# A derivative-based approach for the leading order hadronic contribution to $g_\mu - 2$

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Our method is designed to

- produce a precise, smooth curve for the HVP scalar $\Pi(s)$ in the important low-momentum region
- be completely model-independent
- extract maximal information from the lattice

# Introduction

Standard method:

- Measure HVP tensor on lattice:

$$\Pi_{\mu\nu}(\hat{q}) = \sum_x e^{iq(\Delta x + \frac{a\hat{\mu}}{2})} \langle J_\mu^{\mathrm{CVC}}(x_0) J_\nu^{\mathrm{loc}}(x) \rangle,$$

  with $\hat{q}_\mu = \frac{2}{a} \sin\left(\frac{aq_\mu}{2}\right)$ and $q_\mu = \frac{2\pi n_\mu}{L_\mu}$

- Determine HVP scalar

$$\Pi^f(s) \equiv \Pi^f_{\mu\nu}(\hat{q}) / T_{\mu\nu}(\hat{q})$$

  with the Euclidean momentum tensor $T_{\mu\nu}(\hat{q}) \equiv \left(\hat{q}_\mu \hat{q}_\nu - \hat{q}^2 \delta_{\mu\nu}\right)$ and $s = \hat{q}^2$.

- Convert to a smooth function $\Pi^{\mathrm{sm}}(s)$
- Integrate

$$a_\mu^{\mathrm{had,LO}} = \frac{\alpha}{\pi} \int_0^\infty ds\, f(s) \Pi_p^{\mathrm{sm}}(s),$$

  with the kernel function

$$f(s) = \frac{m_\mu^2 s Z(s)^3 \left(1 - sZ(s)\right)}{1 + m_\mu^2 s Z(s)^2} \quad \text{where} \quad Z = -\frac{s - \sqrt{s^2 + 4m_\mu^2 s}}{2m_\mu^2 s}.$$

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## Use Taylor expansion for $\Pi^{\mathrm{sm}}(s)$

- Determine $\Pi(s_i)$ and a sufficient number of its derivatives at some values of the lattice momentum squared, $s_i$
- Near $s_i$

$$\Pi^{\mathrm{sm}}(s) = \sum_n (s - s_i)^n \frac{1}{n!} \frac{d^n \Pi}{ds^n}\Big|_{s_i}.$$

# Derivatives of $\Pi(s)$ for $s > 0$

Determine derivatives of $\Pi_{\mu\nu}$ with moments of the correlators. Specifically,

$$\begin{aligned}
\Pi_{\mu\nu}(\hat{q}) &= \sum_x e^{iq(\Delta x + \frac{a\hat{\mu}}{2})} \langle J_\mu^{\rm CVC}(x_0) J_\nu^{\rm loc}(x)\rangle \\
\frac{\partial \Pi_{\mu\nu}(\hat{q})}{\partial q_{\alpha_1}} &= i \sum_x (\Delta x_{\alpha_1} + \frac{\delta_{\mu\alpha_1}}{2}) e^{iQ(\Delta x + \frac{a\hat{\mu}}{2})} \langle J_\mu^{\rm CVC}(x_0) J_\nu^{\rm loc}(x)\rangle \\
\frac{\partial^2 \Pi_{\mu\nu}(\hat{q})}{\partial q_{\alpha_1} \partial q_{\alpha_2}} &= -\sum_x (\Delta x_{\alpha_1} + \frac{\delta_{\mu\alpha_1}}{2})(\Delta x_{\alpha_2} + \frac{\delta_{\mu\alpha_2}}{2}) e^{iQ(\Delta x + \frac{a\hat{\mu}}{2})} \langle J_\mu^{\rm CVC}(x_0) J_\nu^{\rm loc}(x)\rangle \\
&= \cdots \\
\frac{\partial^n \Pi_{\mu\nu}(\hat{q})}{\partial q_{\alpha_1} \cdots \partial q_{\alpha_n}} &= i^n \sum_x \left[ \prod_\rho^n (\Delta x_{\alpha_\rho} + \frac{\delta_{\mu\alpha_\rho}}{2}) \right] e^{iQ(\Delta x + \frac{a\hat{\mu}}{2})} \langle J_\mu^{\rm CVC}(x_0) J_\nu^{\rm loc}(x)\rangle .
\end{aligned}$$

