Hamiltonian simulation of lattice gauge theories

Boye Buyens

Ghent University - Quantum Group

Introduction

Hamiltonian formalism

Spatial discretization

Time continuous

$$i\partial_t |\Psi\rangle = H |\Psi\rangle$$

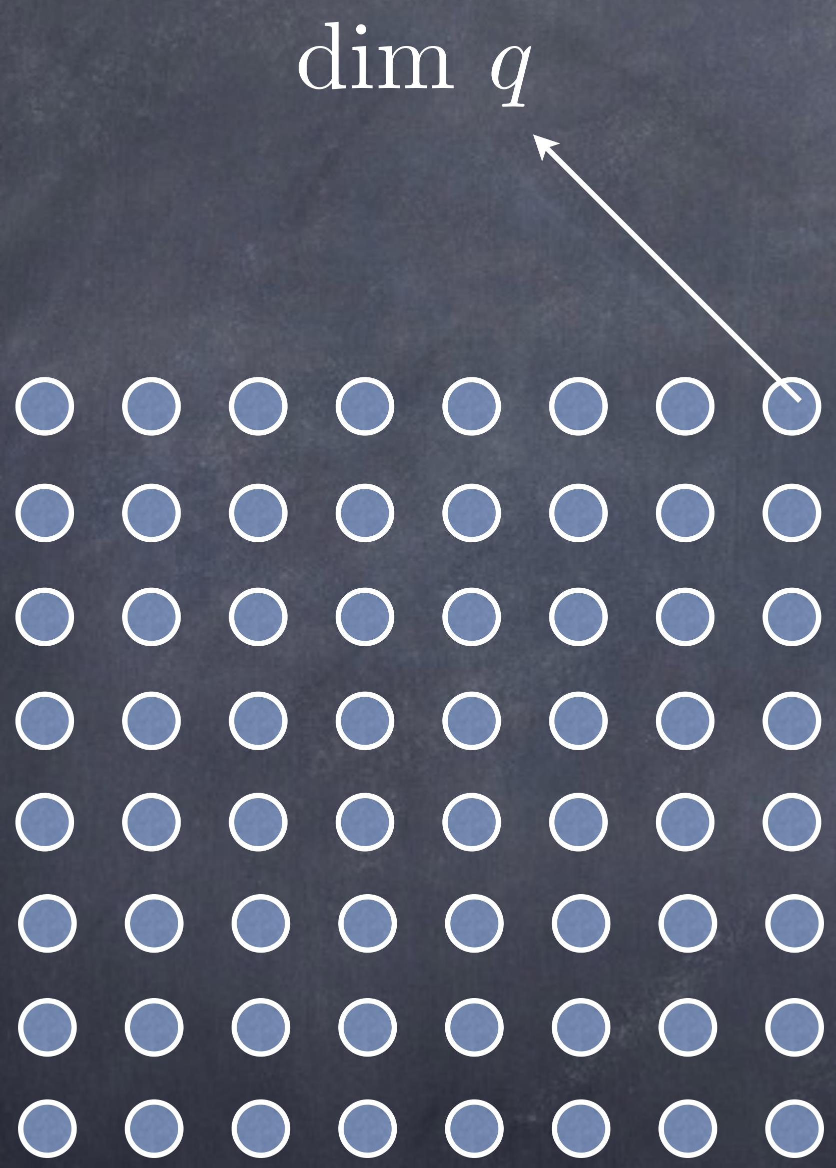
$|\Psi\rangle \in \mathcal{H}$

Hilbert space

$H \in \mathcal{L}(\mathcal{H})$

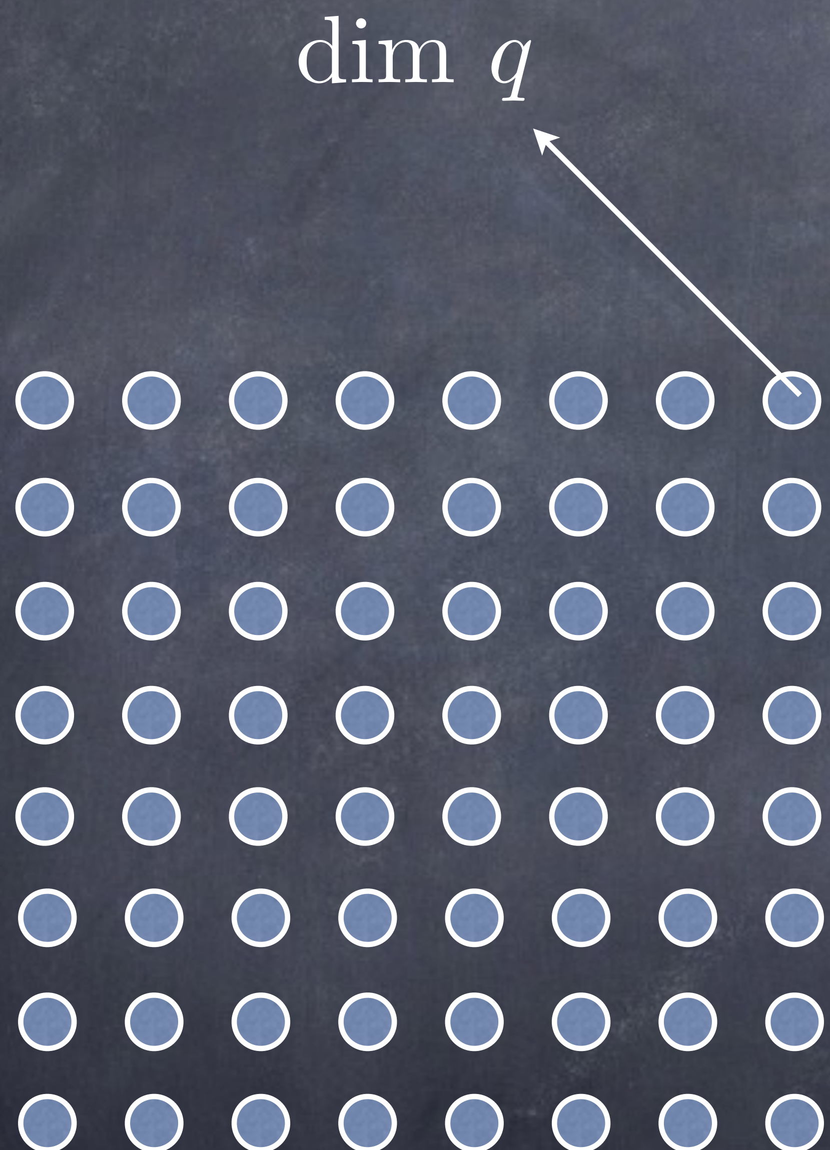
Hamiltonian

Many body problem



$$\mathcal{H}_{loc} \cong \mathbb{C}^q$$

Many body problem



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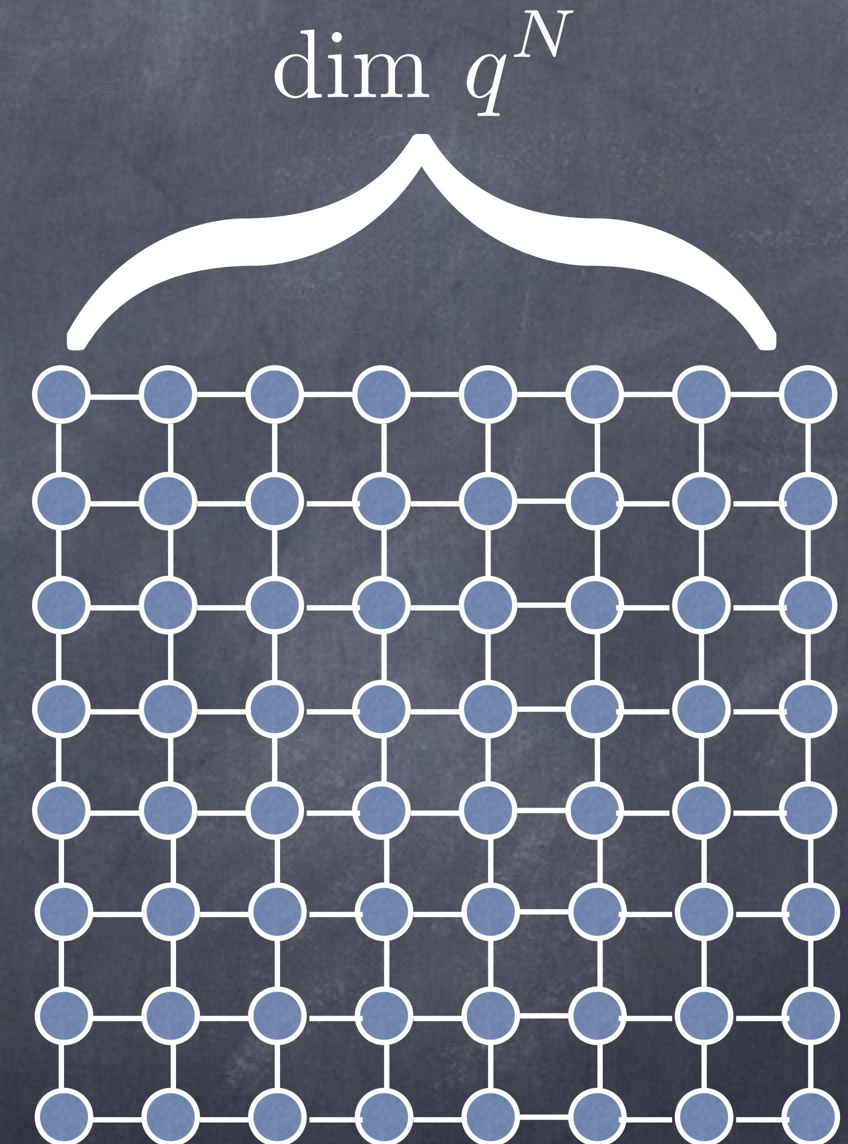
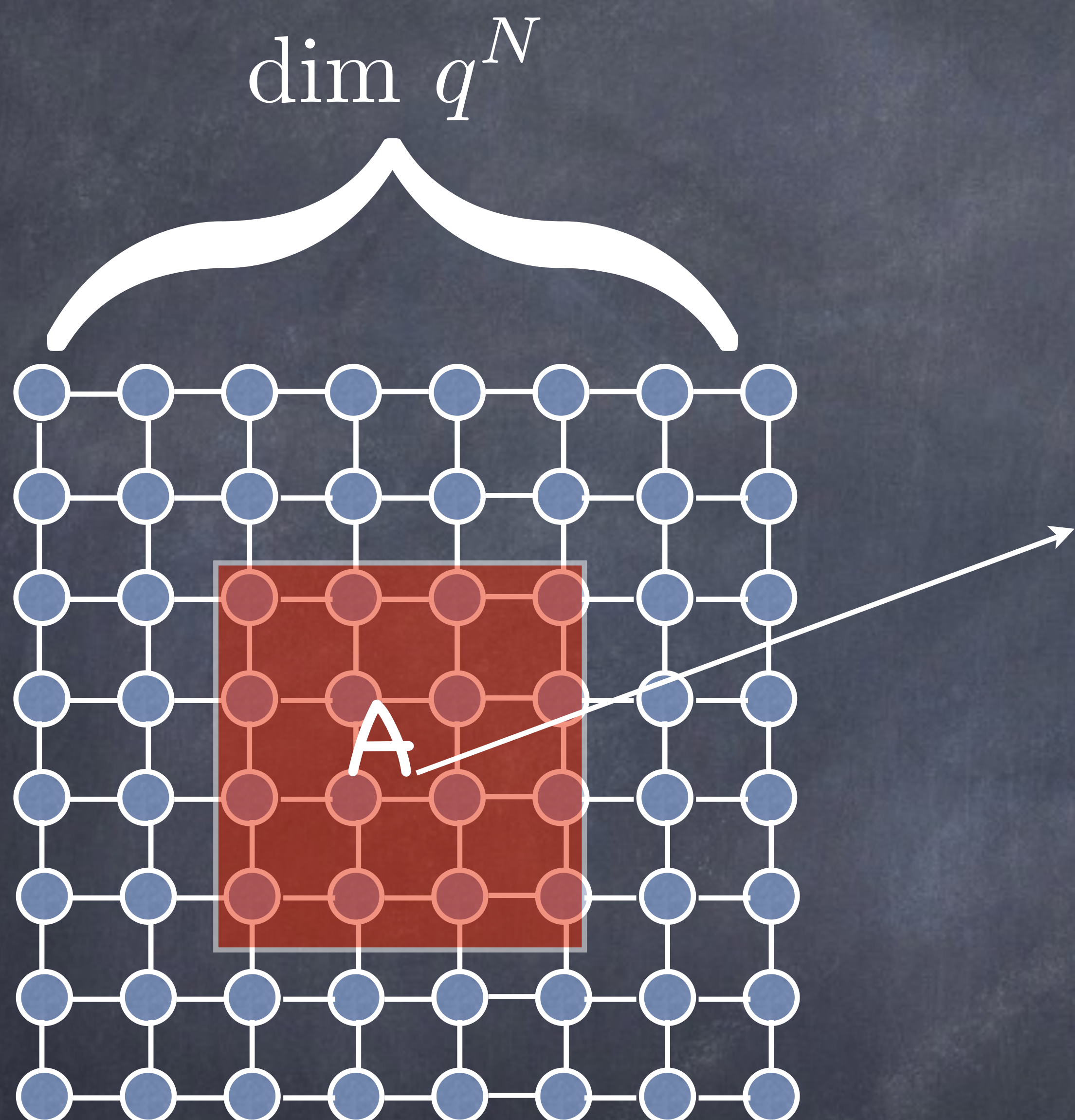


$$\mathcal{H} \cong \bigotimes_{n=1}^{N} \mathcal{H}_{loc} \cong \mathbb{C}^{q^N}$$

