Isospin splitting in Wilson χPT for tmLQCD with 3 quark flavours

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Topic

Motivation

Isospin splitting

$\Delta m = m_\mathrm{d} - m_\mathrm{u} \quad \longrightarrow \quad M^2_{\pi^\pm} - M^2_{\pi^0}\,, \quad M^2_{K^\pm} - M^2_{K^0}$

being investigated in numerical simulations of lattice QCD [2, 3, 4]

Aim: Calculation of isospin splittings in chiral perturbation theory for twisted mass lattice QCD with $N_\mathrm{f} = 1+1+1$ quark flavours

Previous work:

$N_\mathrm{f} = 1+1$, $N_\mathrm{f} = 2+1$ [5], $\quad N_\mathrm{f} = 2+1+1$ [6]

Twisted mass lattice QCD

Frezzotti, Grassi, Sint, Weisz, Rossi

Consider $N_f = 2, \qquad m_u = m_d \equiv m_q$

Mass term $\bar\psi\, M(\omega)\, \psi$ with

$$M(\omega) = m_q\, \mathrm{e}^{\mathrm{i}\,\omega\gamma_5\tau_3} = m + \mathrm{i}\,\mu\,\gamma_5\tau_3$$

$m = m_q \cos(\omega),\;\; \mu = m_q \sin(\omega)$

Continuum: no effect,

remove ω by chiral rotation: $\psi = \mathrm{e}^{-\mathrm{i}\omega\gamma_5\tau_3/2}\,\psi'$

Lattice: dependence on ω due to explicit breaking of chiral symmetry through Wilson term

Promising approach for reducing lattice effects in numerical simulations

Full $\mathcal{O}(a)$ improvement for $\omega = \frac{\pi}{2}$ (Frezzotti, Rossi)

Wilson χ PT

Low-energy effective chiral Lagrangian for pseudoscalar mesons

$$\text{in terms of} \qquad U \doteq \exp\left[\frac{\mathrm{i}}{F_0}\sum_{a=1}^{8} \lambda_a \phi_a\right]$$

$$\mathcal{L}_2 = \frac{F_0^2}{4}\,\mathrm{Tr}\left(\partial_\mu U^\dagger\,\partial^\mu U\right) - \frac{F_0^2}{4}\,\mathrm{Tr}\left(\chi U^\dagger + U\chi^\dagger\right) - \frac{F_0^2}{4}\,\mathrm{Tr}\left(\rho\, U^\dagger + U\rho^\dagger\right)$$

$$\text{where} \quad \chi = 2B_0 M\,, \quad M = \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_d & 0 \\ 0 & 0 & m_s \end{pmatrix}, \quad \rho = 2W_0\, a$$

Wilson χPT for tmLQCD

Twist in the u-d-sector implemented with λ_3

$\longrightarrow$ mass splitting implemented with λ_1

$$M = \begin{pmatrix} \hat m & \frac12\Delta m & 0 \\ \frac12\Delta m & \hat m & 0 \\ 0 & 0 & m_\mathrm{s} \end{pmatrix}, \qquad \hat m \doteq \frac{m_\mathrm{u} + m_\mathrm{d}}{2}$$

Mass twisting:

$$M \;\to\; \exp\left[\mathrm{i}\gamma_5\lambda_3\frac{\omega}{2}\right] M \exp\left[\mathrm{i}\gamma_5\lambda_3\frac{\omega}{2}\right] = \begin{pmatrix} \tilde m + \mathrm{i}\gamma_5\mu & \frac12\Delta m & 0 \\ \frac12\Delta m & \tilde m - \mathrm{i}\gamma_5\mu & 0 \\ 0 & 0 & m_\mathrm{s} \end{pmatrix}$$

$\mu = \hat m\,\sin(\omega)\,, \quad \tilde m = \hat m\cos(\omega)\,, \quad \omega = \arctan\left(\frac{\mu}{\tilde m}\right)$

Physical basis: $\quad M$ unrotated, ρ rotated

$$\rho(\omega) = \exp\left[\mathrm{i}\lambda_3\frac{\omega}{2}\right]\rho\exp\left[\mathrm{i}\lambda_3\frac{\omega}{2}\right] = \begin{pmatrix} \tilde\rho + \mathrm{i}\rho_3 & 0 & 0 \\ 0 & \tilde\rho - \mathrm{i}\rho_3 & 0 \\ 0 & 0 & \rho_0 \end{pmatrix}$$

$\tilde\rho = \rho_0\cos(\omega)\,, \qquad \rho_3 = \rho_0\sin(\omega)$

Next-to-leading order: terms of higher dimension,

Gasser-Leutwyler coefficients L_i, lattice coefficients W_i, W_i'

$$\text{Non-trivial minimum:} \quad U_0 \doteq \exp\left[\frac{\mathrm{i}}{F_0}\lambda_3\check\phi_3\right]$$

$$\check\phi_3 = F_0\rho_3\,\frac{F_0^2 + 8W_6(2\hat\chi + \chi_\mathrm{s}) + 8W_8\hat\chi}{F_0^2\hat\chi + 16L_6(2\hat\chi + \chi_\mathrm{s})\hat\chi + 16L_8(\hat\chi^2 + \Delta\chi^2)} + \mathcal{O}(a^2)$$

