NLO and NNLO Low Energy Constants for $SU(2)$ Chiral Perturbation Theory

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Motivation

- Light pseudoscalar mesons — π, K, η — can be understood as pseudo-Goldstone bosons of chiral symmetry breaking
- Separation of scales suggests effective field theory description where degrees of freedom are pGB's (χPT)
- *To construct* $\mathcal{L}_{\chi\mathrm{PT}}$: pick a power-counting scheme and write down, order-by-order, most general Lagrangian containing all $SU(N_f)_L \times SU(N_f)_R$ invariant operators parametrized by unknown coefficients ("low energy constants")
    - LECs must be determined by fits to lattice or experimental data
    - Non-renormalizable theory with many new LECs at each order!
    - General N_f case: 2 (LO), 11 (NLO), 112 (NNLO), ...
- This talk: fits of RBC-UKQCD DWF data for m_{xy}^2 and f_{xy} to $SU(2)$ NLO and NNLO PQχPT
    - Include full NNLO corrections (Fortran routines provided by J. Bijnens)
    - Determine LECs
    - Probe hierarchy of terms and region of applicability of χPT
- Will also discuss χPT predictions from the NNLO fits, and an extension which includes lattice data for the $I = 2$ $\pi\pi$ scattering length to better constrain LECs

Procedure for $SU(2)$ Global Fits

1. Simultaneous chiral/continuum fit to m_π^2, m_K^2, f_π, f_K, and m_Ω
   - Ansätze are expansion in m_q, a^2, L about chiral, continuum, infinite-volume limit:
     - **m_π^2, f_π: (NNLO continuum PQχPT) + ($\Delta_{\mathrm{FV}}^{\mathrm{NLO}}$) + ($c_a a^2$)**
     - m_K^2, f_K: (NLO heavy-meson PQχPT) + ($\Delta_{\mathrm{FV}}^{\mathrm{NLO}}$) + ($c_a a^2$)
     - m_Ω: (linear ansatz in $\tilde{m}_q$) + (ca^2)
   - Work in terms of total quark masses $a\tilde{m}_q = am_q^{\mathrm{bare}} + am_{\mathrm{res}}$
2. Match to continuum scaling trajectory by numerically inverting fit to determine m_l^{phys}, m_s^{phys} such that m_π/m_Ω and m_K/m_Ω take their physical values
3. Extract lattice scales from $a = m_\Omega/m_{\Omega^-}^{\mathrm{PDG}}$, after correcting m_Ω to m_s^{phys}

More detail: [Blum et. al, arXiv:1411.7017]

Procedure for $SU(2)$ Global Fits (cont'd.)

- Poorly conditioned correlation matrix forces us to perform uncorrelated fits
- Potentially important systematic:
  - PQ measurements on the same set of field configurations are highly correlated
  - Our PQ data mostly comes from heavy ensembles
- To bound this systematic, we consider weighted fits:

$$\chi_e^2 = \alpha_e \sum_i \left(\frac{y_e^i - f_e^i}{\sigma_e^i} \right)^2, \quad \chi^2 = \sum_e \chi_e^2$$

- Each global fit is performed twice:
  - $\alpha_e = 1$ for all ensembles (PQ measurements treated as uncorrelated)
  - $\alpha_e = 1/N_e$, where N_e is the number of non-degenerate PQ measurements on ensemble e (PQ measurements treated as completely correlated)
- True correlated fit lies between these extremes
- Assign systematic errors using shifts in central values between two fits