But we need derivs WRT $s = \hat{q}^2$

# Derivatives of $\Pi(s)$ for $s > 0$

Generally, linear expressions relate derivatives of $\Pi(s)$ and $\Pi_{\mu\nu}(q)$:

$$\frac{\partial^n \Pi_{\mu\nu}}{\partial q_{\alpha_1} \cdots \partial q_{\alpha_n}}(q) = \sum_{m=0}^{n} A^{\{\alpha\}}_{\mu\nu}{}^{n}_{m}(q) \frac{d^m \Pi(s)}{ds^m}$$

with recursion expressions relating the $A^n_m$ to $A_{\mu\nu}{}^0_0(q) = T_{\mu\nu}(q)$.

$$\begin{aligned} A^{\{\alpha\}}_{\mu\nu}{}^{n}_{0}(q) &= \partial_{\alpha_n} \cdots \partial_{\alpha_1} A_{\mu\nu}{}^0_0(q) \\ &= \partial_{\alpha_n} \cdots \partial_{\alpha_1} T_{\mu\nu}(q) \end{aligned}$$

$$A^{\{\alpha\}}_{\mu\nu}{}^{n}_{0}(q) = \begin{cases} T_{\mu\nu}(q) & = q_\mu q_n u - q^2 \delta_{\mu\nu} & \text{for } n = 0 \\ \frac{\partial T_{\mu\nu}}{\partial q_{\alpha_1}} & = \delta_{\mu\alpha_1} q_\nu + \delta_{\nu\alpha_1} q_\mu - 2\delta_{\mu\nu} q_{\alpha_1} & \text{for } n = 1 \\ \frac{\partial^2 T_{\mu\nu}}{\partial q_{\alpha_1} \partial q_{\alpha_2}} & = \delta_{\mu\alpha_1}\delta_{\nu\alpha_2} + \delta_{\mu\alpha_2}\delta_{\nu\alpha_1} + 2\delta_{\mu\nu}\delta_{\alpha_1\alpha_2} & \text{for } n = 2 \\ \frac{\partial^n T_{\mu\nu}}{\partial q_{\alpha_1} \cdots \partial q_{\alpha_n}} & = 0 & \text{for } n < 2 \end{cases}$$

We also see that

$$A^{\{\alpha\}}_{\mu\nu}{}^{n}_{n}(q) = \begin{cases} 2q_{\alpha_n} A^{\{\alpha\}}_{\mu\nu}{}^{n-1}_{n-1} & \text{for } n < 3 \\ 0 & \text{for } n \geq 3, \end{cases}$$

and generally

$$A^{\{\alpha\}}_{\mu\nu}{}^{n}_{m} = 2q_{\alpha_n} A^{\{\alpha\}}_{\mu\nu}{}^{n-1}_{m-1} + \partial_{q_{\alpha_n}} A^{\{\alpha\}}_{\mu\nu}{}^{n-1}_{m}.$$

For $n > 3$ we use a recursive script to calculate needed values of $A^{\{\alpha\}}_{\mu\nu}{}^{n}_{m}$.

# How do we use it?

For $s > 0$:

$$\frac{\partial^n \Pi_{\mu\nu}}{\partial q_{\alpha_1} \cdots \partial q_{\alpha_n}}(q) = \sum_j^n A_{\mu\nu}^{\{\alpha\}}{}^n{}_m(q) \frac{d^m \Pi(s)}{ds^m}$$

- From lattice calculation, with transformations

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- From lattice calculation, with transformations
- Calculate coefficients with script

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- From lattice calculation, with transformations
- Calculate coefficients with script
- Solve system for $\frac{d^m \Pi(s)}{ds^m}$
- Use in Taylor expansion to extrapolate to non-lattice values of $s$.

$$\Pi^{\mathrm{sm}}(s) = \sum_n (s - s_i)^n \frac{1}{n!} \frac{d^n \Pi}{ds^n}\Big|_{s_i}$$

# Special (important!) case: $s = 0$?

$$\frac{\partial^n \Pi_{\mu\nu}}{\partial q_{\alpha_1} \cdots \partial q_{\alpha_n}}(q) = \sum_{m=0}^{n} A^{\{\alpha\}}_{\mu\nu}{}^{n}_{m}(q) \frac{d^m \Pi(s)}{ds^m}$$

For $s = 0$ complications occur because of the $q$ factors in the coefficients.

$$A_{\mu\nu}{}^{0}_{0}(q) = T_{\mu\nu}(q).$$

$$A^{\{\alpha\}}_{\mu\nu}{}^{n}_{m} = 2q_{\alpha_n} A^{\{\alpha\}}_{\mu\nu}{}^{n-1}_{m-1} + \partial_{\alpha_n} A^{\{\alpha\}}_{\mu\nu}{}^{n-1}_{m}$$

