Three particles in a finite volume

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Plan

- Introduction: loosely bound states in a finite volume
- The quantization condition for the three-body systems: essentials
- Spectrum of the three-body bound state in a finite volume
- Conclusions, outlook

The nature of the bound states of quarks and gluons

Is a given state . . .

- a "standard" bound state of quarks: $q\bar{q}$ or qqq ?
- a hadronic molecule (a bound state of hadrons)?
- an "exotic" state: glueball, tetraquark, pentaquark, . . .?

$\hookrightarrow$ What are the relevant degrees of freedom to describe such states?

$\hookrightarrow$ Model independence: how does one formulate the criteria in terms of the observables?

Example: Compositeness condition or pole counting criterion

S. Weinberg, D. Morgan and M.R. Pennington, N.A. Törnqvist, . . .

How does one formulate such criteria on the lattice?

Lüscher's approach

M. Lüscher, lectures given at Les Houches (1988); NPB 364 (1991) 237, $\cdots$

- Lattice simulations are done at a finite volume (box size L)

Lüscher's approach: $R^{-1}L \simeq ML \gg 1$

R: the range of the interaction

- Momenta are small: $p \simeq 2\pi/L \ll$ the lightest mass
- Finite-volume corrections to the energy levels *above threshold* are only power-suppressed in L; exponential suppression *below threshold*
- Studying the dependence of the energy spectrum on L gives the scattering phase in the infinite volume
- The shallow bound states: characteristic bound-state momenta $\kappa \ll R^{-1}$; exponential factor $\exp(-\kappa L)$ not negligible

$\Rightarrow$ The Lüscher approach can be used to study the volume-dependence of the shallow bound states as well.

Example: two-body bound state in a finite volume

Infinite volume, shallow bound state:

$$\left(-\sum_{i=1}^{2}\frac{\nabla_i^2}{2m}+V(\mathbf{r}_1-\mathbf{r}_2)\right)\Psi_B(\mathbf{r}_1,\mathbf{r}_2)=-E_B\Psi_B(\mathbf{r}_1,\mathbf{r}_2)$$

$$\Psi_B=A\sqrt{8\pi\kappa}\,\frac{\exp(-\kappa r)}{4\pi r}\,,\qquad E_B=\frac{\kappa^2}{m}\,,\quad r\gg R$$

$1-Z\propto A^2$: A is the asymptotic normalization coefficient

Finite volume, periodic boundary conditions:

$$V_L(\mathbf{x})=\sum_{\mathbf{n}\in\mathbb{Z}^3}V(\mathbf{x}+\mathbf{n}L)$$

$$H_L\Psi_L\doteq\left(-\sum_{i=1}^{2}\frac{\nabla_i^2}{2m}+V_L(\mathbf{r}_1-\mathbf{r}_2)\right)\Psi_L(\mathbf{r}_1,\mathbf{r}_2)=-E_L\Psi_L(\mathbf{r}_1,\mathbf{r}_2)$$

Energy shift in a finite volume

Periodic ansatz for the wave function:

$$\Psi_0(\mathbf{r}_1 - \mathbf{r}_2) = \sum_{\mathbf{n}\in\mathbb{Z}^3} \Psi_B(\mathbf{r}_1 - \mathbf{r}_2 + \mathbf{n}L)$$

Perturbative solution:

$$(H_L + E_B)\Psi_0 = \eta\,, \qquad \Delta E = E_B - E_L = \langle\Psi_0|\eta\rangle / \langle\Psi_0|\Psi_0\rangle + \cdots$$

$$\Delta E = 6 \int d^3\mathbf{x}\Psi_B(\mathbf{x})V(\mathbf{x})\Psi_B(\mathbf{x} + \mathbf{e}L) = -24\pi A^2 \frac{\exp(-\kappa L)}{mL} + \cdots$$

$$\mathbf{e} = (0, 0, 1)$$

... up to the exponentially suppressed terms at $\kappa L \gg 1$

The asymptotic normalization coefficient

Use non-relativistic EFT:

$$V(\mathbf{p},\mathbf{q}) = C_0 + \frac{1}{2}C_1(\mathbf{p}^2+\mathbf{q}^2) + \cdots$$

$$(\mathbf{p}^2+\kappa^2)\Psi_B(\mathbf{p}) = \int \frac{d^d\mathbf{q}}{(2\pi)^d}\, V(\mathbf{p},\mathbf{q})\Psi_B(\mathbf{q}) \quad \text{with:} \quad \int \frac{d^d\mathbf{q}}{(2\pi)^d}\, |\Psi_B(\mathbf{q})|^2 = 1$$

$$\hookrightarrow \quad \Psi_B(\mathbf{p}) = \sqrt{8\pi\kappa}\,\frac{A}{\mathbf{p}^2+\kappa^2} + Q_0 + Q_1\mathbf{p}^2 + \cdots$$

$$\hookrightarrow \quad \Psi_B(\mathbf{r}) = \underbrace{\sqrt{8\pi\kappa}\,\frac{A\,\exp(-\kappa r)}{4\pi r}}_{\text{long}-\text{range}} + \underbrace{Q_0\delta^3(\mathbf{r}) - Q_1\nabla^2\delta^3(\mathbf{r}) + \cdots}_{\text{short}-\text{range}}$$

