Understanding the problem with logarithmic singularities in the complex Langevin method

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based on the work in collaboration with

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(arXiv:1504.08359 [hep-lat])

Complex Langevin Method (CLM)

[Parisi 83][Klauder 84]

- A promising method for evading the sign problem in the path integral with a complex weight
- e.g.) finite density QCD, real time dynamics,⋯
- sometimes works, sometimes fails
- recent progress : better understanding of the cause of the failure

Recent understanding

CLM fails when

- the probability distribution has large excursions into the imaginary direction [Aarts, Seiler, Stamatescu 09] [Aarts, James, Seiler, Stamatescu 09 11]

- the action involves "log" : $S = \log \Delta(\phi) + \cdots$

  [Mollgaard, Splittorff 13][Greensite 14]

  - CLM gives wrong results when the phase of $\Delta(\phi)$ rotates frequently during the Langevin process

  - why? ambiguity of log branch ? ⋯ not clear

Our understanding

[Nishimura-SS 15]

- log branch is not the cause of the problem
- Rather, the singularity in the drift term causes the problem
- Indeed, the problem occurs even when the action involves non-logarithmic singularity
- (ex) $$S = \frac{\beta}{(x+i\alpha)^2} + \frac{x^2}{2}$$

→ gauge cooling is useful here as well

(Nagata's talk)

one variable case

[Nishimura-SS 15]

- $Z = \int_{-\infty}^{\infty} dx (x + i\alpha)^p e^{-x^2/2}$ ($\alpha \in \mathbf{R}$, $p \in \mathbf{R}$)
- action $S = -\log(x + i\alpha)^p + x^2/2$ "log" or "Log"
- drift term $-\frac{dS}{dx} = \frac{p}{x + i\alpha} - x$
- Langevin eq. $x(t) \to z(t) = x(t) + iy(t)$

$$\frac{dz(t)}{dt} = \frac{p}{z(t) + i\alpha} - z(t) + \eta(t)$$

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Defining the action is not necessary in formulating CLM

one variable case

[Nishimura-SS 15]

- $Z = \int_{-\infty}^{\infty} dx\, w(x)$
- drift term $w(x)^{-1} \frac{dw(x)}{dx}$
- Langevin eq. $x(t) \to z(t) = x(t) + iy(t)$

$$\frac{dz(t)}{dt} = w(z)^{-1} \frac{dw(z)}{dz} + \eta(t)$$

- The problem is whether the ensemble average can reproduce the path integral with the complex weight $w(x)$

one variable case

[Nishimura-SS 15]

- $$Z = \int_{-\infty}^{\infty} dx (x + i\alpha)^p e^{-x^2/2}$$ ($\alpha \in \mathbf{R}$, $p \in \mathbf{R}$)

- Langevin eq.