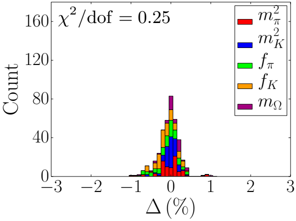
Summary of Ensembles

Ensemble	Action	β	$L^3 \times T \times L_s$	am_l	am_h	m_π (MeV)	a^{-1} (GeV)	L (fm)
24I	DWF+I	2.13	$24^3 \times 64 \times 16$	0.005	0.04	339.6(1.2)	1.784(5)	2.650(7)
		2.13	$24^3 \times 64 \times 16$	0.01	0.04	432.2(1.4)		
32I	DWF+I	2.25	$32^3 \times 64 \times 16$	0.004	0.03	302.0(1.1)	2.382(8)	2.647(9)
		2.25	$32^3 \times 64 \times 16$	0.006	0.03	359.7(1.2)		
		2.25	$32^3 \times 64 \times 16$	0.008	0.03	410.8(1.5)		
32ID	DWF+ID	1.75	$32^3 \times 64 \times 32$	0.001	0.046	172.7(9)	1.378(7)	4.573(22)
		1.75	$32^3 \times 64 \times 32$	0.0042	0.046	250.1(1.2)		
32I-fine	DWF+I	2.37	$32^3 \times 64 \times 12$	0.0047	0.0186	370.1(4.4)	3.144(17)	2.005(11)
48I	MDWF+I	2.13	$48^3 \times 96 \times 24$	0.00078	0.0362	139.1(4)	1.729(4)	5.468(12)
64I	MDWF+I	2.25	$64^3 \times 128 \times 12$	0.000678	0.02661	139.0(5)	2.357(7)	5.349(16)
32ID-M1	MDWF+ID	1.633	$32^3 \times 64 \times 24$	0.00022	0.0596	117.3(4.4)	0.981(39)	6.429(260)
32ID-M2	MDWF+ID	1.943	$32^3 \times 64 \times 12$	0.00478	0.03297	401.0(2.3)	2.055(11)	3.067(16)

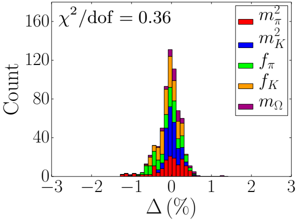
- (M)DWF: (Möbius) domain wall fermions ($N_f = 2 + 1$)
- I(D): Iwasaki (+DSDR) gauge action
- PQ measurements with $(21, 21, 28)$ different m_q^{val} pairs on (24I, 32I, 32ID)
- Fit to subsets of this data: vary cuts on heaviest pseudoscalar mass

Summary of Fits

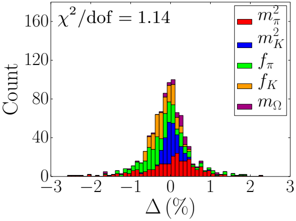
Stacked histograms of $\Delta_i \equiv 200 \times (y_i - f_i)/(y_i + f_i)$ (% dev. between fit and data):



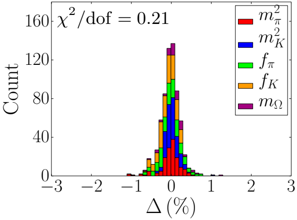
(a) NLO, 260 MeV cut



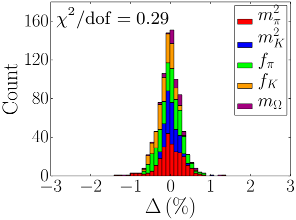
(b) **NLO, 370 MeV cut**



(c) NLO, 450 MeV cut



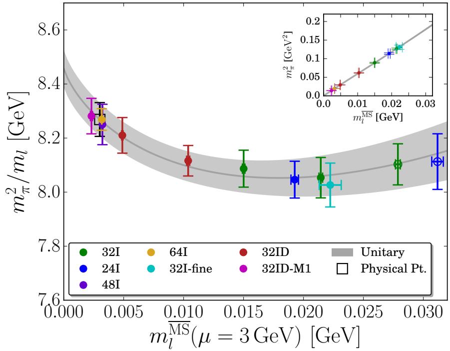
(d) NNLO, 370 MeV cut



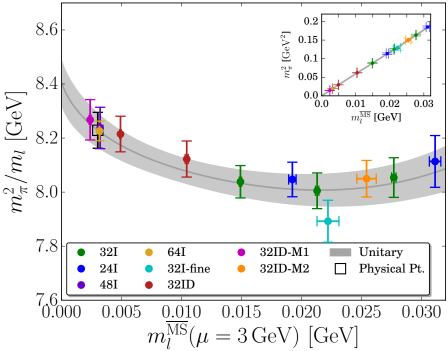
(e) **NNLO, 450 MeV cut**

- Generally observe sub-percent accuracy
- "Bad" fit (c) has $\mathcal{O}(2-3\%)$ outliers