Calculation

Method

Leading order propagator $\quad \mathrm{i}G^{-1}(p^2) = p^2\mathbf{1} + \mathfrak{M}_\mathrm{LO}$

$\mathfrak{M}_\mathrm{LO}$ contains non-diagonal terms:

mixings in the π^0-η sector and among the kaons

Next-to-leading order $\quad \mathrm{i}G^{-1}(p^2) = p^2\mathbf{1} + \mathfrak{M}_\mathrm{LO} - A + p^2 B$

Relative diagonalisation

$$\mathrm{i}G^{-1}(p^2) = \left(\mathbf{1} + \frac{B}{2}\right)\left[p^2\mathbf{1} + \mathfrak{M}_\mathrm{LO} - A - \frac12\{B, \mathfrak{M}_\mathrm{LO}\}\right]\left(\mathbf{1} + \frac{B}{2}\right) + \mathcal{O}(p^6)$$

Diagonalisation of the resulting mass matrix $\longrightarrow$ masses

Calculation of $\mathfrak{M}_\mathrm{LO}$, A, B is quite tedious

LO masses

$$\bar m_\pi^2 = 2B_0\hat m + 2aW_0\cos(\omega) + \frac{a^2W_0^2\sin^2(\omega)}{B_0\hat m}$$

$$\bar m_K^2 = B_0(\hat m + m_\mathrm{s}) + aW_0(1 + \cos(\omega)) + \frac12\frac{a^2W_0^2\sin^2(\omega)}{B_0\hat m}$$

$$\bar m_\eta^2 = \frac13\left[2B_0(\hat m + 2m_\mathrm{s}) + 2aW_0(2 + \cos(\omega)) + \frac{a^2W_0^2\sin^2(\omega)}{B_0\hat m}\right] + \mathcal{O}(a^3)$$

NLO pion mass

Mass eigenstates: π_1, π_2,π_3

$\pi^0 \equiv \pi_1, \quad \pi^\pm \sim$ linear combinations of π_2 and π_3

$$\begin{aligned}
M^2_{\pi_2} &= \bar m_\pi^2 \\
&+ \frac{32B_0^2}{F_0^2}\Big[\big(-2L_4^\mathrm{r} - L_5^\mathrm{r} + 4L_6^\mathrm{r} + 2L_8^\mathrm{r}\big)\hat m^2 + \big(-L_4^\mathrm{r} + 2L_6^\mathrm{r}\big)\hat m\, m_\mathrm{s}\Big] \\
&+ \frac{32aB_0W_0}{F_0^2}\Big[\big(-W_4^\mathrm{r} + W_6^\mathrm{r}\big)\hat m + \big(-L_4^\mathrm{r} + W_6^\mathrm{r}\big)m_\mathrm{s}\cos(\omega) \\
&\qquad + \big(-2L_4^\mathrm{r} - L_5^\mathrm{r} - 2W_4^\mathrm{r} - W_5^\mathrm{r} + 4W_6^\mathrm{r} + 2W_8^\mathrm{r}\big)\hat m\cos(\omega)\Big] \\
&+ \frac{32a^2W_0^2}{F_0^2}\Big[-2W_4^\mathrm{r} - W_5^\mathrm{r} + 4W_6'^\mathrm{r} + 2W_8'^\mathrm{r} \\
&\qquad + \big(-W_4^\mathrm{r} + 2W_6'^\mathrm{r}\big)\cos(\omega) \\
&\qquad + (-4L_6^\mathrm{r} - 2L_8^\mathrm{r} + 4W_6^\mathrm{r} + 2W_8^\mathrm{r} - 4W_6'^\mathrm{r} - 2W_8'^\mathrm{r})\sin^2(\omega) \\
&\qquad + \big(-L_4^\mathrm{r} - 2L_6^\mathrm{r} + 2W_6^\mathrm{r}\big)\left[\frac{m_\mathrm{s}}{2\hat m}\right]\sin^2(\omega)\Big] \\
&+ \frac{\bar m_\pi^2}{32\pi^2F_0^2}\left[\bar m_\pi^2\ln\left(\frac{\bar m_\pi^2}{\Lambda^2}\right) - \frac{\bar m_\eta^2}{3}\ln\left(\frac{\bar m_\eta^2}{\Lambda^2}\right)\right] + \mathcal{O}(\Delta m^3, a^3)
\end{aligned}$$

Automatic $\mathcal{O}(a)$-improvement at full-twist is lost due to mixing with η meson