Area-law



Area-law for entropy
low-energy states

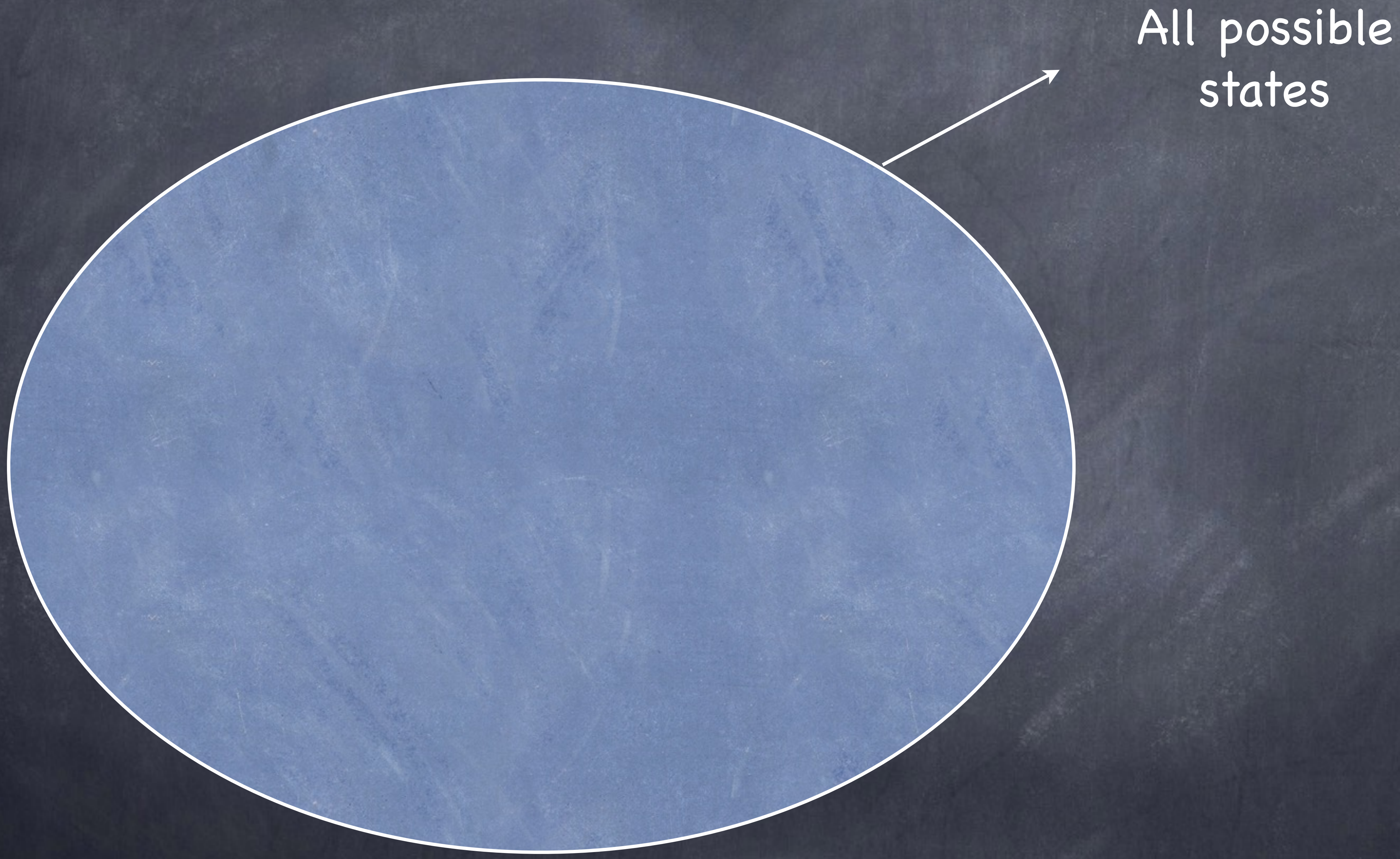
$$S_A = \mathrm{tr}(\rho_A \log \rho_A) \propto \partial A$$

$$\rho_A = \mathrm{tr}_{\bar{A}} \left|\Psi\right\rangle \left\langle\Psi\right|$$

(Hastings '07)

Variational class of states obeying area-law

Tiny corner



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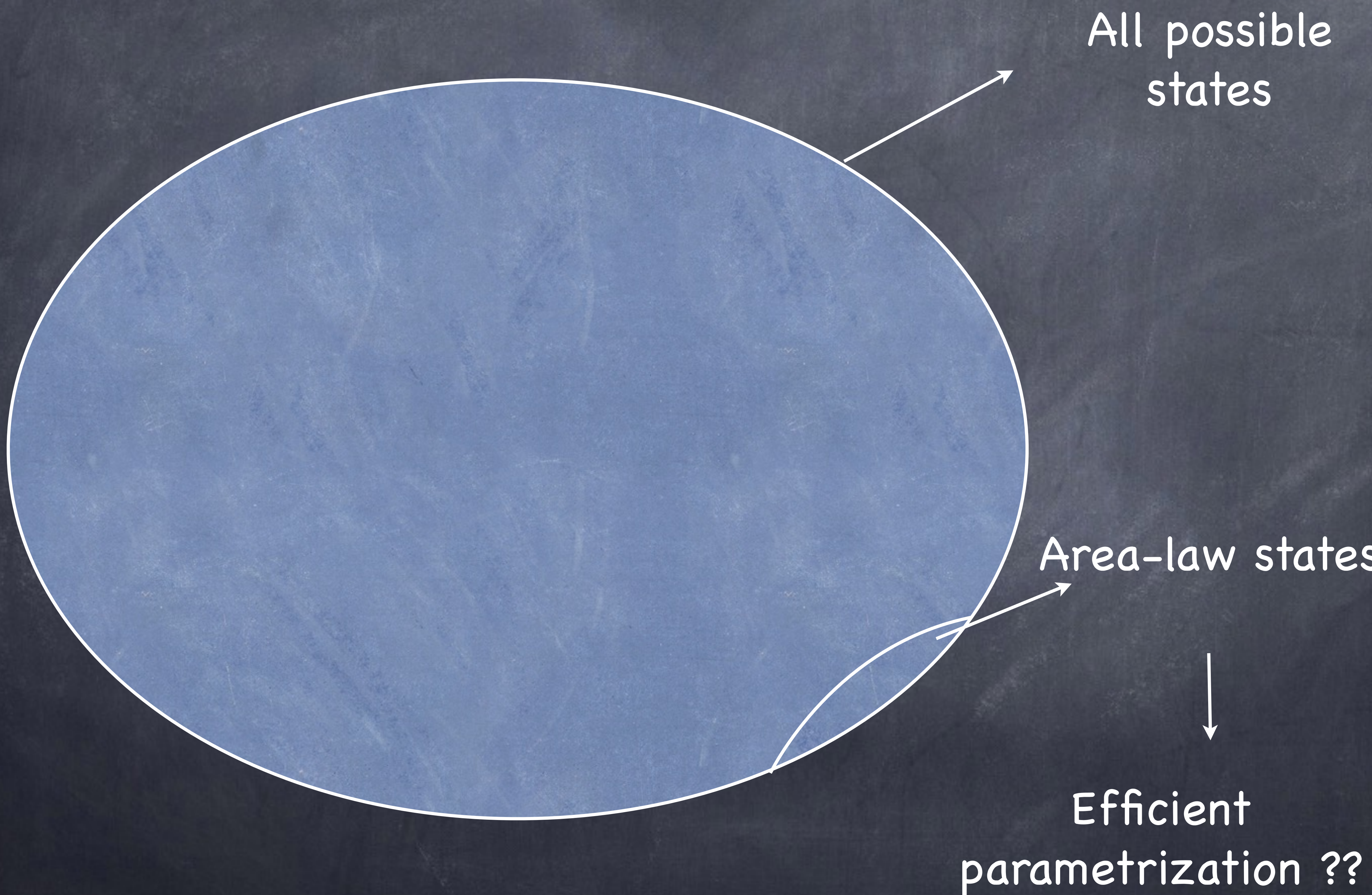
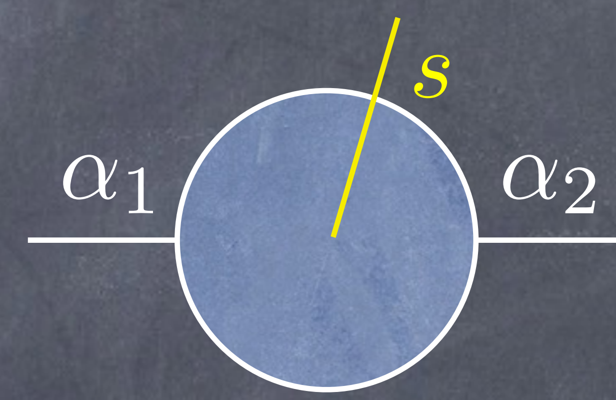

All possible states
Area-law states
Efficient parametrization ??

Matrix product states (MPS)

(Fannes, Nachtergaele, Werner ’92)

$$[A^s]_{\alpha_1,\alpha_2} =$$

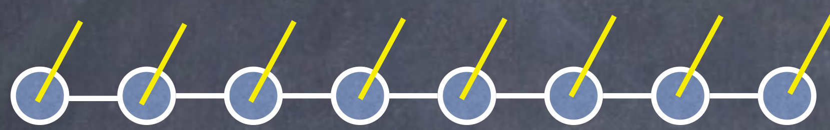


$\alpha_i = 1 \ldots D$ Virtual indices

$s = 1 \ldots q$ Physical indices

Matrix product states (MPS)

(Fannes, Nachtergaele, Werner '92)



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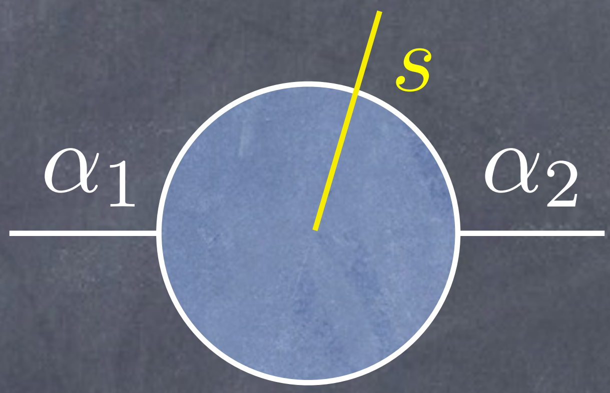

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$$|\Psi[A]\rangle = \sum_{\{s_n\}=1}^{q} \operatorname{tr}\left(A_1^{s_1} \ldots A_N^{s_N}\right) |s_1\rangle \ldots |s_N\rangle$$

Matrix product states (MPS)

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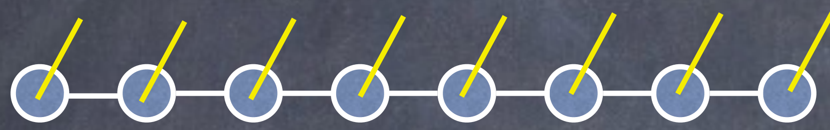

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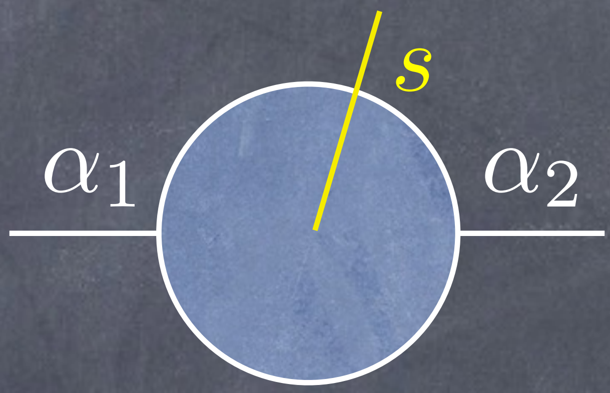