$$\frac{dx(t)}{dt} = \mathrm{Re}\frac{p}{z(t) + i\alpha} - x(t) + \eta(t)$$

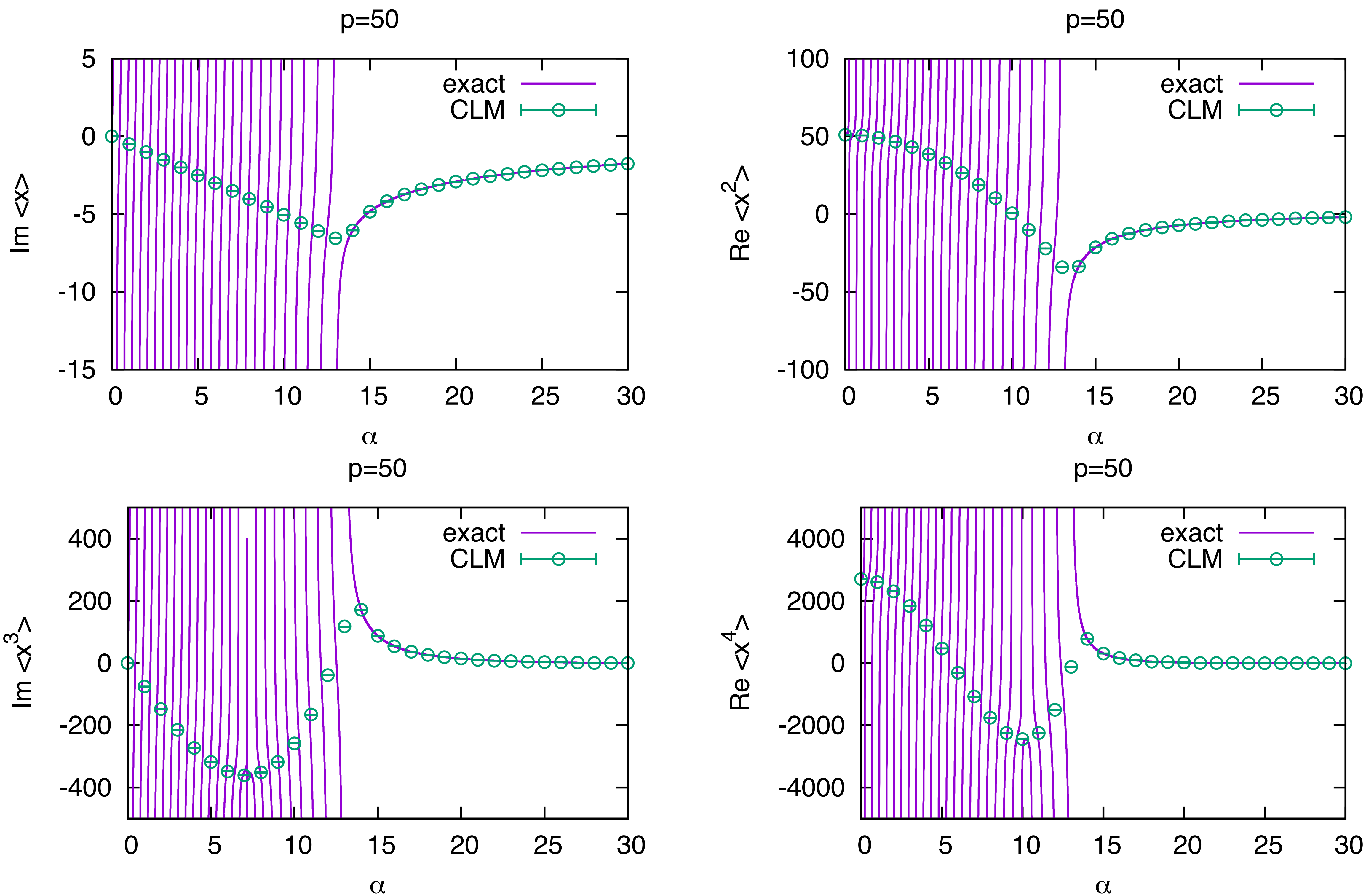
$$\frac{dy(t)}{dt} = \mathrm{Im}\frac{p}{z(t) + i\alpha} - y(t)$$

