Aspects of topological actions on the lattice

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"Topological" action

- Invariant under smooth deformation of the field

Example: $O(N)$ spin model, $S = \sum_{\langle i,j \rangle} R_\theta(\vec{\sigma}_i \cdot \vec{\sigma}_j)$
$R_\theta(x) = 0$ if $x > \cos\theta$, else $+\infty$
i.e. "flat" action with constraint

- Interesting because:

- no classical continuum limit
- nevertheless, as constraint becomes tighter, fluctuations are reduced:

$$\vec{\sigma}_i \cdot \vec{\sigma}_j \to 1 \;\Rightarrow\; \xi/a \to \infty$$

- empirically, continuum limit ok, and *improved scaling*
- constraint tight enough $\Rightarrow$ no "dislocations"

i.e. topology freezes, topological sectors well defined
$O(2)$ spin system: angle$(\vec{\sigma}_i, \vec{\sigma}_j) < \frac{\pi}{2} \;\Rightarrow$ No vortices!

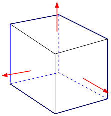
- So far spin models: W. Bietenholz et al, 1009.2146; 1212.0579

Here, $4d$ compact $U(1)$ gauge theory

"Topological" action for $U(1)$ LGT

$$e^{-S} = \begin{cases} 1 & \mathrm{ReTr}\, U_P > \cos\delta_{max} \quad \forall P \\ 0 & \text{else} \end{cases}$$

- Gauge-invariant: no constraint on link angles



- Monopoles (DeGrand-Toussaint): flux 2π through $3d$ cube:
  cannot exist if $\delta_{max} < \pi/3$
  cf. Lüscher's *admissibility condition*:
  $\mathrm{Tr U_P} > 0.97$ for $SU(2)$ $\rightarrow$ topological sectors
  [Schwinger model (constraint but not flat action): hep-lat/0305004]

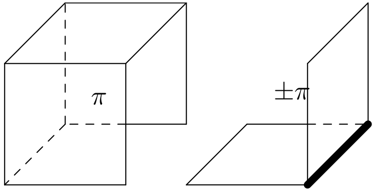
- Recall $U(1)$ phase diagram (Wilson action):
  - Coulomb phase at weak coupling
  - confining phase with monopole condensation at strong coupling

Wilson action: first-order PT at $\beta_c \approx 1$ Here, transition at $\delta_{max} = \frac{\pi}{3}$??

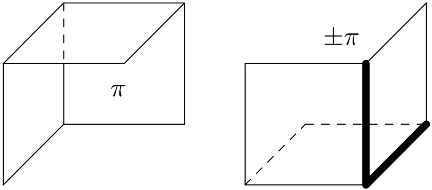
Monte Carlo with topological action

- Careful with Metropolis update:
  allowed link-angle region may not be connected $\rightarrow$ non-ergodicity

- Single-link update changes 2 plaquettes in a cube $\rightarrow$ requires $\delta_{max} \geq \frac{\pi}{2}$ to create monopole



- Double-link update needed to create/destroy monopoles down to $\delta_{max} = \frac{\pi}{3}$



Renormalized coupling via helicity modulus

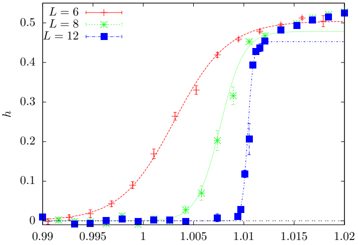
- Introduce external magnetic flux ϕ with $\cos(\theta_P) \to \cos(\theta_P + \phi)$, $P \in$ stack

$$\boxed{h \equiv \left.\frac{\partial^2 f(\phi)}{\partial \phi^2}\right|_{\phi=0}} \qquad h(L=\infty) = \begin{cases} 0 & \text{confined} \\ \neq 0 & \text{Coulomb} \end{cases}$$

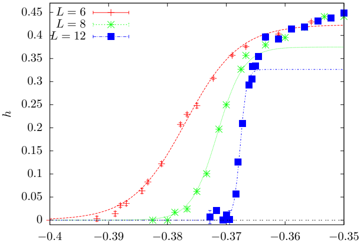
- Wilson action: $h = \beta\left(\langle \cos\theta_P \rangle - \beta\left\langle \left(\sum_{\text{stack}} \sin\theta_P\right)^2 \right\rangle\right)$ hep-lat/0311006
- Topological action: make ϕ *dynamical* $\to$ distribution $p(\phi) = \exp(-f(\phi))$ 1212.0579