Unitary Chiral Extrapolation of m_π^2

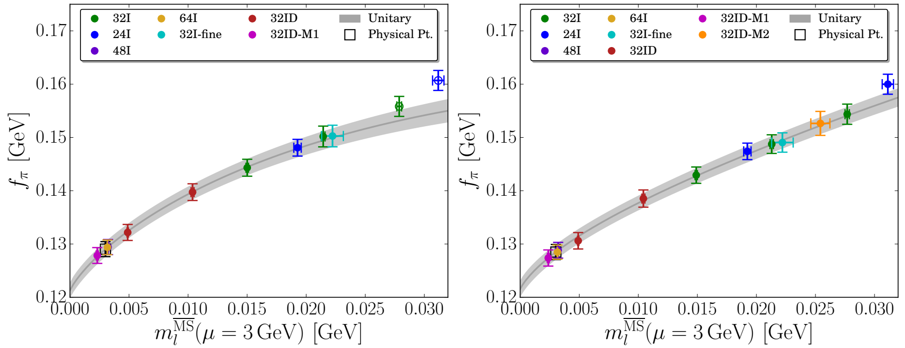


(a) NLO, 370 MeV cut



(b) NNLO, 450 MeV cut

- Unitary m_π^2 data, corrected to $m_s^{\rm phys}$, infinite-volume, and continuum using fit (open symbols are excluded points)
- NLO and NNLO fits completely consistent within statistics
- Need lighter pions and better statistics to unambiguously see chiral logs (still not there)

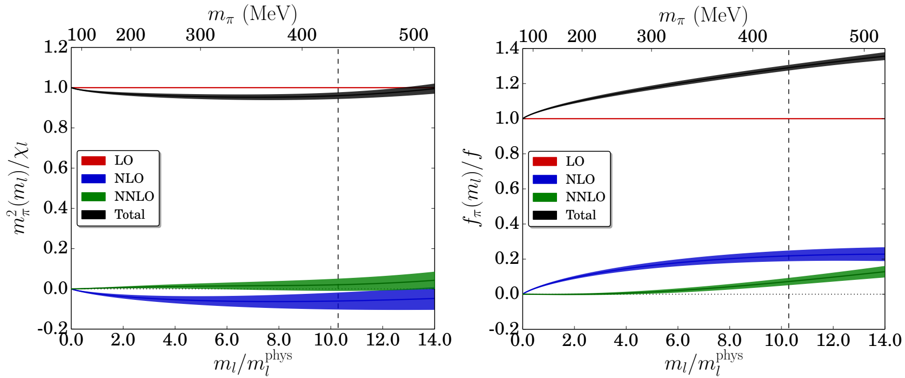


(a) NLO, 370 MeV cut (b) NNLO, 450 MeV cut

- Tension between NLO fit and heaviest ensembles
- Suggests NLO χPT consistent with lattice data up to $m_\pi \sim \mathcal{O}(350\text{MeV})$
- NNLO χPT consistent with full data set

Hierarchy of Terms up to NNLO

Decompose m_π^2 and f_π by chiral order, normalized by LO:



- Rapidly convergent at physical point:

$$\frac{m_\pi^2}{\chi_l} = 1.0000 - 0.0245(41) + 0.0034(10)$$
$$\frac{f_\pi}{f} = 1.0000 + 0.0586(35) - 0.0011(7)$$

- NLO and NNLO contributions to f_π comparable for $m_\pi \sim \mathcal{O}(500\,\text{MeV})$
- Indication that NNLO χPT has become unreliable

Partially Quenched $SU(2)$ LECs

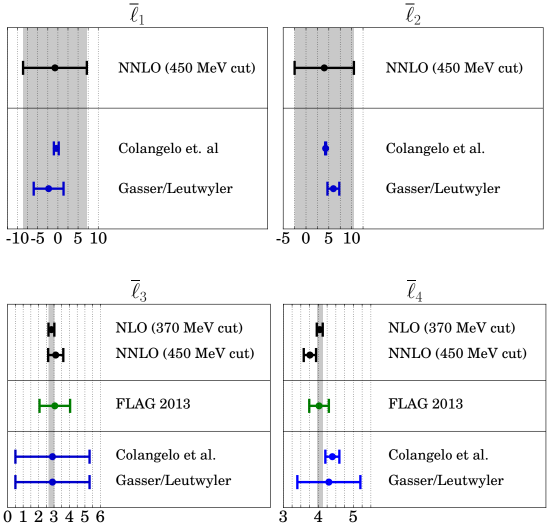
Statistical errors only, from unweighted fits:

Order	LEC	NLO (370 MeV cut)	NNLO (450 MeV cut)
LO	$B^{\overline{\text{MS}}}(\mu = 2\,\text{GeV})$	2.804(34) GeV	2.787(39) GeV
	f	121.3(1.5) MeV	121.5(1.6) MeV
NLO	$10^3\,\hat{L}_0^{(2)}$	—	1.0(1.1)
	$10^3\,\hat{L}_1^{(2)}$	—	-0.62(52)
	$10^3\,\hat{L}_2^{(2)}$	—	0.06(74)
	$10^3\,\hat{L}_3^{(2)}$	—	-1.56(87)
	$10^3\,\hat{L}_4^{(2)}$	-0.211(79)	-0.56(22)
	$10^3\,\hat{L}_5^{(2)}$	0.438(72)	0.60(28)
	$10^3\,\hat{L}_6^{(2)}$	-0.175(48)	-0.38(10)
	$10^3\,\hat{L}_7^{(2)}$	—	-0.75(27)
	$10^3\,\hat{L}_8^{(2)}$	0.594(36)	0.69(13)