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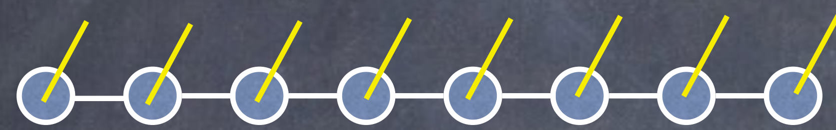
Proven that ground state of local gapped Hamiltonian can be approximated efficiently by a MPS (i.e. with 'reasonable' D)

Matrix product states (MPS)

(Fannes, Nachtergaele, Werner '92)



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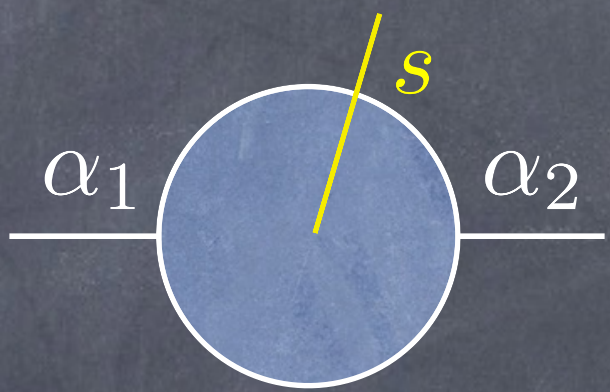

$\alpha_i = 1 \dots D$ Virtual indices

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$$|\Psi[A]\rangle = \sum_{\{s_n\}=1}^{q} \mathrm{tr}\left(A_1^{s_1} \dots A_N^{s_N}\right) |s_1\rangle \dots |s_N\rangle$$

Expectation values can be computed by matrix multiplications

$$\langle \Psi[\bar{A}] | \, O \, | \Psi[A] \rangle \sim \mathcal{O}(D^3)$$

Schwinger model

(1+1)-D QED

Schwinger model

$$\mathcal{L} = \bar{\psi}\left(\gamma^\mu(i\partial_\mu + gA_\mu) - m\right)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

Kogut-Susskind discretization + JW transformation

(Kogut, Susskind '75, Banks '78)

$$H = \frac{g}{2\sqrt{x}}\Bigg(\sum_{n\in\mathbb{Z}} L(n)^2 + \frac{\mu}{2}\sum_{n\in\mathbb{Z}}(-1)^n\sigma_z(n) +$$

$$x\sum_{n\in\mathbb{Z}}(\sigma^+(n)e^{i\theta(n)}\sigma^-(n+1) + h.c.)$$

$$x = \frac{1}{g^2a^2} \quad \mu = 2\frac{m}{g}\frac{1}{ga}$$

Ground state MPS ansatz

$$|\Psi[A]\rangle = \sum_{\{s,p\}} \mathrm{tr}[A_1^{s_1,p_1} \ldots A_{2N}^{s_{2N},p_{2N}}]\,|s_1, p_1, \ldots, s_{2N}, p_{2N}\rangle$$

We can take symmetries into account!

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We can take symmetries into account!

Translation symmetry (over two sites)

$$N \to \infty$$
$$A_{2n-1} = A_1, A_{2n} = A_2, \forall n \in \mathbb{Z}$$

Ground state MPS ansatz

$$|\Psi[A]\rangle = \sum_{\{s,p\}} \mathrm{tr}\left[\prod_{n\in\mathbb{Z}} A_1^{s_{2n-1},p_{2n-1}} A_2^{s_{2n},p_{2n}}\right] |\{s_n, p_n\}\rangle$$

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We can take symmetries into account!

CT symmetry

$$N \to \infty$$

$$A_{2n-1}^{s,p} = A^{-s,-p}, A_{2n} = A^{s,p}, \forall n \in \mathbb{Z}$$

Ground state MPS ansatz

$$|\Psi[A]\rangle = \sum_{s_n, p_n} \mathrm{tr}\left[\prod_{n\in\mathbb{Z}} A^{s_n, p_n}\right] |\{(-1)^n s_n, (-1)^n p_n\}\rangle$$

We can take symmetries into account!

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Ground state MPS ansatz

$$|\Psi[A]\rangle = \sum_{\{s,p\}} \mathrm{tr}[A_1^{s_1,p_1} \ldots A_{2N}^{s_{2N},p_{2N}}]\,|s_1, p_1, \ldots, s_{2N}, p_{2N}\rangle$$

We can take symmetries into account!

Gauge symmetry (Gauss' law)

$$\partial_x E = \rho$$

Ground state MPS ansatz

$$|\Psi[A]\rangle = \sum_{\{s,p\}} \mathrm{tr}[A_1^{s_1,p_1} \dots A_{2N}^{s_{2N},p_{2N}}]\,|s_1,p_1,\dots,s_{2N},p_{2N}\rangle$$

We can take symmetries into account!

Gauge symmetry (Gauss' law)

$$L(n) - L(n-1) = \frac{\sigma_z(n) + (-1)^n}{2}$$

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$$|\Psi[A]\rangle = \sum_{\{s,p\}} \mathrm{tr}[A_1^{s_1,p_1} \dots A_{2N}^{s_{2N},p_{2N}}]\,|s_1, p_1, \dots, s_{2N}, p_{2N}\rangle$$

We can take symmetries into account!

Gauge symmetry (Gauss' law)

$$L(n) - L(n-1) = \frac{\sigma_z(n) + (-1)^n}{2}$$

$$[A_n^{s,p}]_{(q\alpha);(r\beta)} = \delta_{p,q+(s+(-1)^n)/2}\delta_{p,r}[a_n^{s,p}]_{\alpha,\beta}$$

$$[A_n^{s,p}]_{(q\alpha);(r\beta)} = \delta_{p,q+(s+(-1)^n)/2}\,\delta_{p,r}\,[a_n^{s,p}]_{\alpha,\beta}$$

E.g. n = 1, s = 1: p = q - 1; r = p

$$
\begin{array}{c|c|c|c|c|c}
q\backslash r & \cdots & p-1 & p & p+1 & \cdots \\
\hline
\vdots & \ddots & \vdots & \vdots & \vdots & \iddots \\
\hline
p-1 & \cdots & \begin{smallmatrix}0&\cdots&0\\ \vdots&\ddots&\vdots\\ 0&\cdots&0\end{smallmatrix} & \begin{smallmatrix}0&\cdots&0\\ \vdots&\ddots&\vdots\\ 0&\cdots&0\end{smallmatrix} & \begin{smallmatrix}0&\cdots&0\\ \vdots&\ddots&\vdots\\ 0&\cdots&0\end{smallmatrix} & \cdots \\
\hline
p & \cdots & \begin{smallmatrix}0&\cdots&0\\ \vdots&\ddots&\vdots\\ 0&\cdots&0\end{smallmatrix} & \begin{smallmatrix}0&\cdots&0\\ \vdots&\ddots&\vdots\\ 0&\cdots&0\end{smallmatrix} & \begin{smallmatrix}0&\cdots&0\\ \vdots&\ddots&\vdots\\ 0&\cdots&0\end{smallmatrix} & \cdots \\
\hline
p+1 & \cdots & \begin{smallmatrix}0&\cdots&0\\ \vdots&\ddots&\vdots\\ 0&\cdots&0\end{smallmatrix} & a_1^{1,p} & \begin{smallmatrix}0&\cdots&0\\ \vdots&\ddots&\vdots\\ 0&\cdots&0\end{smallmatrix} & \cdots \\
\hline
p+2 & \cdots & \begin{smallmatrix}0&\cdots&0\\ \vdots&\ddots&\vdots\\ 0&\cdots&0\end{smallmatrix} & \begin{smallmatrix}0&\cdots&0\\ \vdots&\ddots&\vdots\\ 0&\cdots&0\end{smallmatrix} & \begin{smallmatrix}0&\cdots&0\\ \vdots&\ddots&\vdots\\ 0&\cdots&0\end{smallmatrix} & \cdots \\
\hline
\vdots & \iddots & \vdots & \vdots & \vdots & \ddots
\end{array}
$$

Application of MPS to (1+1)-D QED

1) Ground state + 1-particle excitations
2) Real-time evolution
3) Confinement and string breaking
4) Thermal states

BB, J. Haegeman, K. Van Acoleyen, H. Verschelde, F. Verstraete, PRL 113, 2014, arXiv:1312.6654
BB, K. Van Acoleyen, J. Haegeman, F. Verstraete, PoS(LATTICE2014)308, 2014, arXiv:1411.0020

Application of MPS to (1+1)-D QED

This talk

1) Ground state + 1-particle excitations
2) Real-time evolution
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4) Thermal states (see also talk by Dr H. Saito)

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1) Ground state, 1-P excited states

$$|\Psi[A]\rangle = \sum_{\{s,p\}} \mathrm{tr}[A_1^{s_1,p_1} \dots A_{2N}^{s_{2N},p_{2N}}] \, |s_1, p_1, \dots, s_{2N}, p_{2N}\rangle$$

Minimisation w.r.t. tensors A of

$$\frac{\langle \Psi[\bar{A}] | \, H \, | \Psi[A] \rangle}{\langle \Psi[\bar{A}] | \Psi[A] \rangle}$$

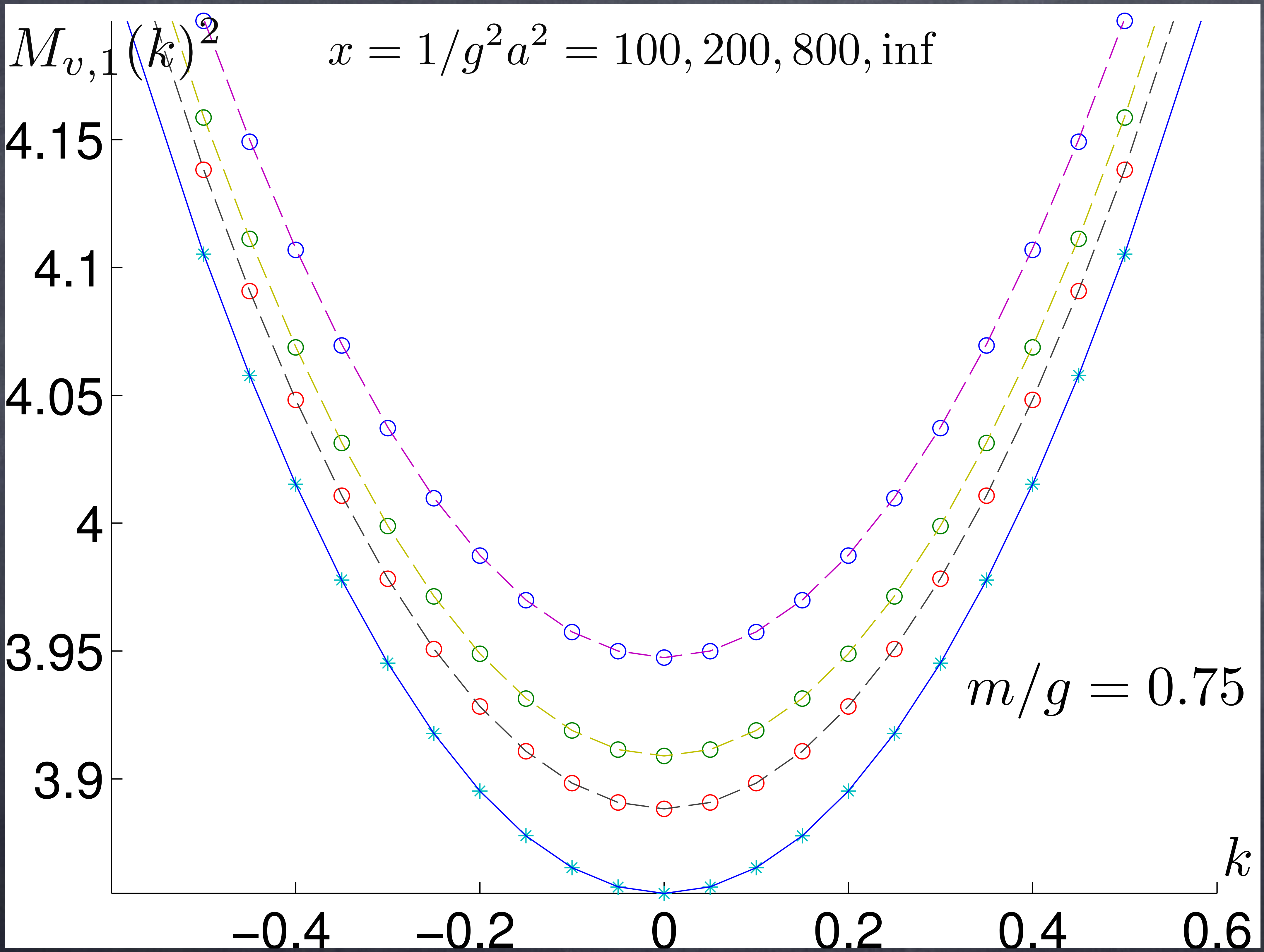
DMRG (White '92),
TDVP (Haegeman '13 '14)

