q'} e A_{(2)}}]^{-1}$ . The bottom diagram is a quark line with a photon attached to the middle vertex, labeled  $D [e^{-i Q_\mu e A_{(1)}} e^{-i Q_\mu e A_{(2)}}]^{-1}$ . To the right of the brackets is the product  $(U_{(1)}, A_{(1)}) \cdot (U_{(2)}, A_{(2)})$ .

The leading-order diagrams from  $\mathcal{K}_D$  correspond exactly to the **full set of the unwanted diagrams**, including multiplicity:

- Let's focus on the quark part which involves two independent QCD averages.
- $\langle j_\mu \rangle_{\text{QCD}} = 0$ . For the QCD average to be nontrivial at the leading order of  $\alpha$ , **one additional photon must be supplied** by each of the QED averages.

## On $\mathcal{K}_D$

- That photon can emerge either from the valence quark loop (1st term on RHS) or from the sea quark loop (2nd term), giving the whole **hadronic vacuum polarization function** (diagrams with  $O(a)$  QED vertices are not shown here)

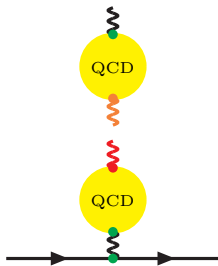
The diagram shows the decomposition of the hadronic vacuum polarization function. On the left, a photon line (wavy) enters a yellow circle labeled 'QCD', and a photon line (wavy) exits. This is equal to the sum of two terms. The first term is a photon line entering a black loop with a quark line (double line) and a gluon line (wavy) loop, with a photon line exiting from the quark line. The second term is a photon line entering a black loop with a quark line and a gluon line loop, with a photon line exiting from the gluon line. The second term is highlighted in red.

$$\text{Photon} \text{---} \text{QCD} \text{---} \text{Photon} = \text{Photon} \text{---} \text{Quark Loop} \text{---} \text{Photon} + \text{Photon} \text{---} \text{Gluon Loop} \text{---} \text{Photon}$$

where **the red stuffs are generated by the ensemble average.**

## On $\mathcal{K}_D$

- The situation reached so far for  $\mathcal{K}_D$  at the leading order is



- These two photons are the quanta of different  $U(1)$  gauge fields so that they cannot be identical with each other.
- Hence, both photon lines must be stuck somewhere on the muon lines.

On  $\mathcal{K}_D$

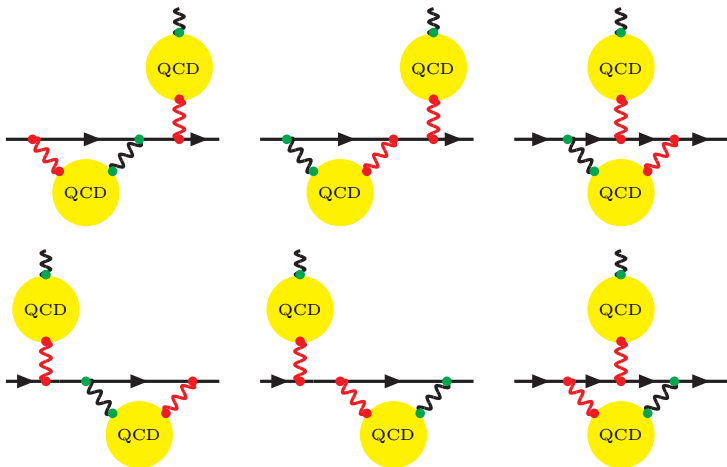


Figure:  $\mathcal{K}_D$  at the leading-order, which coincides with the set of *unwanted* diagrams

# Summary

- The VEV of four EM currents calculated by lattice QCD simulation inevitably contains the field-theoretically disconnected components.
- To invent the method to calculate the hadronic light-by-light scattering diagrams, it is necessary to specify the way to subtract  $O(\alpha^3)$  hadronic vacuum polarization contribution (HVP) caused by these disconnected components.
- In the context of nonperturbative QED method, it is possible to add a term  $(-\mathcal{K}_D)$  to subtract the unwanted  $O(\alpha^3)$  HVP contribution

$$\frac{1}{3} \{ (\mathcal{M}_C - \mathcal{S}_C) + (\mathcal{M}_{C'} - \mathcal{S}_{C'}) + (\mathcal{M}_D - \mathcal{S}_D) - \mathcal{K}_D \} .$$