Coefficients $A^{\{\alpha\}}_{\mu\nu}{}^{n}_{m}(q)$ have $(2-n)+2m$ powers of momentum.
So for any value of $m$, $n = 2+2m$ gives a constant coefficient. Then we can solve:

$$\left.\frac{d^m \Pi}{ds^m}\right|_{s=0} = \frac{1}{A^{\{\alpha\}}_{\mu\nu}{}^{(2+2m)}_{m}} \left.\frac{\partial^{(2+2m)} \Pi_{\mu\nu}}{\partial \hat{q}_{\alpha_1} \cdots \partial \hat{q}_{\alpha_{2+2m}}}\right|_{\hat{q}=0}$$

$$\Pi(0) = \frac{1}{A^{\{\alpha\}}_{\mu\nu}{}^{2}_{0}} \frac{\partial^2 \Pi_{\mu\nu}(0)}{\partial_{\hat{q}_{\alpha_1}} \partial_{\hat{q}_{\alpha_2}}}$$

$$\frac{d\Pi}{ds}(0) = \frac{1}{A^{\{\alpha\}}_{\mu\nu}{}^{4}_{1}} \frac{\partial^4 \Pi_{\mu\nu}(0)}{\partial_{\hat{q}_{\alpha_1}} \partial_{\hat{q}_{\alpha_2}} \partial_{\hat{q}_{\alpha_3}} \partial_{\hat{q}_{\alpha_4}}}$$

$$\frac{d^2\Pi}{ds^2}(0) = \frac{1}{A^{\{\alpha\}}_{\mu\nu}{}^{6}_{2}} \frac{\partial^6 \Pi_{\mu\nu}(0)}{\partial_{\hat{q}_{\alpha_1}} \partial_{\hat{q}_{\alpha_2}} \partial_{\hat{q}_{\alpha_3}} \partial_{\hat{q}_{\alpha_4}} \partial_{\hat{q}_{\alpha_5}} \partial_{\hat{q}_{\alpha_6}}}$$

$$\cdots$$

## Example $A^2_0 \longrightarrow \Pi(0)$:

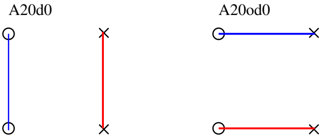
For $(n = 2, m = 0)$ we have

$$A^{\{\alpha\}}_{\mu\nu}{}^{2}_{0} = (\delta_{\alpha_1\mu}\delta_{\alpha_2\nu} - 2\delta_{\mu\nu}\delta_{\alpha_1\alpha_2\nu}) \quad (1)$$

We see that combinations of the indicies $\mu, \nu, \alpha_1, \alpha_2$ can produce two different non-zero values of $A^{\{\alpha\}}_{\mu\nu}{}^{2}_{0}$:

$$A^{\{\alpha\}}_{\mu\nu}{}^{2}_{0} = \begin{cases} -2 & \text{for } \mu = \nu,\ \alpha_1 = \alpha_2,\ \alpha_1 \neq \mu \\ 1 & \text{for } \mu = \alpha_1,\ \nu = \alpha_2,\ \mu \neq \nu \end{cases} \quad (2)$$

| $A^2_0$ | label | $N_{\mathrm{comb}}$ | $N_{cl}$ | $N_{cc}$ |
|---|---|---|---|---|
| -2 | A20d0 | 12 | 12 | 12 |
| 1 | A20od0 | 24 | 12 | 6 |
| | total | 36 | 24 | 18 |

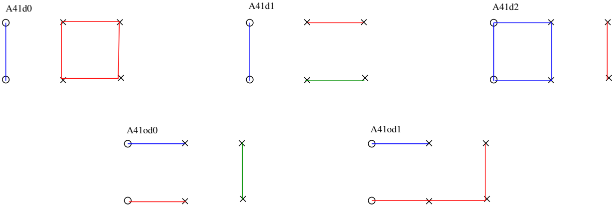


# Find cases where the constant $A_m^n \neq 0$

$A_1^4$: 6 permutations of

$$2\delta_{\alpha_3\alpha_4}\left(\delta_{\alpha_1\mu}\delta_{\alpha_2\nu} + \delta_{\alpha_2\mu}\delta_{\alpha_1\nu} - 2\delta_{\mu\nu}\delta_{\alpha_1\alpha_2}\right)$$

| $A_1^4$ | label | $N_{\mathrm{comb}}$ | $N_{cl}$ | $N_{cc}$ |
|---|---|---|---|---|
| -24 | A41d0 | 12 | 12 | 12 |
| -8 | A41d1 | 72 | 12 | 12 |
| -4 | A41d2 | 72 | 12 | 12 |
| +2 | A41od0 | 288 | 24 | 12 |
| +6 | A41od1 | 96 | 24 | 12 |
| | total | 540 | 84 | 60 |