Order	LEC	NNLO (450 MeV cut)
NNLO	$10^6\left(\hat{K}_{17}^{(2)} - \hat{K}_{39}^{(2)}\right)$	-7.6(1.1)
	$10^6\left(\hat{K}_{18}^{(2)} + 6\hat{K}_{27}^{(2)} - \hat{K}_{40}^{(2)}\right)$	19.2(4.7)
	$10^6\,\hat{K}_{19}^{(2)}$	-0.9(4.2)
	$10^6\,\hat{K}_{20}^{(2)}$	-3.2(2.8)
	$10^6\left(\hat{K}_{21}^{(2)} + 2\hat{K}_{22}^{(2)}\right)$	4.9(4.1)
	$10^6\,\hat{K}_{23}^{(2)}$	-2.8(1.4)
	$10^6\,\hat{K}_{25}^{(2)}$	1.3(1.7)
	$10^6\left(\hat{K}_{26}^{(2)} + 6\hat{K}_{27}^{(2)}\right)$	11.2(3.6)

- Values for NLO and NNLO LECs are quoted at $\Lambda_\chi = 1\,\text{GeV}$
- LECs have expected hierarchy of sizes
- New LECs which enter into NNLO fits generally have $\mathcal{O}(25\%)$ or larger statistical errors, but most are resolvable from zero

Unquenched NLO $SU(2)$ LECs



$$\bar{\ell}_i \equiv \gamma_i l_i(\mu) - \log\left(\frac{m_\pi^2}{\mu}\right)$$

LEC	NLO (370 MeV cut)	NNLO (450 MeV cut)
$\bar{\ell}_1$	—	-0.7(7.2)(2.5)
$\bar{\ell}_2$	—	4.0(6.2)(2.1)
$\bar{\ell}_3$	2.83(19)(2)	3.11(49)(3)
$\bar{\ell}_4$	4.04(8)(2)	3.76(16)(8)
$10^3 l_7$	—	6.6(5.4)(0.1)

- Errors are statistical (left) and correlated systematic (right)
- l_7 is scale independent, and (I believe) previously undetermined
- Gasser/Leutwyler estimate $l_7 \sim 5 \times 10^{-3}$ from $\pi^0 - \eta$ mixing

FLAG 2013: [Eur.Phys.J. **C74**, 2890 (2014)]
Colangelo et al.: [Nucl.Phys. **B603**, 125-179 (2001)]
Gasser/Leutwyler: [Annals Phys. **158**, 142 (1984)]

Some One-Loop $SU(2)$ Predictions

- NNLO fits to m_π^2 and f_π determine $\bar{\ell}_1$, $\bar{\ell}_2$, and l_7
  - $\bar{\ell}_1, \bar{\ell}_2 \xrightarrow{\text{NLO}} \pi\pi$ scattering
  - $l_7 \xrightarrow{\text{NLO}} \pi^\pm - \pi^0$ mass splitting due to up/down mass difference
- Can predict $\pi\pi$ scattering lengths (a_ℓ^I) and slopes (b_ℓ^I), as well as the dimensionless ratio $(m_{\pi^\pm}^2 - m_{\pi^0}^2)/(m_d - m_u)^2$
- For simplicity, focus on s-wave ($\ell = 0$) scattering lengths

	Prediction	Expt. [Ref.]
$m_\pi \, a_0^0$	0.192(14)(7)	0.221(5)
$m_\pi \, a_0^2$	-0.042(5)(1)	-0.043(5)
$\left[\frac{\left(m_{\pi^\pm}^2 - m_{\pi^0}^2\right)}{\left(m_d - m_u\right)^2}\right]_{\text{QCD}}$	31.5(17)(0.4)	—

- Taking, e.g. $m_d - m_u = 2.5\,\text{MeV}$, we predict $\sim 15\%$ of $\pi^\pm - \pi^0$ mass splitting is due to purely QCD effects
- Dominant contribution is from QED at $\mathcal{O}(m_d - m_u)$

Ref.: [B. Bloch-Devaux, PoS **KAON09**, 033 (2009)]