- Predominately molecular state: $A \simeq 1$ and $Q_0, Q_1, \ldots \simeq 0$
- A tight compound: $A \simeq 0$ and $Q_0, Q_1, \ldots \neq 0$

Weinberg's compositeness condition

A bound-state pole in the elastic T-matrix:

$$T(p) = \frac{1}{p\cot\delta(p) - ip} \to \frac{2\kappa(Z-1)}{p^2+\kappa^2} + \text{regular}$$

$$p\cot\delta(p) = \frac{1}{a} + \frac{1}{2}rp^2 + \cdots\,, \qquad Z = 1 - |A|^2 = -\frac{r\kappa}{1-r\kappa} + \cdots$$

$\hookrightarrow$ For shallow bound states, Z can be expressed in terms of the effective-range expansion parameters (Weinberg):

$$a\kappa = \frac{2(1-Z)}{2-Z}\,, \qquad r\kappa = -\frac{Z}{1-Z}$$

- $Z \simeq 1$: a tight quark compound
- $Z \simeq 0$: a large moleqular component

Is it possible to formulate the compositeness condition on the lattice?

A. Martinez Torres *et al.*, PRD 85 (2012) 014027; T. Sekihara and T. Hyodo, PRC 87 (2013) 045202; D. Agadjanov, F.-K. Guo, G. Rios and AR, JHEP 1501 (2015) 118

Three and more particles in a finite volume

The problem: volume dependence of the observables in the multiparticle systems

- Properties of the hadron resonances (e.g., the Roper resonance)
- Nuclear physics from lattice QCD

Finite-volume energy spectrum in three-particle systems:

K. Polejaeva and AR, EPJA 48 (2012) 67

P. Guo, arXiv:1303.3349 [hep-lat]

R. A. Briceno and Z. Davoudi, PRD 87 (2013) 094507

M. Hansen and S. Sharpe, arXiv:1311.4848; PRD 11 (2014) 116003; arXiv:1504.04248

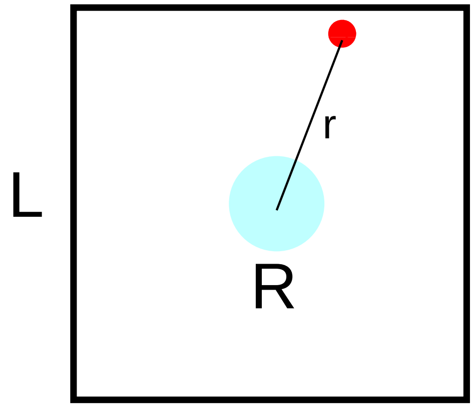
S. Bour *et al.,* PRD 84 (2011) 091503; PRC 86 (2012) 034003

S. Kreuzer *et al.,* PLB 673 (2009) 260; EPJA 43 (2010) 229; PLB 694 (2011) 624

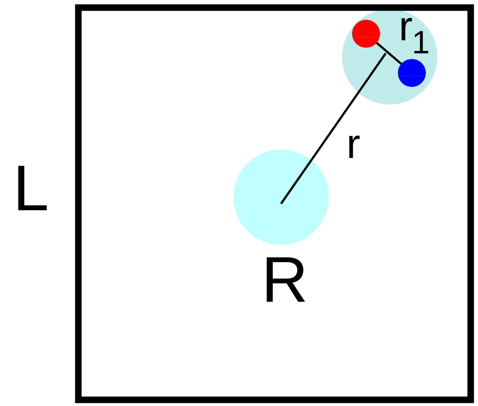
S. König *et al.,* PRL 107 (2011) 112001, Ann. Phys. 327 (2012) 1450

U.-G. Meißner, G. Rios and AR, PRL 114 (2015) 9, 091602

Physical background



2 particles



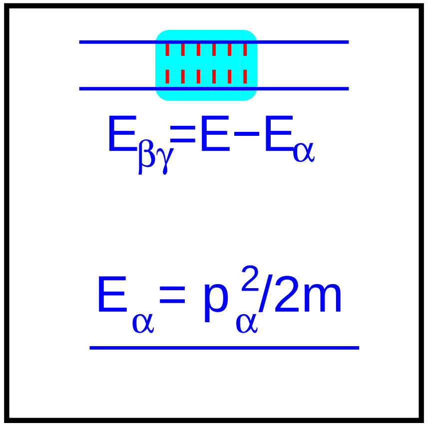
3 particles

- In case of 2 particles: $r \gg R$, when particles are near the walls
- In case of 3 particles: it may happen that $r \gg R, \quad r_1 \simeq R$, when the particles are near the walls

The problem with the disconnected contributions: is the finite-volume spectrum in the 3-particle case determined solely through the on-shell scattering matrix?

Faddeev resummation in a finite volume

K. Polejaeva and AR, EPJA 48 (2012) 67



Pair T−matrices below threshold!