Extract h as before, or better, fit $\beta_R \equiv \frac{1}{e_R^2}$ to classical ansatz

$$f(\phi) = -\log \sum_k e^{-\frac{\beta_R}{2}(\phi - 2\pi k)^2}$$

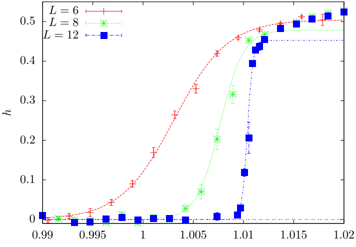


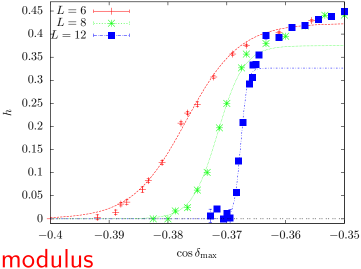
Wilson action



Topological action

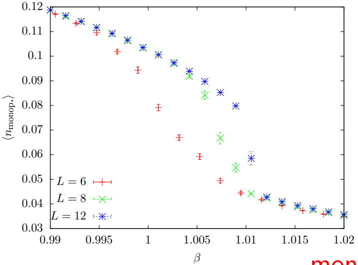
Wilson action vs Topological action

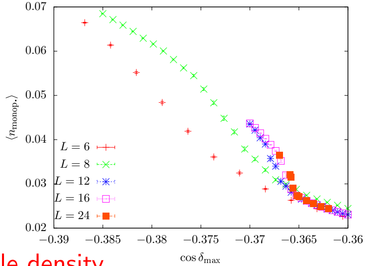




helicity modulus

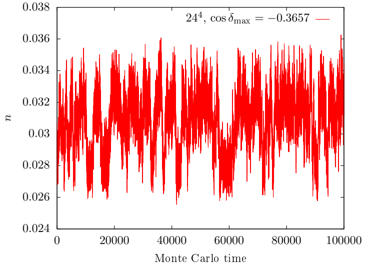
- Similar first-order transition, with almost identical values h_c
- Transition NOT at $\delta_{max} = \frac{\pi}{3}$
- Much weaker jump in monop. dens.

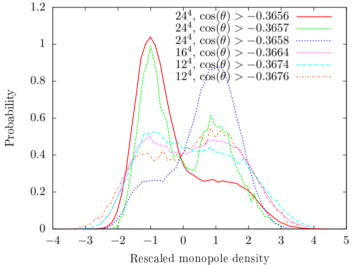




monopole density

Is the top. action transition really first-order?





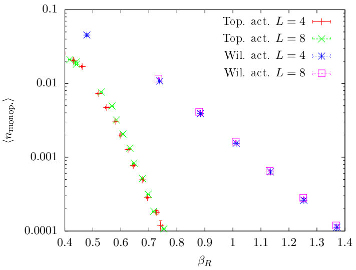
- Large 24^4 lattice: no doubt
  - Monte Carlo metastability of monopole density
  - clear two-peak distribution
  - Creutz ratios consistent with area resp. Coulomb law in resp. phases

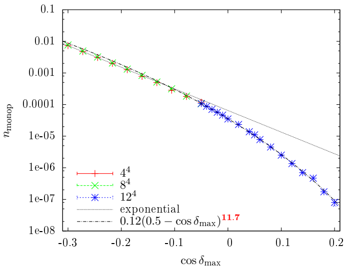
Why does transition occur when $\delta_{max} > \frac{\pi}{3}$?

- Transition triggered by monopole condensation
- Need non-zero monopole density to "avalanche" and condense

New question: what happens as $\delta_{max} \searrow \frac{\pi}{3}$?

Monopole density in Coulomb phase





- Not too far from the first-order transition (*left plot*):
  - Wilson: $n_{\rm monop} \sim \exp(-c\beta_R)$, because *classical* monopole mass $\sim 1/e_R^2$
  - Top. action: also $\sim \exp(-c\beta_R)$!!; heavier monop. ($\rightarrow$ improved action)

- As δ_{max} approaches $\frac{\pi}{3}$ (*right plot*):
  - $n_{\rm monop}$ vanishes with power law behaviour
  - $n_{\rm monop} \sim (\delta_{max} - \frac{\pi}{3})^{\mathbf{11.7}}$?? See later.

Is there another phase transition when $\delta_{max} = \frac{\pi}{3}$?

Phase transition at topological freezing point?

- Monopole density is identically zero for $\delta_{max} < \frac{\pi}{3}$, then non-zero $\implies$ non-analytic

- Consider an extended action, also topological: $S = S_t + \lambda n_{\mathrm{monop}}$,

  where S_t is our top. action, and $\lambda \geq 0$ suppresses monopoles (1212.0579)

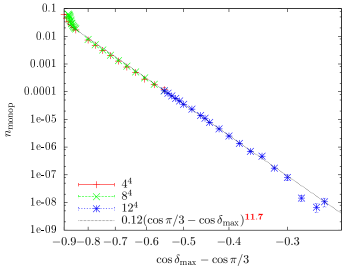
  Then $\langle n_{\mathrm{monop}} \rangle = - \left. \frac{d \log Z}{d\lambda} \right|_{\lambda=0}$ $\rightarrow$ free energy is singular