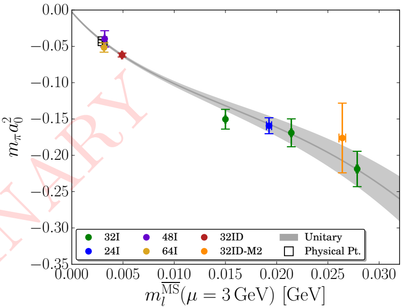
NNLO Fit with Partial a_0^2 Data Set

- Repeat NNLO fit with 450 MeV cut, adding a_0^2 measurements on a subset of ensembles
- $m_\pi a_0^2$ ansatz:

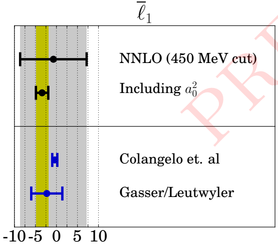
$$(\text{NLO continuum } \chi\text{PT}) + (c_a a^2) + (c_h \tilde{m}_h)$$

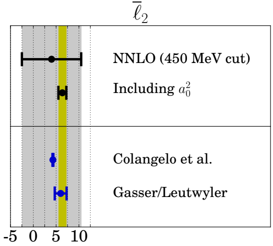
- Factor of 5.4 (7.5) reduction in error on $\bar{\ell}_1$ ($\bar{\ell}_2$)

	NNLO (450 MeV cut)	Including a_0^2
$\bar{\ell}_1$	-0.7(7.2)(2.5)	-3.4(1.5)(0.0)
$\bar{\ell}_2$	4.0(6.2)(2.1)	6.37(86)(11)
$m_\pi a_0^0$ (χPT)	0.192(14)(7)	0.199(5)(1)
$m_\pi a_0^2$ (Data)	-0.042(5)(1)	-0.041(2)(1)



- Fits suggest $\mathcal{O}(5\%)$ discretization error for $a^{-1} = 2\,\text{GeV}$





- We are able to fit $SU(2)$ PQχPT with the full NNLO corrections to our data for m_π^2 and f_π without the need for additional terms or constraints
- We determine 9 NLO LECs, and 8 independent linear combinations of NNLO LECs, in the PQ theory
- Good agreement between unquenched LECs and lattice/phenomenological results reported in the literature
- The $SU(2)$ chiral expansion appears to be very robust:
  - At the physical light quark mass NLO corrections are $\mathcal{O}(5\%)$, and NNLO corrections are $\mathcal{O}(.5\%)$
  - Lattice data for f_π starts to systematically disagree with NLO χPT for $m_\pi \sim \mathcal{O}(350\,\text{MeV})$, although the fit remains consistent with the data to $\mathcal{O}(2-3\%)$
  - NLO and NNLO contributions to f_π become comparable in size around $m_\pi \sim \mathcal{O}(500\,\text{MeV})$, likely indicating onset of N^{3}LO terms
- Paper based on this work to appear on the arXiv soon

Thank you for your attention!

$\pi\pi$ Scattering on the Lattice

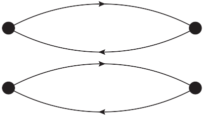
- a_0^I characterizes low energy $\pi - \pi$ interaction of isospin I state
- Lüscher's quantization condition (box size L):

$$E_{\pi\pi}^I - 2m_\pi = -\frac{4\pi a_0^I}{m_\pi L^3}\left[1 + c_1\left(\frac{a_0^I}{L}\right) + c_2\left(\frac{a_0^I}{L}\right)^2 + c_3\left(\frac{a_0^I}{L}\right)^3\right] + \mathcal{O}(L^{-7})$$

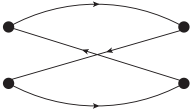
- Two diagrams contribute to $E_{\pi\pi}^2$

$$\langle O_{\pi\pi}^2(t)^\dagger O_{\pi\pi}^2(0)\rangle = 2\,(D(t) - C(t))$$
$$\overset{t\to\infty}{\simeq} A\left(e^{-E_{\pi\pi}^2 t} + e^{-E_{\pi\pi}^2 (T-t)} + C\right)$$

- $I = 0$ case harder (noisy vacuum diagram)
- At NLO in χPT, $m_\pi a_0^2 \propto (\bar{\ell}_1 + \bar{\ell}_2)$
  - Sharpen lattice predictions for $\bar{\ell}_1$, $\bar{\ell}_2$
  - Stabilize NNLO fits with light mass cuts



(a) D ("direct")



(b) C ("cross")