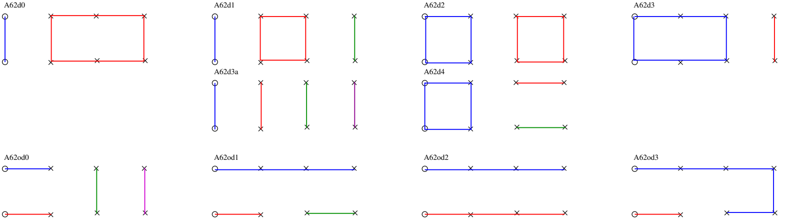


# Find cases where the constant $A_m^n \neq 0$

$A_2^6$: 45 terms, permutations of

$$4\delta_{\alpha_1\alpha_2}\delta_{\alpha_3\alpha_4}\left(\delta_{\alpha_5\mu}\delta_{\alpha_6\nu}+\delta_{\alpha_6\mu}\delta_{\alpha_5\nu}-2\delta_{\mu\nu}\delta_{\alpha_5\alpha_6}\right).$$

| $A_2^6$ | label | $N_{\mathrm{comb}}$ | $N_{cl}$ | $N_{cc}$ |
|---|---|---|---|---|
| -360 | A62d0 | 12 | 12 | 12 |
| -72 | A62d1 | 360 | 24 | 24 |
| -48 | A62d2 | 180 | 12 | 12 |
| -24 | A62d3 | 180 | 12 | 12 |
| -24 | A62d3a | 360 | 4 | 4 |
| -16 | A62d4 | 1080 | 12 | 12 |
| +4 | A62od0 | 2160 | 12 | 6 |
| +12 | A62od1 | 3600 | 48 | 24 |
| +36 | A62od2 | 240 | 12 | 6 |
| +60 | A62od3 | 144 | 24 | 12 |
| | total | 8316 | 172 | 124 |



# Find cases where the constant $A^n_m \neq 0$

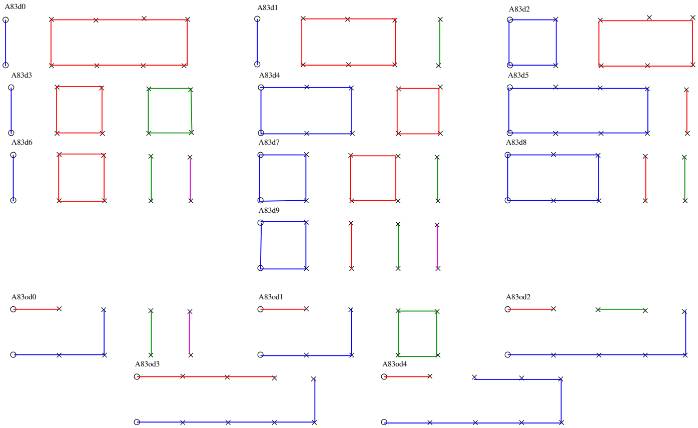
$A^8_3$: 420 terms like:

$$8\delta_{\alpha_1\alpha_2}\delta_{\alpha_3\alpha_4}\delta_{\alpha_5\alpha_6}\left(\delta_{\alpha_7\mu}\delta_{\alpha_8\nu}+\delta_{\alpha_8\mu}\delta_{\alpha_7\nu}-2\delta_{\mu\nu}\delta_{\alpha_7\alpha_8}\right)$$