Results: GS and excitations

Ground state energy density and 1-particle excitations

m/g	ω_0	$M_{v,1}$	$M_{s,1}$	$M_{v,2}$
0	-0.318320(4)	0.56418(2)		
0.125	-0.318319(4)	0.789491(8)	1.472(4)	2.10 (2)
0.25	-0.318316(3)	1.01917 (2)	1.7282(4)	2.339(3)
0.5	-0.318305(2)	1.487473(7)	2.2004 (1)	2.778 (2)
0.75	-0.318285(9)	1.96347(3)	2.658943(6)	3.2043(2)
1	-0.31826(2)	2.44441(1)	3.1182 (1)	3.640(4)

Earlier studies: Byrnes ’02, Banuls ‘13


$M_{v,1}(k)^2$
$x = 1/g^2a^2 = 100, 200, 800, \text{inf}$
4.15
4.1
4.05
4
3.95
3.9
$m/g = 0.75$
k
−0.4
−0.2
0
0.2
0.4
0.6

$$E(k)^2 = E(0)^2 + k^2$$

Application of MPS to (1+1)-D QED

This talk

1) Ground state + 1-particle excitations
2) Real-time evolution
3) Confinement and string breaking
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2) Real-time evolution

Real-time evolution induced by background electric field

$$H = \frac{g}{2\sqrt{x}}\left(\sum_{n\in\mathbb{Z}} L^2(n) + \frac{\mu}{2}\sum_{n\in\mathbb{Z}}(-1)^n(\sigma_z(n) + (-1)^n) + x\sum_{n\in\mathbb{Z}}(\sigma^+(n)e^{i\theta(n)}\sigma^-(n+1) + h.c.)\right)$$

$$L(n) \to L(n) + \alpha$$

2) Real-time evolution

Real-time evolution induced by background electric field

$$H_\alpha = \frac{g}{2\sqrt{x}}\left(\sum_{n\in\mathbb{Z}}[L(n)+\alpha]^2 + \frac{\mu}{2}\sum_{n\in\mathbb{Z}}(-1)^n(\sigma_z(n)+(-1)^n) + x\sum_{n\in\mathbb{Z}}(\sigma^+(n)e^{i\theta(n)}\sigma^-(n+1)+h.c.)\right)$$

Related: semi-classical studies: Kluger ’92, Hebenstreit ‘13

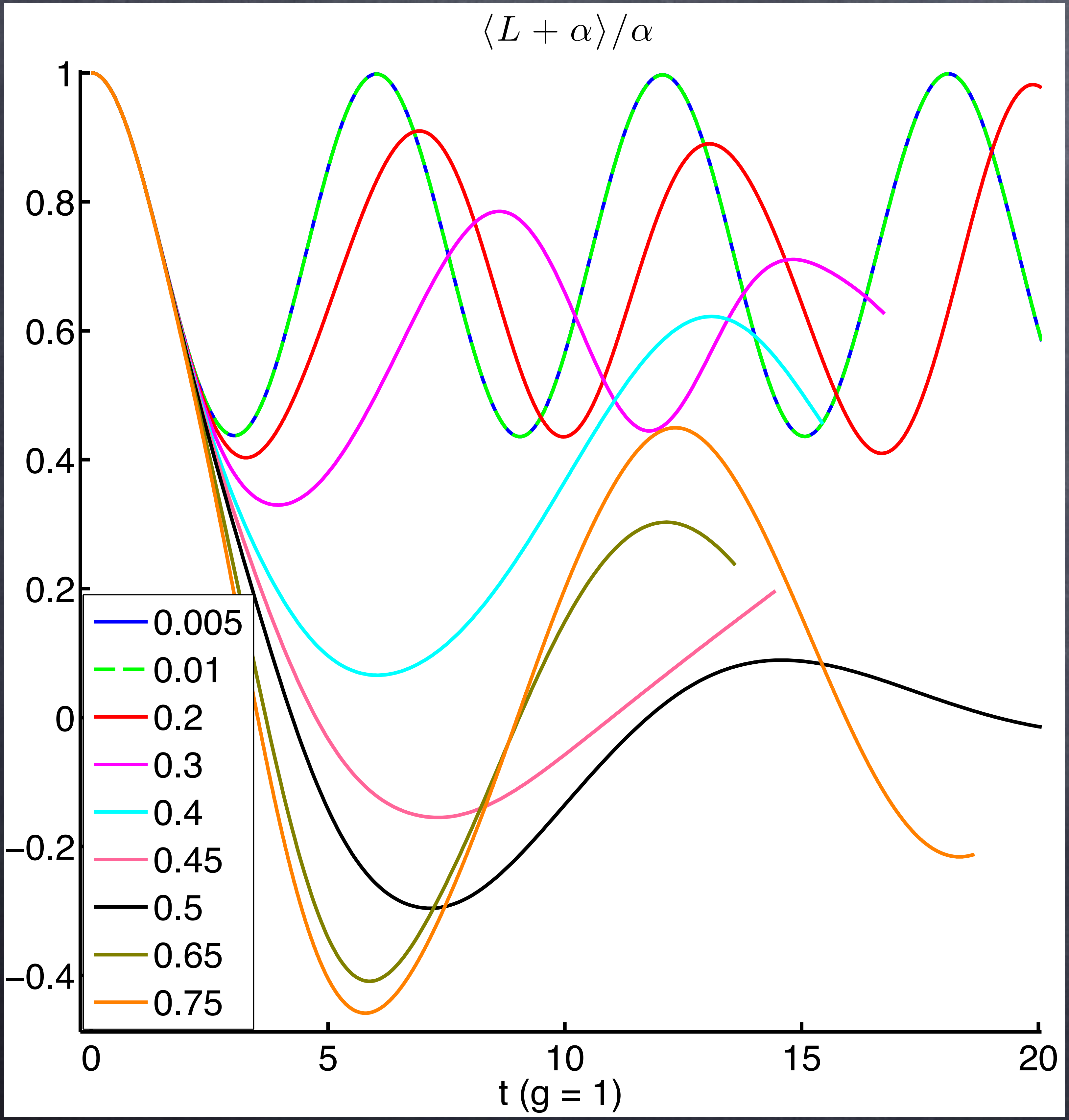
iTEBD

$$|\Psi[A]\rangle = \sum_{\{s,p\}} \mathrm{tr}[A_1^{s_1,p_1} \ldots A_{2N}^{s_{2N},p_{2N}}]\,|s_1,p_1,\ldots,s_{2N},p_{2N}\rangle$$

$$\exp(-iH_\alpha t)\,|\Psi[A]\rangle \approx |\Psi[A(t)]\rangle$$ ITEBD (Vidal '07)

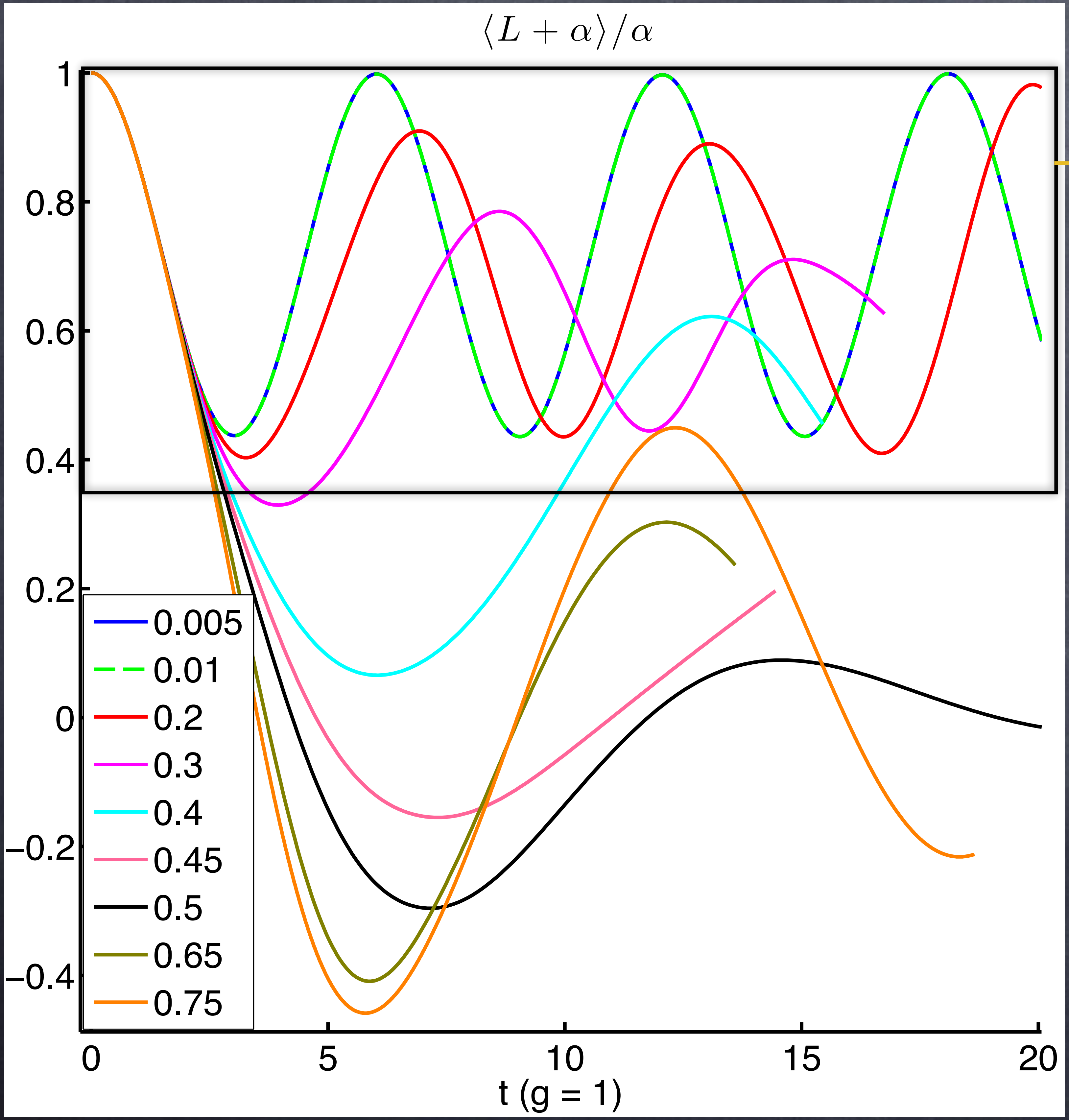
$$|\Psi[A(t)]\rangle = \sum_{\{s,p\}} \mathrm{tr}\,[A_1^{s_1,p_1}(t) \ldots A_{2N}^{s_{2N},p_{2N}}(t)]\,|s_1,p_1,\ldots,s_{2N},p_{2N}\rangle$$

Results



$\langle L + \alpha \rangle / \alpha$
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
0
5
10
15
20
t (g = 1)
0.005
0.01
0.2
0.3
0.4
0.45
0.5
0.65
0.75