Mass splittings

Pion mass splittings

$$\begin{aligned}
M^2_{\pi_1} - M^2_{\pi_2} = B_0\Delta m^2\Big\{ & \frac{4B_0}{3F_0^2}(-3L_4^\mathrm{r} + 2L_5^\mathrm{r} + 6L_6^\mathrm{r} - 78L_7^\mathrm{r} - 18L_8^\mathrm{r}) \\
& + \frac{4B_0}{F_0^2}\left(\frac{\hat m}{m_\mathrm{s} - \hat m}\right)(-3L_4^\mathrm{r} - 2L_5^\mathrm{r} + 6L_6^\mathrm{r}) \\
& + \frac{1}{4(m_\mathrm{s} - \hat m)}\left[1 - \frac{aW_0(1 - \cos\omega)}{B_0(m_\mathrm{s} - \hat m)}\right] \\
& + \frac{4aW_0}{F_0^2(m_\mathrm{s} - \hat m)}\Big[(-L_5^\mathrm{r} - 3W_4^\mathrm{r} - W_5^\mathrm{r} + 3W_6^\mathrm{r}) \\
&\qquad + (L_4^\mathrm{r} + L_5^\mathrm{r} + 20L_7^\mathrm{r} + 4L_8^\mathrm{r} + 2W_4^\mathrm{r} \\
&\qquad\quad + W_5^\mathrm{r} - 3W_6^\mathrm{r} - 10W_7^\mathrm{r} - 2W_8^\mathrm{r})(1 - \cos\omega) \\
&\qquad + \left(\frac{\hat m}{m_\mathrm{s} - \hat m}\right)(3L_4^\mathrm{r} + L_5^\mathrm{r} + 4L_8^\mathrm{r} + W_5^\mathrm{r} - 3W_6^\mathrm{r} - 2W_8^\mathrm{r}) \\
&\qquad\qquad (1 - \cos\omega)\Big] \\
& - \frac{B_0}{64\pi^2F_0^2}\left[\ln\left(\frac{\bar m_K^2}{\Lambda^2}\right) + 1\right] \\
& - \frac{1}{192\pi^2F_0^2(m_\mathrm{s} - \hat m)}\left[1 - \frac{aW_0(1 - \cos\omega)}{B_0(m_\mathrm{s} - \hat m)}\right] \\
&\qquad \Big[\bar m_\pi^2\ln\left(\frac{\bar m_\pi^2}{\Lambda^2}\right) - 3\bar m_\eta^2\ln\left(\frac{\bar m_\eta^2}{\Lambda^2}\right) \\
&\qquad\quad - \frac12(3\bar m_\eta^2 - \bar m_\pi^2)\ln\left(\frac{\bar m_K^2}{\Lambda^2}\right) + \frac32(\bar m_\pi^2 + \bar m_\eta^2)\Big]\Big\} \\
& + \mathcal{O}(\Delta m^3, a^2)
\end{aligned}$$

From the mass twist:

$$M^2_{\pi_3} - M^2_{\pi_2} = \frac{32a^2W_0^2}{F_0^2}\sin^2(\omega)\,(-4L_6^\mathrm{r} - 2L_8^\mathrm{r} + 4W_6^\mathrm{r} + 2W_8^\mathrm{r} - 4W_6'^\mathrm{r} - 2W_8'^\mathrm{r}) + \mathcal{O}(a^3)$$

Kaon Mass Splitting

$$\begin{aligned}
M^2_{K^0} - M^2_{K^\pm} = B_0\Delta m\Big\{1 + & \frac{16B_0}{F_0^2}\Big[\big(-2L_4^\mathrm{r} - L_5^\mathrm{r} + 4L_6^\mathrm{r} + 2L_8^\mathrm{r}\big)\hat m \\
&\qquad + \big(-L_4^\mathrm{r} - L_5^\mathrm{r} + 2L_6^\mathrm{r} + 2L_8^\mathrm{r}\big)m_\mathrm{s}\Big] \\
& + \frac{8aW_0}{F_0^2}\Big[\big(-2L_5^\mathrm{r} - 6W_4^\mathrm{r} - 2W_5^\mathrm{r} + 6W_6^\mathrm{r} + 4W_8^\mathrm{r}\big) \\
&\qquad + \big(L_5^\mathrm{r} + 4W_4^\mathrm{r} + W_5^\mathrm{r} - 4W_6^\mathrm{r} - 2W_8^\mathrm{r}\big)(1 - \cos\omega)\Big] \\
& + \frac{1}{32\pi^2F_0^2}\Big[\frac{2\bar m_\eta^2}{3}\ln\left(\frac{\bar m_\eta^2}{\Lambda^2}\right) \\
&\qquad + \frac{1}{B_0(m_\mathrm{s} - \hat m)}\left[1 - \frac{aW_0(1 - \cos\omega)}{B_0(m_\mathrm{s} - \hat m)}\right] \\
&\qquad\quad \frac{\bar m_K^2}{2}\left(\bar m_\eta^2\ln\left(\frac{\bar m_\eta^2}{\Lambda^2}\right) - \bar m_\pi^2\ln\left(\frac{\bar m_\pi^2}{\Lambda^2}\right)\right)\Big]\Big\} \\
& + \mathcal{O}(\Delta m^2, a^2)
\end{aligned}$$

References

References

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