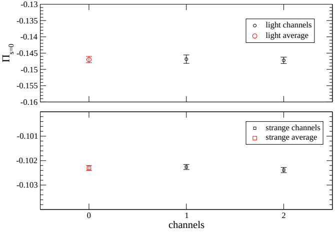
| $A^6_2$ | label | $N_{\mathrm{comb}}$ | $N_{cl}$ | $N_{cc}$ |
|---|---|---|---|---|
| -6720 | A83d0 | 12 | 12 | 12 |
| -960 | A83d1 | 672 | 24 | 24 |
| -720 | A83d2 | 336 | 12 | 12 |
| -576 | A83d3 | 840 | 12 | 12 |
| -288 | A83d4 | 840 | 12 | 12 |
| -240 | A83d5 | 336 | 12 | 12 |
| -192 | A83d6 | 5040 | 12 | 12 |
| -144 | A83d7 | 10080 | 12 | 12 |
| -96 | A83d8 | 5040 | 12 | 12 |
| -48 | A83d9 | 10080 | 4 | 4 |
| +24 | A83od0 | 60480 | 24 | 12 |
| +72 | A83od1 | 26880 | 48 | 24 |
| +120 | A83od2 | 9408 | 48 | 24 |
| +360 | A83od3 | 1344 | 24 | 12 |
| +840 | A83od4 | 192 | 24 | 12 |
| | total | 131580 | 292 | 208 |

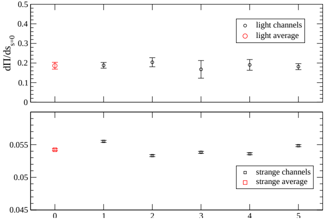
## Find cases where the constant $A^n_m \neq 0$

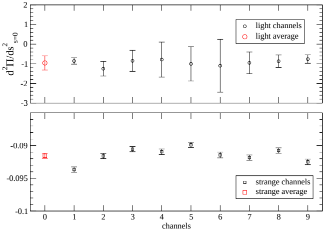
$A^8_3$:

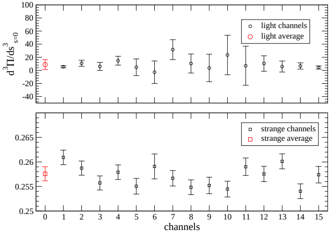


# Are different channels consistent?

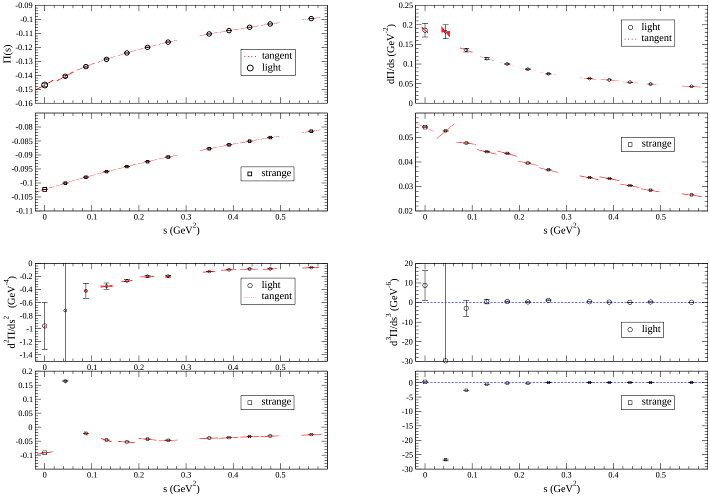


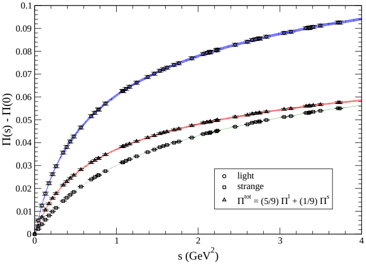



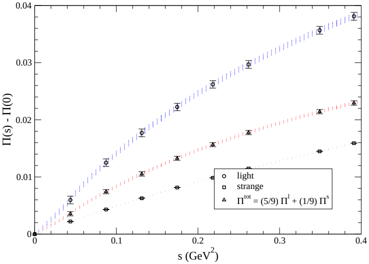



# Sanity check: tangents



# Sample smooth curves for $\Pi(s)$



# Use Taylor expansion for $\Pi^{\rm sm}(s)$

For $s$ between two lattice momenta values $s_i < s < s_{i+1}$, make "lower" and "upper" estimates,

$$\Pi^{\rm low}(s) = \sum_n (s - s_i)^n \, \frac{1}{n!} \frac{d^n \Pi}{ds^n}\Big|_{s_i}$$

$$\Pi^{\rm up}(s) = \sum_n (s - s_{i+1})^n \, \frac{1}{n!} \frac{d^n \Pi}{ds^n}\Big|_{s_{i+1}}$$

Combine with a weighted average:

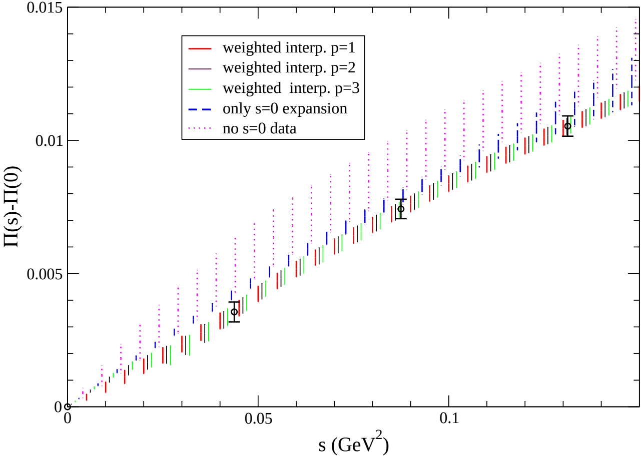
$$\Pi_p^{\rm wt}(s) = \frac{\Pi^{\rm low}(s) w^{\rm low}(s) + \Pi^{\rm up}(s) w^{\rm up}(s)}{w^{\rm low}(s) + w^{\rm up}(s)}$$

with

$$w^{\rm low}(s) = \frac{1}{\left|(s - s_i)\, \sigma\left(\frac{d\Pi}{ds}\Big|_{s_i}\right)\right|^p} \quad \text{and} \quad w^{\rm up}(s) = \frac{1}{\left|(s - s_{i+1})\, \sigma\left(\frac{d\Pi}{ds}\Big|_{s_{i+1}}\right)\right|^p}.$$

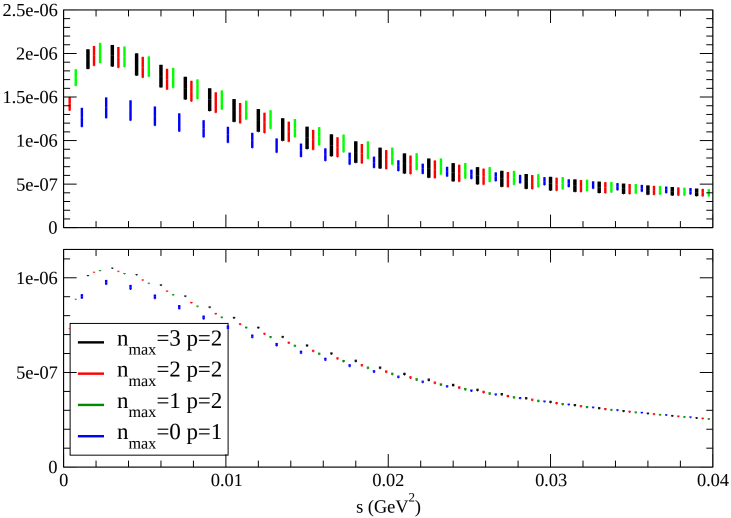
$\sigma\left(\frac{d\Pi}{ds}\right)$ is a proxy for the uncertainty in $\Pi^{\rm low/up}$.

Different values of interpolation parameter $p$



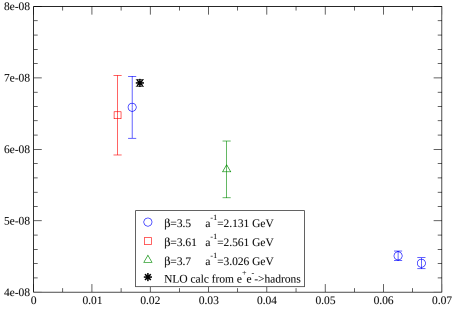
## Integrand: $n_{\max}$ systematics

Dependence on maximum Taylor expansion order $n_{\max}$:



# Preliminary results

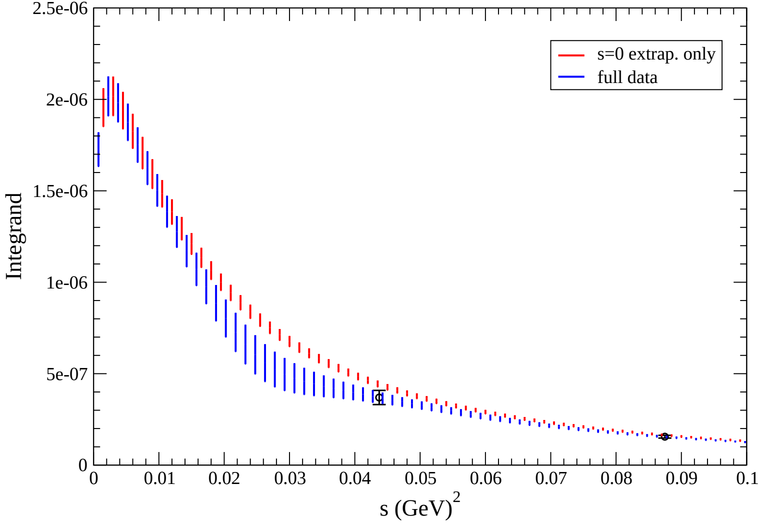
| 2-HEX ($N_f = 2+1$) | | | | |
|---|---|---|---|---|
| $am_{ud}^{\text{bare}}$ | $am_s^{\text{bare}}$ | volume | # cfgs | $M_\pi$ (GeV) |
| | | $\beta = 3.5,\ a^{-1} = 2.131$ GeV | | |
| -0.05294 | -0.0060 | $64^3 \times 64$ | 1060 | 0.130(2) |
| -0.04900 | -0.0120 | $32^3 \times 64$ | 216 | 0.250(2) |
| -0.04900 | -0.0060 | $32^3 \times 64$ | 110 | 0.258(2) |
| | | $\beta = 3.61,\ a^{-1} = 2.561$ GeV | | |
| -0.03650 | -0.003 | $64^3 \times 72$ | 560 | 0.120(1) |
| | | $\beta = 3.7,\ a^{-1} = 3.026$ GeV | | |
| -0.02700 | -0.00 | $64^3 \times 64$ | 208 | 0.182(2) |



- Using all available index combinations increases the precision of estimate of $\Pi(s)$ near $s = 0$
- No added inversion cost
- No models needed!
- Can be used in combination with other techniques
- $n_{\max} = 3$ seems to be a sufficient number of derivatives of $\Pi(s)$
- Need a more thorough examination of systematic uncertainties

# Spare slides

Is it the *only* important point?



# Preliminary results

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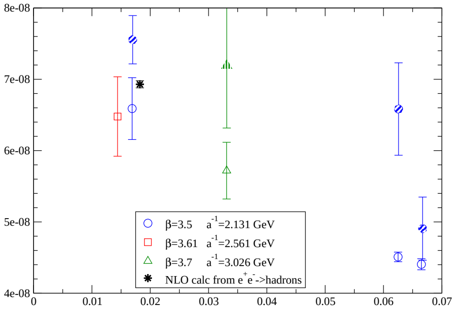
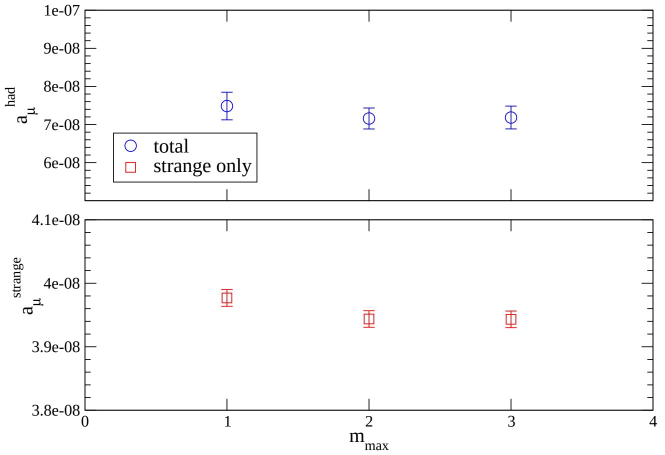


Dependence on maximum Taylor expansion order $n_{\max}$



# Aside: transform $q \to \hat{q}$

Remember, $q \neq \hat{q}$!
Invert $\hat{q}_\mu = \frac{2}{a} \sin\left(\frac{aq_\mu}{2}\right)$ and take derivatives:

$$\begin{aligned}
q_\sigma &= \frac{2}{a} \arcsin\left(\frac{\hat{q}_\sigma a}{2}\right) \\
\frac{dq_\sigma}{d\hat{q}_\rho} &= \frac{\delta_{\sigma\rho}}{\sqrt{1 - \left(\frac{\hat{q}_\sigma a}{2}\right)^2}} \\
\frac{d^2 q_\sigma}{d\hat{q}_\rho d\hat{q}_\tau} &= \frac{\delta_{\sigma\rho}\delta_{\sigma\tau} a^2 \hat{q}_\sigma}{\left(1 - \left(\frac{\hat{q}_\sigma a}{2}\right)^2\right)^{\frac{3}{2}}} \\
\frac{d^3 q_\sigma}{d\hat{q}_\rho d\hat{q}_\tau d\hat{q}_\lambda} &= \delta_{\sigma\rho}\delta_{\sigma\tau}\delta_{\sigma\lambda} \left(\frac{a}{2}\right)^2 \left(\frac{3\hat{q}_\sigma^2 a^2}{4\left(1 - \left(\frac{\hat{q}_\sigma a}{2}\right)^2\right)^{\frac{5}{2}}} + \frac{1}{\left(1 - \left(\frac{\hat{q}_\sigma a}{2}\right)^2\right)^{\frac{3}{2}}}\right) \\
&\cdots
\end{aligned} \tag{3}$$