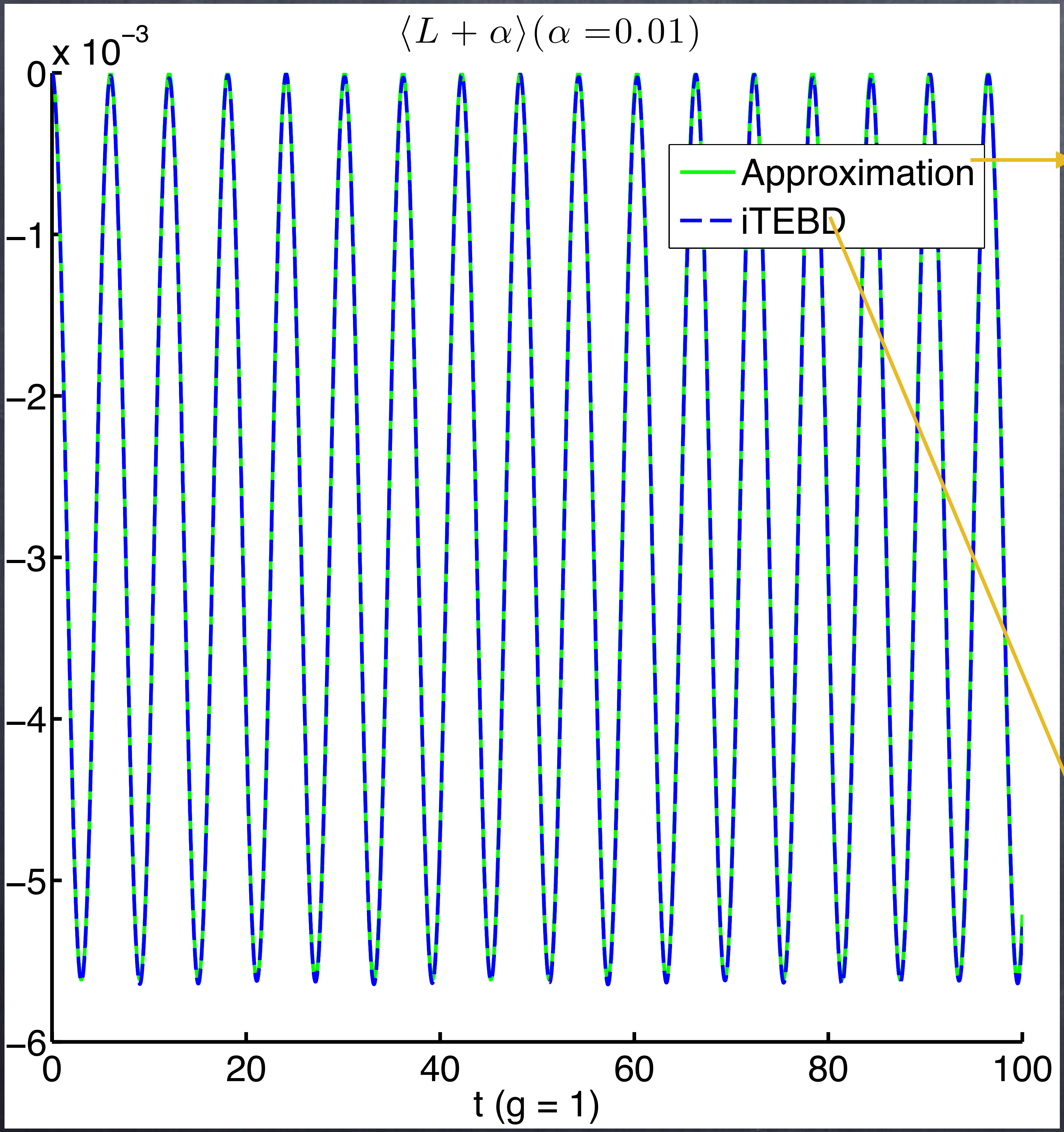
Results



$\alpha \lesssim 0.3$

Weak-field regime

Weak-field regime



$|\Psi[A]\rangle \approx |\Psi_0^\alpha\rangle + \text{small density of 1P on top}$

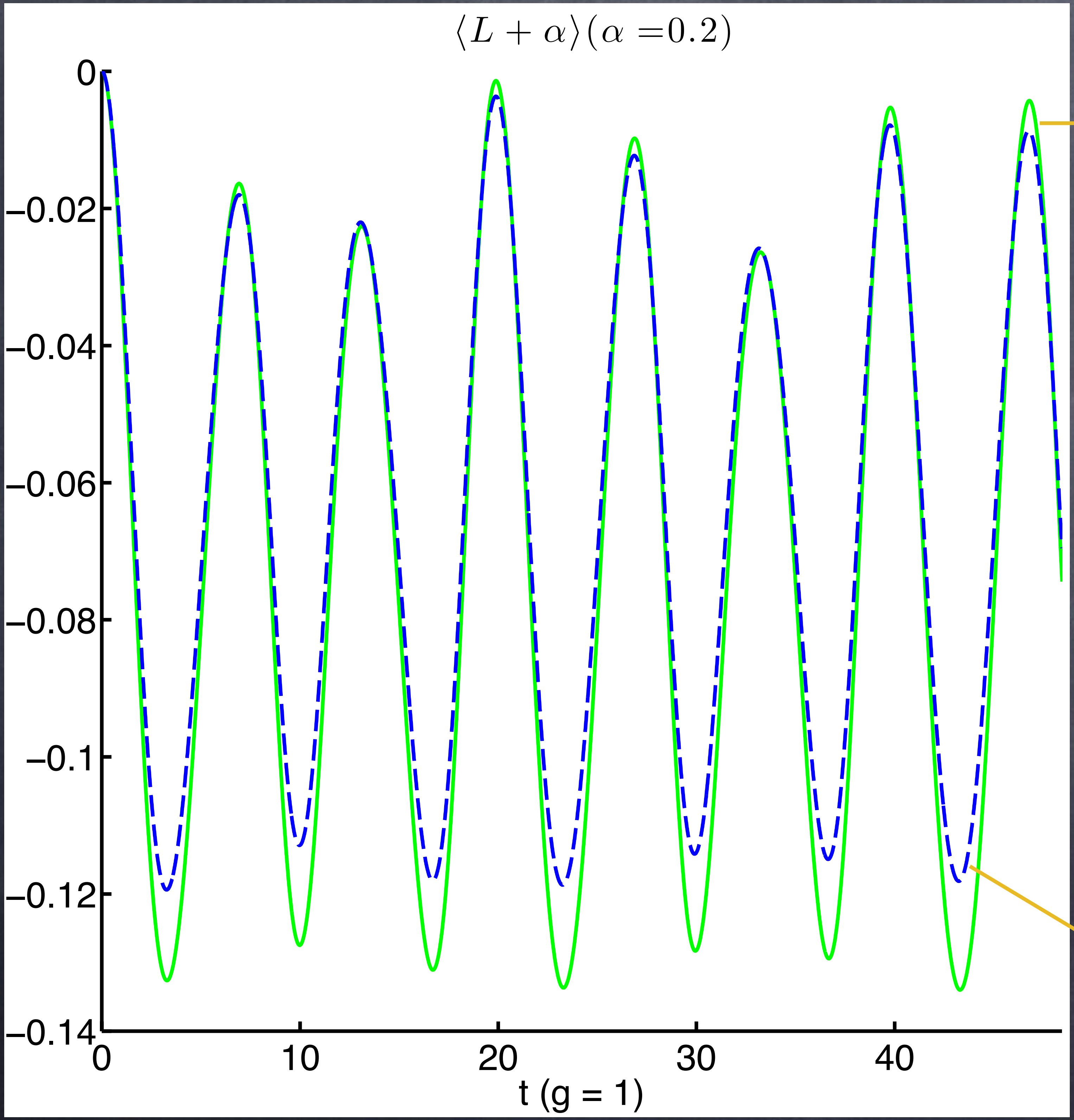
$H_\alpha \approx \text{free 1P Hamiltonian}$

$\alpha = 0.01$

OK

$\exp(-iH_\alpha t)\,|\Psi[A]\rangle \approx |\Psi[A(t)]\rangle$

Weak-field regime



$|\Psi[A]\rangle \approx |\Psi_0^\alpha\rangle + \text{small density of 1P on top}$

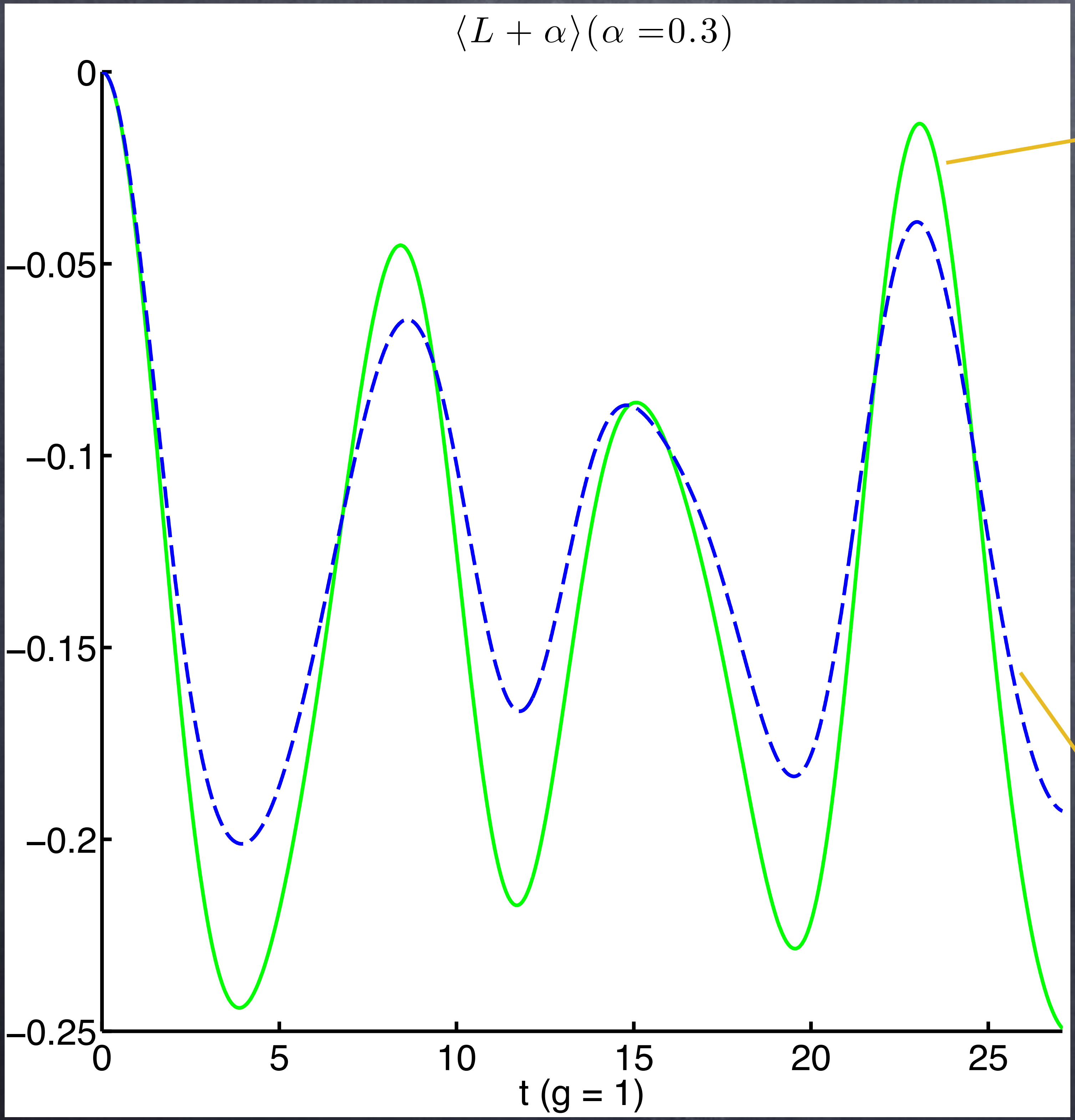
$H_\alpha \approx \text{free 1P Hamiltonian}$

$\alpha = 0.2$

reasonable

$\exp(-iH_\alpha t)\,|\Psi[A]\rangle \approx |\Psi[A(t)]\rangle$

Weak-field regime



$|\Psi[A]\rangle \approx |\Psi_0^\alpha\rangle$ + small density of 1P on top

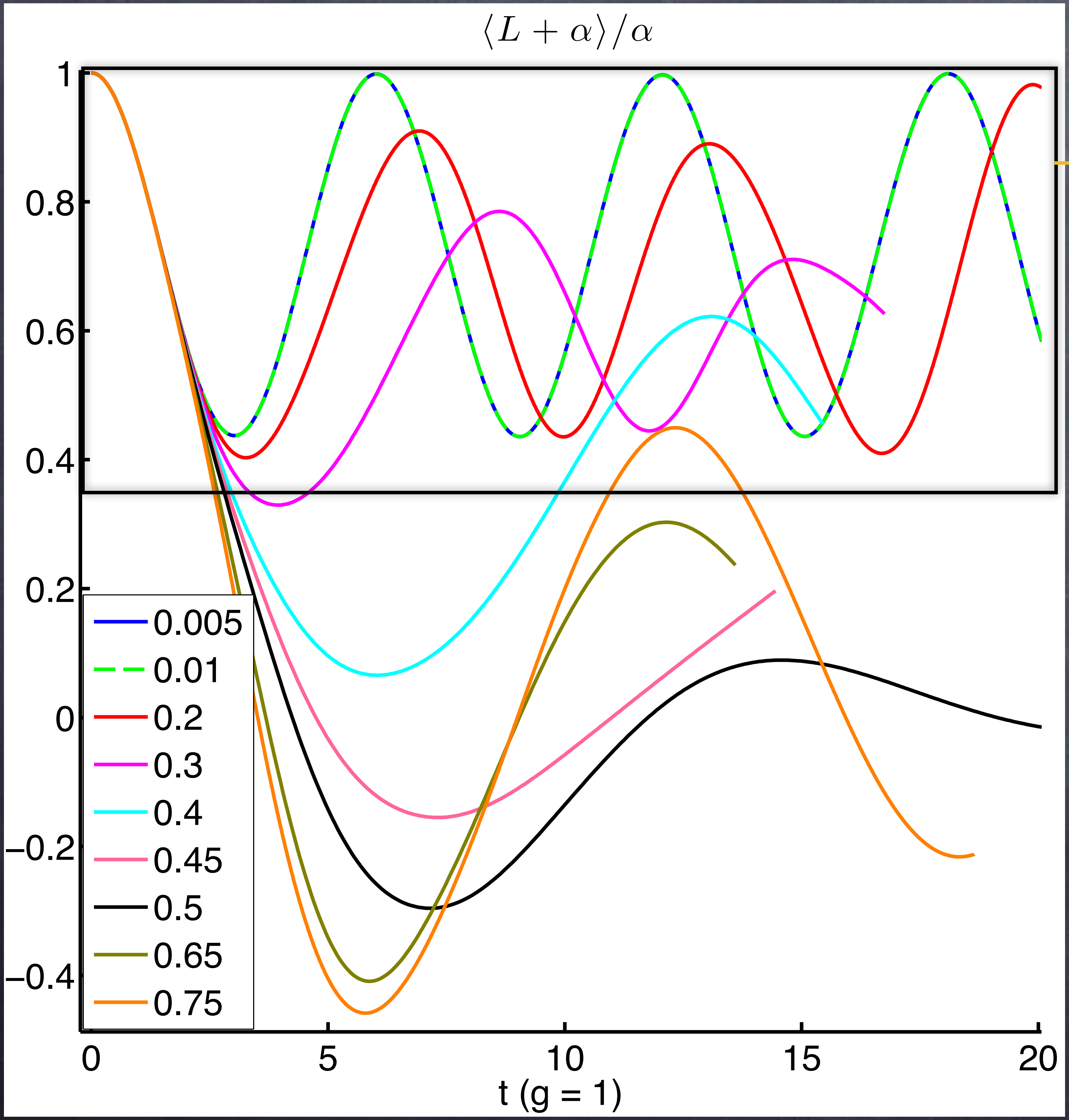
$H_\alpha \approx$ free 1P Hamiltonian

$\alpha \gtrsim 0.3$

fails!