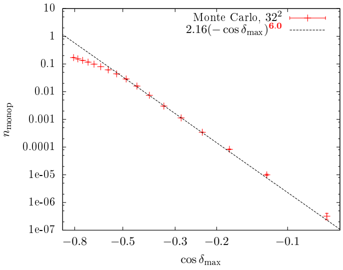
- In practice: vanishingly small monopole density $\rightarrow$ hard to study/detect

- Same argument for vortices in spin system, or topological sectors in Yang-Mills

Critical exponent for defect density

- Similar behaviour for monopoles in $4d$ $U(1)$ and for vortices in $2d$ XY:





- n_{monop} vanishes following a power law at $\delta_{\max} = \frac{\pi}{3}$ $(U(1))$ or $\frac{\pi}{2}$ (XY)
- Crude estimate of critical exponent:
  - XY: vortex-antivortex pair on 2 neighb. plaquettes $\rightarrow$ phase space $(\delta - \frac{\pi}{2})^{\mathbf{2\times 3}}$
  - $U(1)$: monop.-antimonop. in 4 cubes sharing a plaquette $\rightarrow$ $(\delta - \frac{\pi}{3})^{\mathbf{4\times 5}}$

  Both times, neglect constraint on shared link/plaquette $\rightarrow$ overestimate exponent

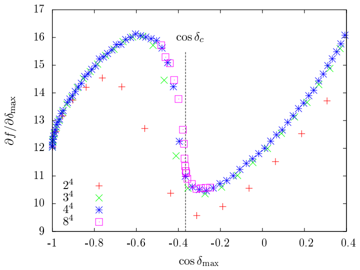
Bonus: free energy for free!

$Z(\delta_{max}) \equiv e^{-\frac{f(\delta_{max})}{V}}$ = number of configurations satisfying the constraint
$\mathrm{Tr}U_P > \cos(\delta_{max})\ \forall P$

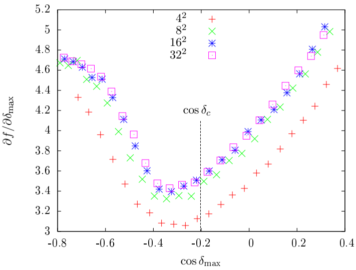
$Z(\delta_2)/Z(\delta_1)$ = proportion of configurations of $Z(\delta_1)$ still belonging to $Z(\delta_2)$

$\Downarrow$

Generate a Monte Carlo sample for $Z(\delta_1)$
and *count* the fraction of survivors versus δ
— possible also with complex action —



$\partial f/\partial\delta_{max}$ for 4d $U(1)$ $\rightarrow$ jump



and for 2d XY $\rightarrow$ no jump

Continuum limit (not rigorous)

- Tighter constraint $\rightarrow$ smeared $\delta(U_P - \mathbf{1}) \Rightarrow \frac{\xi}{a} \rightarrow \infty$, i.e. continuum limit

  Under "favorable conditions", continuum limit is Yang-Mills math-ph/0305058

Idea here: flat distribution unimportant
becomes Gaussian-like for block variables anyway, by central limit theorem

- Example: $1d$ XY unit spin chain, $\text{angle}(s_i, s_{i+1}) \equiv \theta_i \in [-\delta_{max}, +\delta_{max}]$

$$Z = \int_{-\delta_{\max}}^{\delta_{\max}} \prod_{i=1}^{N} \frac{\mathrm{d}\theta_i}{2\delta_{\max}} \delta\left(\exp\left(i\Theta\right) - 1\right), \quad \Theta \equiv \sum_{i=1}^{N} \theta_i = 2\pi m, \ m \text{ winding nb.}$$

θ_i uniform in $[-\delta_{max}, +\delta_{max}]$ (variance $\frac{\delta_{max}^2}{3}$) $\rightarrow$ $\Theta \approx$ Gaussian, variance $N\frac{\delta_{max}^2}{3}$

$$Z \propto \int d\Theta \exp\left(-\frac{\Theta^2}{2N\frac{\delta_{max}^2}{3}}\right) = \sum_m \exp\left(-\frac{3}{2\alpha^2} m^2\right), \quad \alpha^2 = \frac{N\delta_{max}^2}{4\pi^2}$$

Top. susceptibility: $\chi_t = \frac{\langle m^2 \rangle}{Na} \underset{Na \rightarrow \infty}{=} \frac{1}{4\pi^2 I}$, I moment of inertia of quantum rotor

Conclusions

- Topological action: intriguing, valid choice
- Constraint removes large field configurations $\rightarrow$ better scaling
- Free energy for free $\rightarrow$ density of states approach to sign pb?
- Together with topological sectors, new possibilities for θ-term ?
  cf. U.-J. Wiese's 2d $O(3)$ at $\theta = \pi$ & 1204.4913

Thank you for your attention