# Aside: transform $q \to \hat{q}$

Then we can convert using the chain rule:

$$\frac{d\Pi_{\mu\nu}}{d\hat{q}_\alpha} = \frac{d\Pi_{\mu\nu}}{dq_\alpha}\frac{dq_\alpha}{d\hat{q}_\alpha} \quad \text{no sum over } \alpha \tag{4}$$

$$\frac{d^2\Pi_{\mu\nu}}{d\hat{q}_\alpha d\hat{q}_\beta} = \frac{d^2\Pi_{\mu\nu}}{dq_\alpha dq_\beta}\frac{dq_\alpha}{d\hat{q}_\alpha}\frac{dq_\beta}{d\hat{q}_\beta} + \frac{d\Pi_{\mu\nu}}{dq_\alpha}\frac{d^2 q_\alpha}{d\hat{q}_\alpha d\hat{q}_\beta}\delta_{\alpha\beta} \tag{5}$$

$$\begin{aligned}\frac{d^3\Pi_{\mu\nu}}{d\hat{q}_\alpha d\hat{q}_\beta d\hat{q}_\gamma} &= \frac{d^3\Pi_{\mu\nu}}{dq_\alpha dq_\beta dq_\gamma}\frac{dq_\alpha}{d\hat{q}_\alpha}\frac{dq_\beta}{d\hat{q}_\beta}\frac{dq_\gamma}{d\hat{q}_\gamma} \\ &+ \frac{d^2\Pi_{\mu\nu}}{dq_\alpha dq_\beta}\left(\frac{d^2 q_\alpha}{d\hat{q}_\alpha^2}\frac{dq_\beta}{d\hat{q}_\beta}\delta_{\alpha\gamma} + \frac{dq_\alpha}{d\hat{q}_\alpha}\frac{d^2 q_\beta}{d\hat{q}_\beta^2}\delta_{\beta\gamma} + \frac{dq_\alpha}{d\hat{q}_\alpha}\frac{d^2 q_\gamma}{d\hat{q}_\gamma^2}\delta_{\alpha\beta}\right) \\ &+ \frac{d\Pi_{\mu\nu}}{dq_\alpha}\frac{d^3 q_\alpha}{d\hat{q}_\alpha^3}\delta_{\alpha\beta}\delta_{\alpha\gamma}\end{aligned} \tag{6}$$

$$\ldots \tag{7}$$

# Derivatives of $\Pi(s)$ for $s > 0$

Linear expressions relate derivatives of $\Pi(s)$ and $\Pi_{\mu\nu}(q)$, e.g.,

$$\begin{aligned}
\Pi_{\mu\nu}(q) &= T_{\mu\nu}(q)\Pi(s) \\
\frac{\partial \Pi_{\mu\nu}}{\partial q_{\alpha_1}}(q) &= \partial_{q_{\alpha_1}} T_{\mu\nu}(q)\Pi(s) + T_{\mu\nu}\partial_{q_{\alpha_1}}\Pi(s) \\
&= \partial_{q_{\alpha_1}} T_{\mu\nu}(q)\Pi(s) + 2q_{\alpha_1} T_{\mu\nu}\frac{d\Pi(s)}{ds} \\
\frac{\partial^2 \Pi_{\mu\nu}}{\partial q_{\alpha_1}\partial q_{\alpha_2}}(q) &= \partial_{q_{\alpha_2}}\partial_{q_{\alpha_1}} T_{\mu\nu}(q)\Pi(s) + 2\left(\delta_{\alpha_1,\alpha_2} T_{\mu\nu} + q_{\alpha_1}(\partial_{q_{\alpha_2}} T_{\mu\nu})\right)\frac{d\Pi(s)}{ds} + 4q_{\alpha_1}q_{\alpha_1} T_{\mu\nu}\frac{d^2\Pi(s)}{ds^2} \\
&\cdots
\end{aligned}$$

($\hat{q} \to q$ for simplicity.)