$\exp(-iH_\alpha t)\,|\Psi[A]\rangle \approx |\Psi[A(t)]\rangle$

Results



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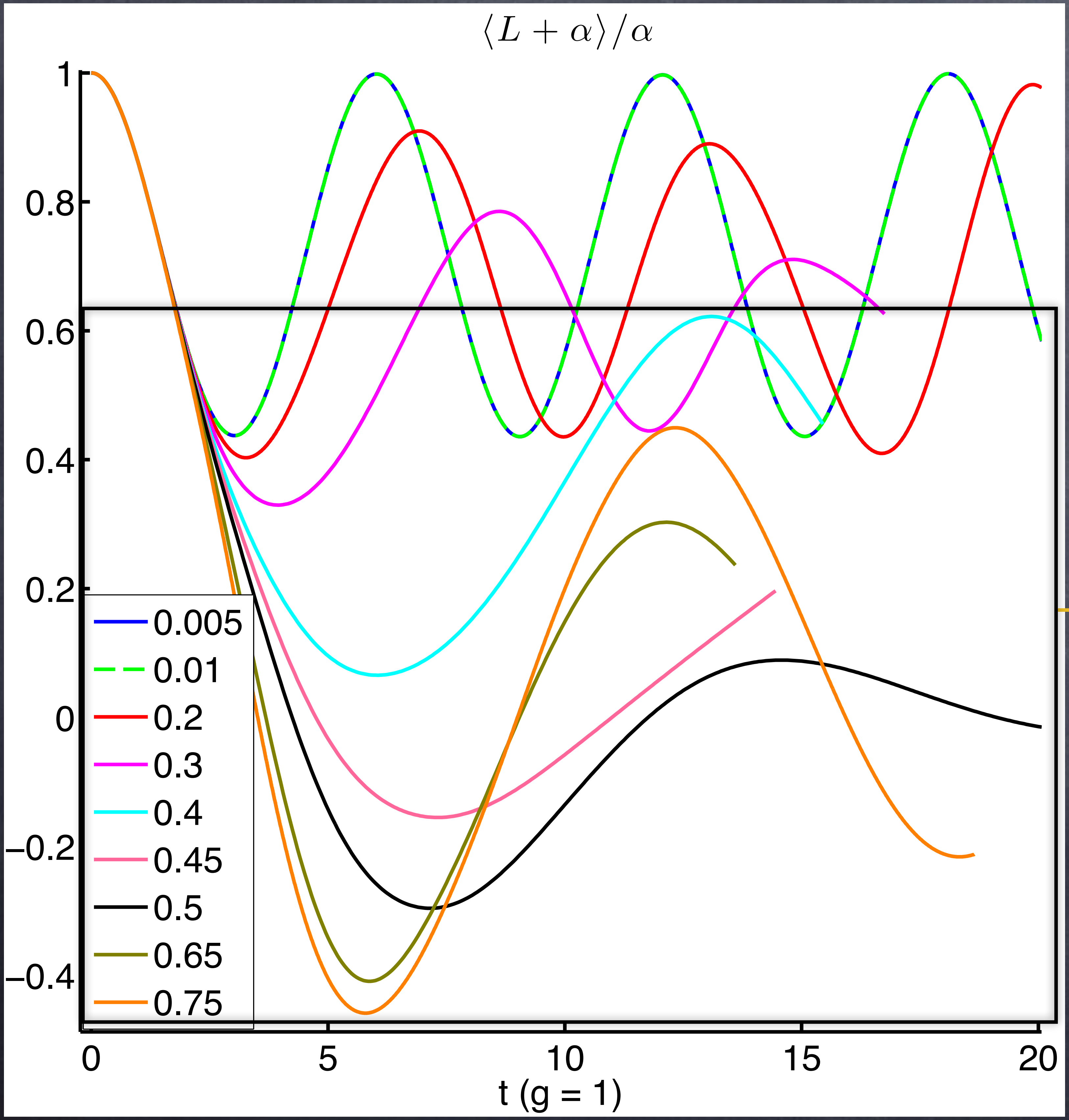
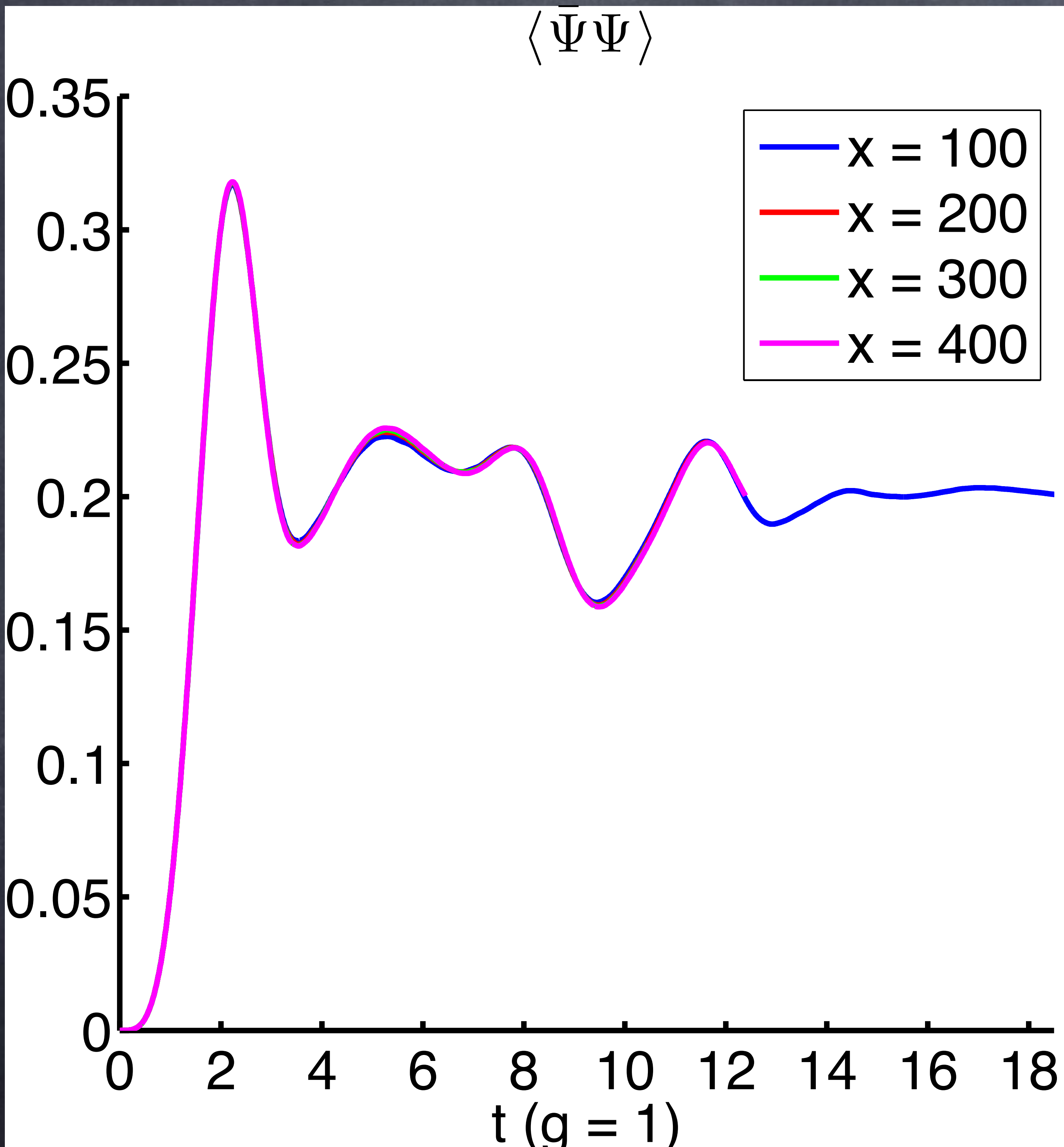