Disconnected potential is volume−dependent:
$\delta(p_\alpha - q_\alpha) \to \delta_{p_\alpha q_\alpha}$

- Naive generalization of Faddeev equations in a finite volume incorrect: Disconnected potential is volume-dependent

  Disconnected terms should be treated properly!

$\hookrightarrow$ The energies of the 3-particle system in a finite box is determined by the on-shell scattering matrix elements in the infinite volume

$\hookrightarrow$ Confirmed by M. Hansen and S. Sharpe: $\rightarrow$ Talk by M. Hansen
Quantization condition for the three-particle system

Three-body bound states in the infinite volume

- Three identical particles
- Unitary limit: two-body scattering length $a \to \infty$, $r \to 0$
- Short-range two-body interactions, no three-body force at this moment

$\hookrightarrow$ Efimov spectrum:

Infinitely many, arbitrarily shallow 3-body bound states. Energy levels $E_T^{(n)}$ accumulate towards the point $E = 0$. Universal relation:

$$E_T^{(n+1)}/E_T^n = 1/515.03 \qquad \text{as} \quad n \to \infty\,, \quad a = \pm\infty$$

Eigenvalue equation:

$$\sin\left\{\frac{1}{2}\, s_0 \ln(mE_T/\kappa_*^2)\right\} = 0\,, \qquad s_0 = 1.00624$$

Schrödinger equation and the wave function

$$\left(-\frac{1}{2m}\sum_{j=1}^{3}\nabla_i^2+\sum_{jk}V(\mathbf{x}_i)\right)\Psi_B(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3)=-E_B\Psi_B(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3)$$

$$\mathbf{x}_i=\mathbf{r}_j-\mathbf{r}_k\,,\qquad \mathbf{y}_i=\frac{1}{\sqrt{3}}(\mathbf{r}_j+\mathbf{r}_k-2\mathbf{r}_i)$$

Hyperspherical coordinates:

$$R^2=\frac{1}{3}(\mathbf{x}_1^2+\mathbf{x}_2^2+\mathbf{x}_3^2)\,,\qquad |\mathbf{x}_i|=\sqrt{2}R\sin\alpha_i$$

Neglecting orbital excitations, the ground-state wave function is:

$$\Psi_B(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3)=\psi_0(R,\alpha_1)+\psi_0(R,\alpha_2)+\psi_0(R,\alpha_3)$$

$$\psi_0(R,\alpha)=A\mathcal{N}\,\frac{\sin\big(s_0(\pi/2-\alpha)\big)}{R^2\sin(2\alpha)}\,K_{is_0}(\sqrt{2}\,\kappa R)\,,\qquad E_T^{(0)}=\frac{\kappa^2}{m}$$

Shift of the three-body energy level

The first non-perturbative explicit result in the 3-particle sector

U.-G. Meißner, G. Rios and AR, PRL 114 (2015) 9, 091602

Generalization to the finite volume:

$$V_L(\mathbf{x}) = \sum_{\mathbf{n}\in\mathbb{Z}^3} V(\mathbf{x}+\mathbf{n}L)$$

$$\left(-\frac{1}{2m}\sum_{j=1}^{3}\nabla_i^2 + \sum_{jk} V_L(\mathbf{r}_{jk})\right)\Psi_L \doteq H_L\Psi_L = -E_L\Psi_L$$

Variational solution:

$$\Psi_0 = \sum_{\mathbf{n},\mathbf{m}} \Psi_B\left(\mathbf{x}_i - (\mathbf{n}+\mathbf{m})L, \mathbf{y}_1 - \frac{1}{\sqrt{3}}(\mathbf{n}-\mathbf{m})L\right)$$

$$(H_L+E_B)\Psi_0 = \eta\,, \qquad \Delta E = \langle\eta|\Psi_0\rangle/\langle\Psi_0|\Psi_0\rangle + \cdots$$

Evaluation of the energy shift

$$\Delta E = 36 \int d^3\mathbf{x} d^3\mathbf{y} \Psi_B(\mathbf{x},\mathbf{y}) V(\mathbf{x}) \Psi_B\left(\mathbf{x} - \mathbf{e}L, \mathbf{y} - \frac{1}{\sqrt{3}}\mathbf{e}L\right) + \cdots$$

Substituting the explicit expression for Ψ_B ...

$$\Delta E = c \cdot (\kappa^2/m)(\kappa L)^{-3/2} |A|^2 \exp(-2\kappa L/\sqrt{3}) + \cdots$$

$$c \simeq -87.886$$

- Valid up to exponentially suppressed terms at $\kappa L \to \infty$
- Qualitatively agrees with the numerical results of:
  S. Kreuzer *et al.*, PLB 673 (2009) 260; EPJA 43 (2010) 229; PLB 694 (2011) 624

Conclusions, outlook

- Using effective theories of QCD in a finite volume facilitates the extraction of physical observables from lattice simulations
- In particular: the volume-dependence of the two- and three-body bound state spectrum is determined by the normalization coefficient; can be used to judge about the nature of the states
- Future developments:
  - Three-particle bound states beyond the infinite scattering length approximation; including partial-wave mixing
  - Three-particles with twisted boundary conditions
  - Relation to the general three-particle quantization condition
  - Reactions in the three-particle systems
  - . . . and many more