$\alpha \gtrsim 0.3$

Strong-field regime

Results



$$m/g = 0.25 \ \alpha = 0.75$$

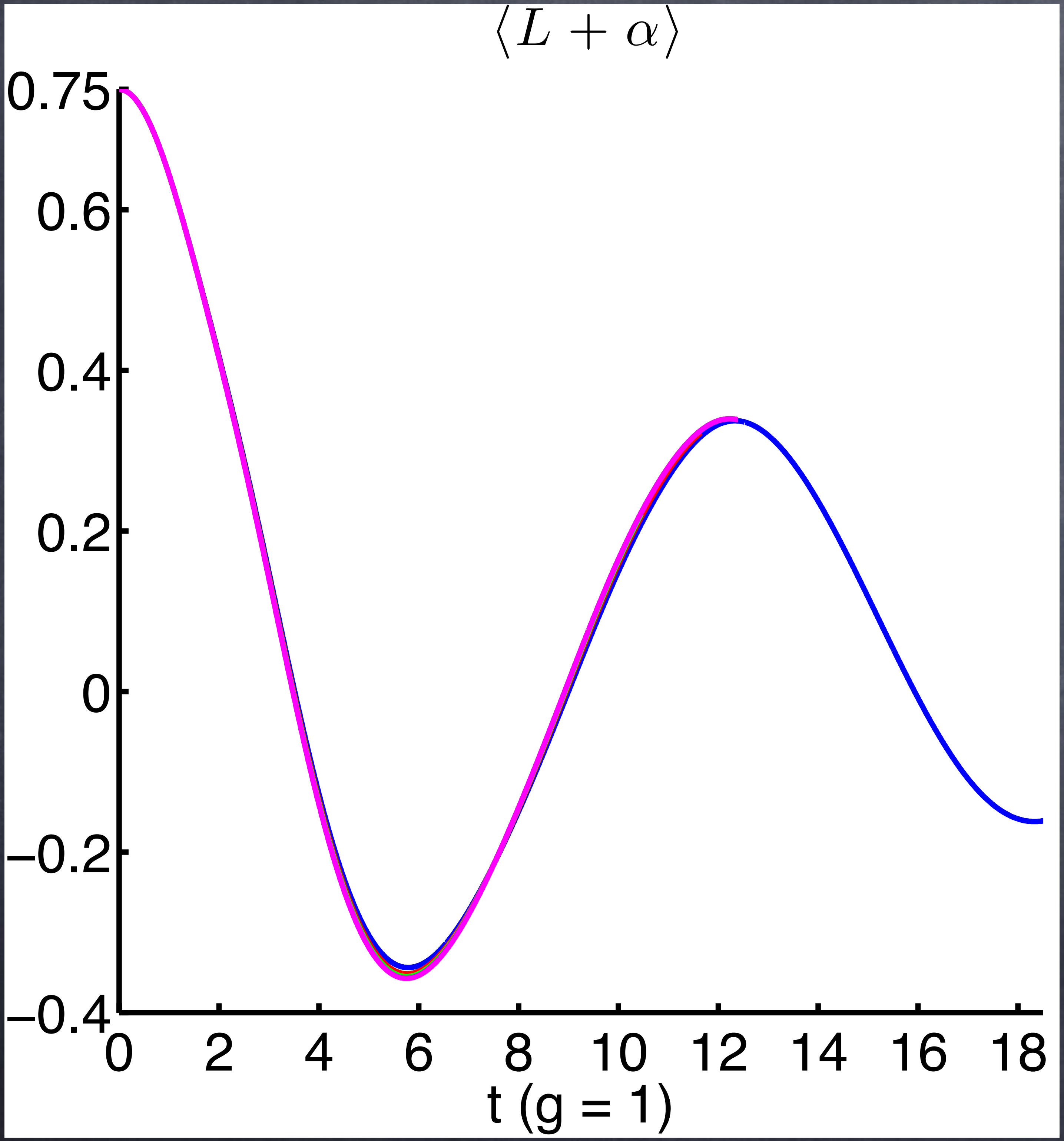
<u>Strong-field regime</u>

$$\alpha \gtrsim 0.3$$

$$\exp(-iH_\alpha t)\,|\Psi[A]\rangle \approx |\Psi[A(t)]\rangle$$

Thermalisation of local observables

Results



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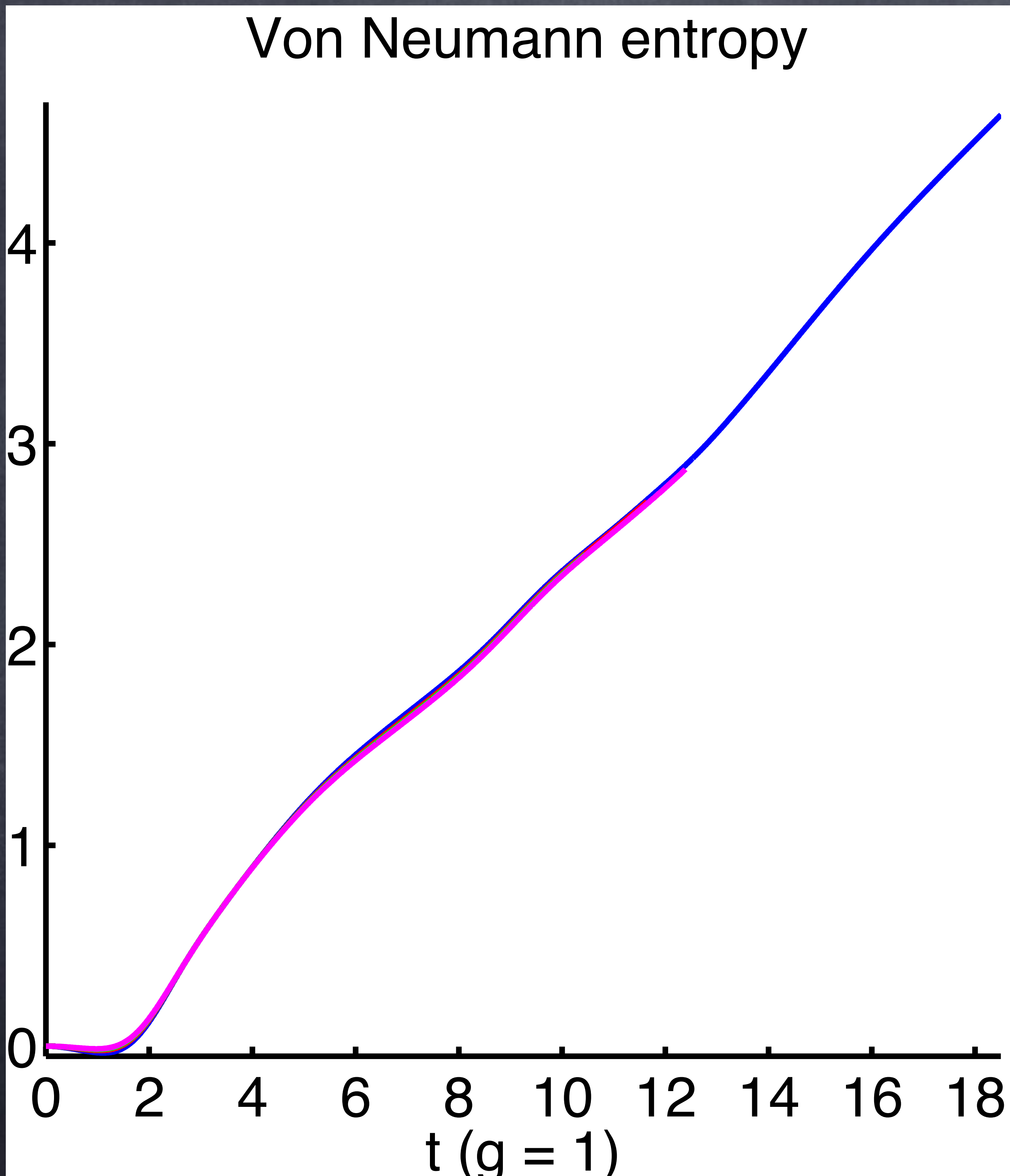

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$$\exp(-iH_\alpha t) \, |\Psi[A]\rangle \approx |\Psi[A(t)]\rangle$$

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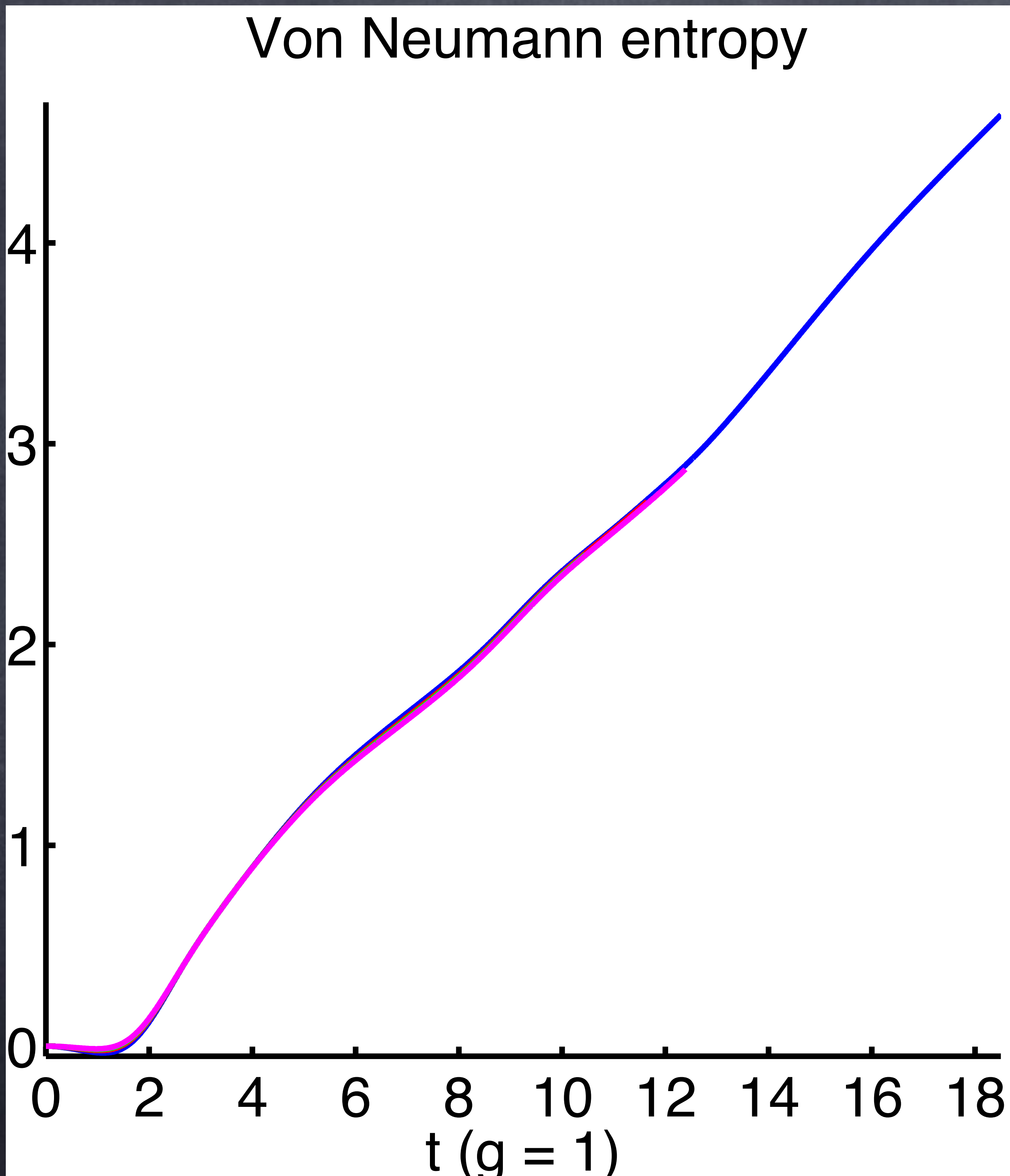
Linear growth of entropy

Results



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Strong-field regime

$\alpha \gtrsim 0.3$

$\exp(-iH_\alpha t)\,|\Psi[A]\rangle \approx |\Psi[A(t)]\rangle$

Thermalisation of local observables

Linear growth of entropy

Exponential increase of variational parameters

Conclusion

MPS is reliable method for (1+1)-D gauge theories

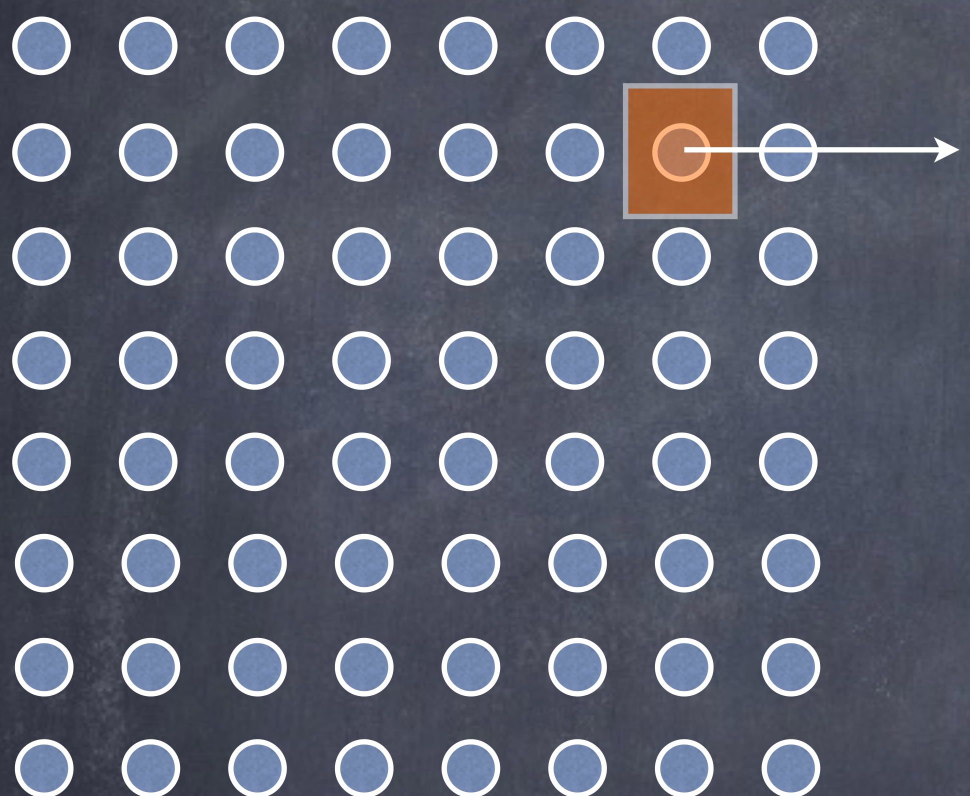
Real-time: despite results, still needs better understanding
-> Including scattering states (Vanderstraeten '14)

Higher dimensions: PEPS is numerically much harder

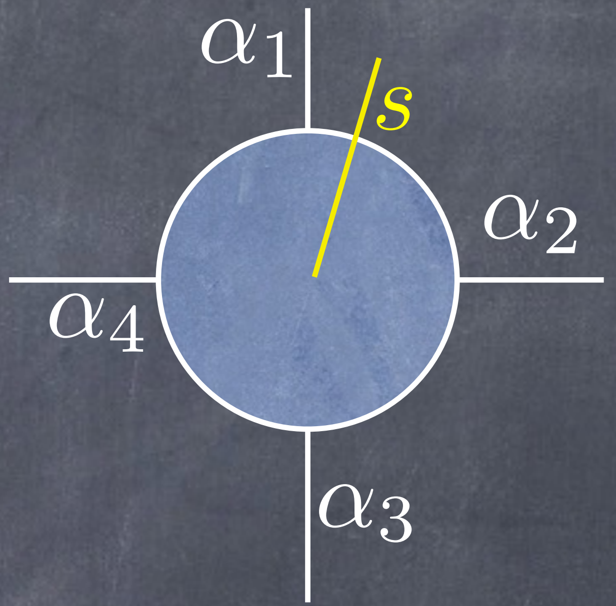
Although: already big advance in this field
- Numerically
- Theoretically

Projected-entangled pair states (PEPS)

(Verstraete, Cirac '05)



$$[A^s]_{\alpha_1,\alpha_2,\alpha_3,\alpha_4} =$$



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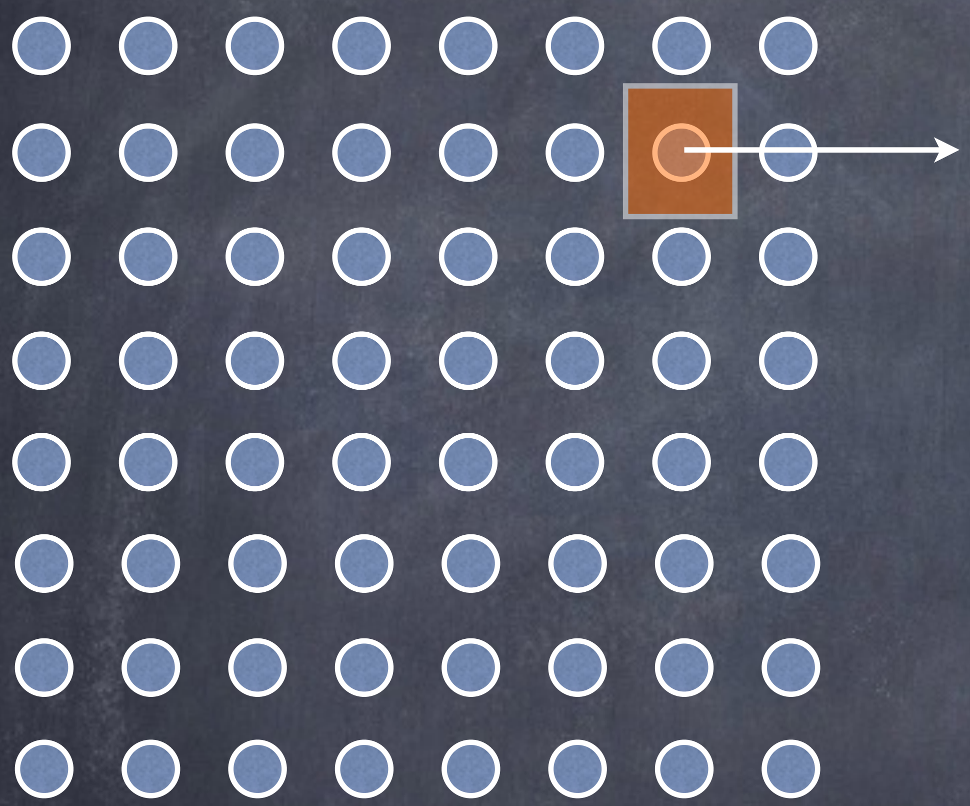
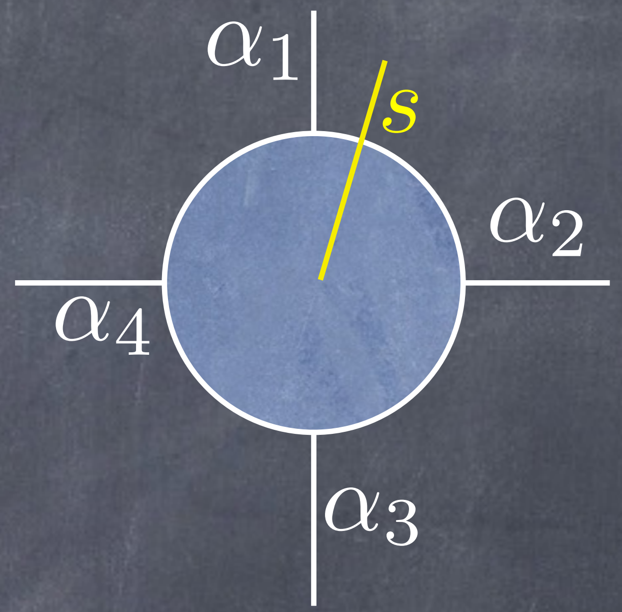
$s = 1 \ldots q$ Physical indices

Projected-entangled pair states (PEPS)



$$[A^s]_{\alpha_1,\alpha_2,\alpha_3,\alpha_4} =$$

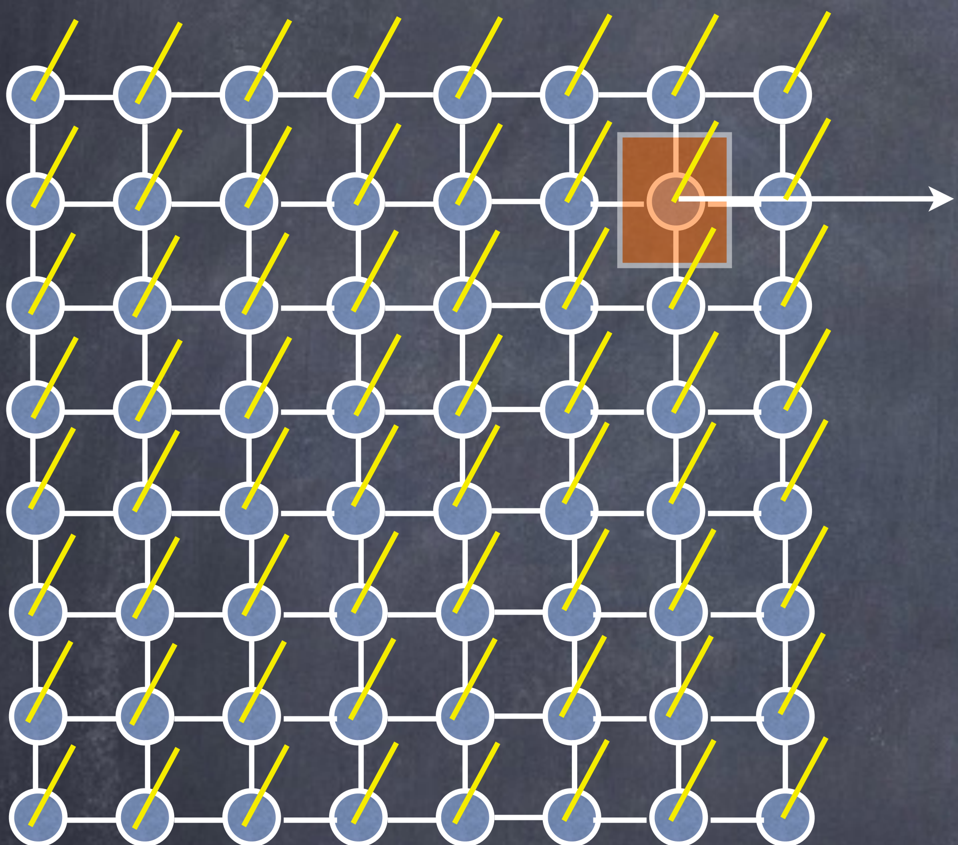


$\alpha_i = 1 \ldots D$ Virtual indices

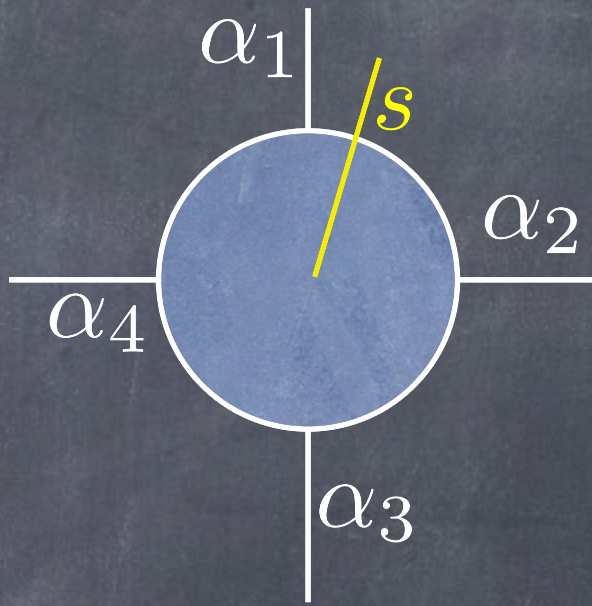
$s = 1 \ldots q$ Physical indices

Contraction of virtual indices along lattice

Projected-entangled pair states (PEPS)



$$[A^s]_{\alpha_1,\alpha_2,\alpha_3,\alpha_4} =$$



$\alpha_i = 1 \ldots D$ Virtual indices

$s = 1 \ldots q$ Physical indices

Satisfy area law!

$$S_A \sim \log(D)\, \partial A$$

Related work

J. Haegeman, K. Van Acoleyen, J.I. Cirac, N. Schuch, F. Verstraete, arXiv:1407.1025

T. M. Byrnes, P. Sriganesh, R.J. Bursill, C.J. Hamer Phys. Rev. D66, 13002 (2002)

T. Sugihara, JHEP 07, 022 (2005)

L. Tagliacozzo and G. Vidal, Phys. Rev. B 83, 115127 (2011)

M.C. Bañuls, K. Cichy, K. Jansen, and J.I. Cirac, JHEP 11, 158 (2013)

M.C Bañuls, K. Cichy, J.I. Cirac, K. Jansen, H. Saito, PoS (Lattice 2013) 332 and PoS (Lattice 2014) 302

E.Rico, T. Pichler, M.Dalmonte, P. Zoller, S.Montangero, PRL 112, 201601 (2014)

P. Silvi, E. Rico, T. Calarco and S. Montangero, New J. Phys. 16,10, 103015 (2014)

L.Tagliacozzo, A. Celi and M. Lewenstein, Phys. Rev X4 (2014) 4,041024

T. Osborne, A. Milsted, https://github.com/tobiasosborne/Latticegauge-theoryandtensor-networks

M.C. Bañuls, K. Cichy, J.I Cirac, K. Jansen, H. Saito, arXiv:1505.00279 (2015)

S. Kühn, E. Zohar, J.I Cirac, M.C. Ba{\~n}uls, arXiv:1505.04441 (2015)

T. Pichler, M. Dalmonte, E. Rico, P. Zoller, S. Montangero, arXiv:1505.04440 (2015)

Back-up slides

Gauge invariant MPS

$$|\Psi[A]\rangle = \sum_{\{s,p\}} \mathrm{tr}[A_1^{s_1,p_1} \ldots A_{2N}^{s_{2N},p_{2N}}]\,|s_1, p_1, \ldots, s_{2N}, p_{2N}\rangle$$

Gauge symmetry (Gauss' law)

$$L(n) - L(n-1) = \frac{\sigma_z(n) + (-1)^n}{2}$$

$$[A_n^{s,p}]_{(q\alpha);(r\beta)} = \delta_{p,q+(s+(-1)^n)/2}\delta_{p,r}[a_n^{s,p}]_{\alpha,\beta}$$

Ansatz 1-particle states

$$|\Psi[A]\rangle = \sum_{\{s,p\}} \mathrm{tr}\left[\prod_{n\in\mathbb{Z}} A_1^{s_{2n-1},p_{2n-1}} A_2^{s_{2n},p_{2n}}\right] |\{s_n,p_n\}\rangle$$

$$|\Phi_k(B,A)\rangle = \sum_{m\in\mathbb{Z}} e^{ikma} \sum_{\{q_n\}} \mathrm{tr}\left[\left(\prod_{n<m} A_1^{s_{2n-1},p_{2n-1}} A_2^{s_{2n},p_{2n}}\right)\right.$$
$$\left. B^{s_{2m-1},p_{2m-1},s_{2m}p_{2m}} \left(\prod_{n>m} A_1^{s_{2n-1},p_{2n-1}} A_2^{s_{2n},p_{2n}}\right)\right] |\{s_n,p_n\}\rangle$$

Ansatz 1-particle states

$$|\Psi[A]\rangle = \sum_{s_n, p_n} \mathrm{tr}\left[\prod_{n\in\mathbb{Z}} A^{s_n, p_n}\right] |\{(-1)^n s_n, (-1)^n p_n\}\rangle$$

$$|\Phi_k(B,A)\rangle = \sum_{m\in\mathbb{Z}} e^{ikma}\gamma^m \sum_{\{q_n\}} \mathrm{tr}\left[\left(\prod_{n<m} A^{s_n, p_n}\right) B^{s_m, p_m} \left(\prod_{n>m} A^{s_n, p_n}\right)\right] |\{(-1)^n s_n, (-1)^n p_n\}\rangle$$

Real-time: weak-field - approximation

$H_\alpha \approx$ free

$$H_\alpha \approx \sum_n \int dk\, \epsilon_n(k) a_n^\dagger(k) a_n(k)$$

$a_n^{(\dagger)}(k)$ annihilates (creates) particle n with momentum k

$$H_0 \approx \sum_{n,m} \int dk\, M_{n,m}(k) a_n^\dagger(k) a_m(k) + \sum_n \left(c_n a_n(0) + \bar{c}_n a_n^\dagger(0)\right)$$

$|\Psi(0)\rangle$ is GS of $H_0 \Rightarrow a_m(k)\,|\Psi(0)\rangle = \delta(k) d_m\,|\Psi(0)\rangle$

$$|\Psi(t)\rangle = \exp(-iH_\alpha t)\,|\Psi(0)\rangle$$

Expand observables O similar like H_0

Evolution of bond dimension