- compute the average of z^n in CLM

$$Z = \int_{-\infty}^{\infty} dx (x + i\alpha)^p e^{-x^2/2}$$

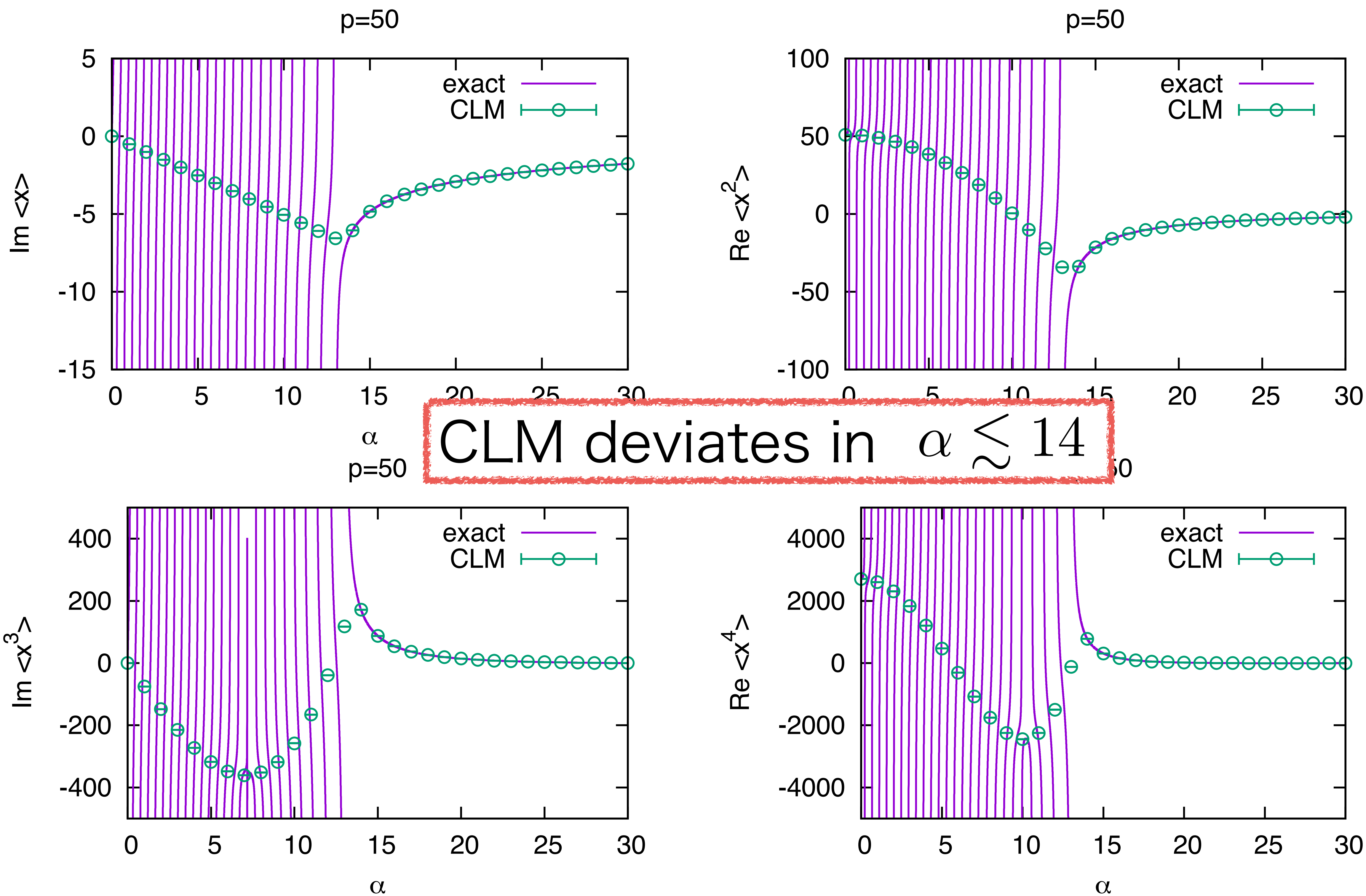
p=50

Result (p=50)



p=50
Im <x>
α
exact
CLM
p=50
Re <x^2>
α
exact
CLM
p=50
Im <x^3>
α
exact
CLM
p=50
Re <x^4>
α
exact
CLM

Result (p=50)



CLM deviates in $\alpha \lesssim 14$

Why does CLM fail?

- For CLM to work :

probability distribution associated with the Langevin process

complex weight

$$\int dx dy \mathcal{O}(x+iy) P(x,y) = \int dx \mathcal{O}(x) e^{-S}$$

- We can show this equality if partial integration on the complex plane is allowed

[Aarts, Seiler, Stamatescu 09][Aarts, James, Seiler, Stamatescu 09 11]

Why does CLM fail?

- For partial integration to be justified :
  - The distribution should have a fast fall-off

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  - The drift term should not have singularities

    [Nishimura-SS 15]

$$\int_{-\infty}^{\infty} dx \frac{1}{x} \frac{d}{dx} e^{-x^2} \neq \int_{-\infty}^{\infty} dx \frac{1}{x^2} e^{-x^2}$$

Why does CLM fail?

- For partial integration to be justified :
  - The distribution should have fast fall-off

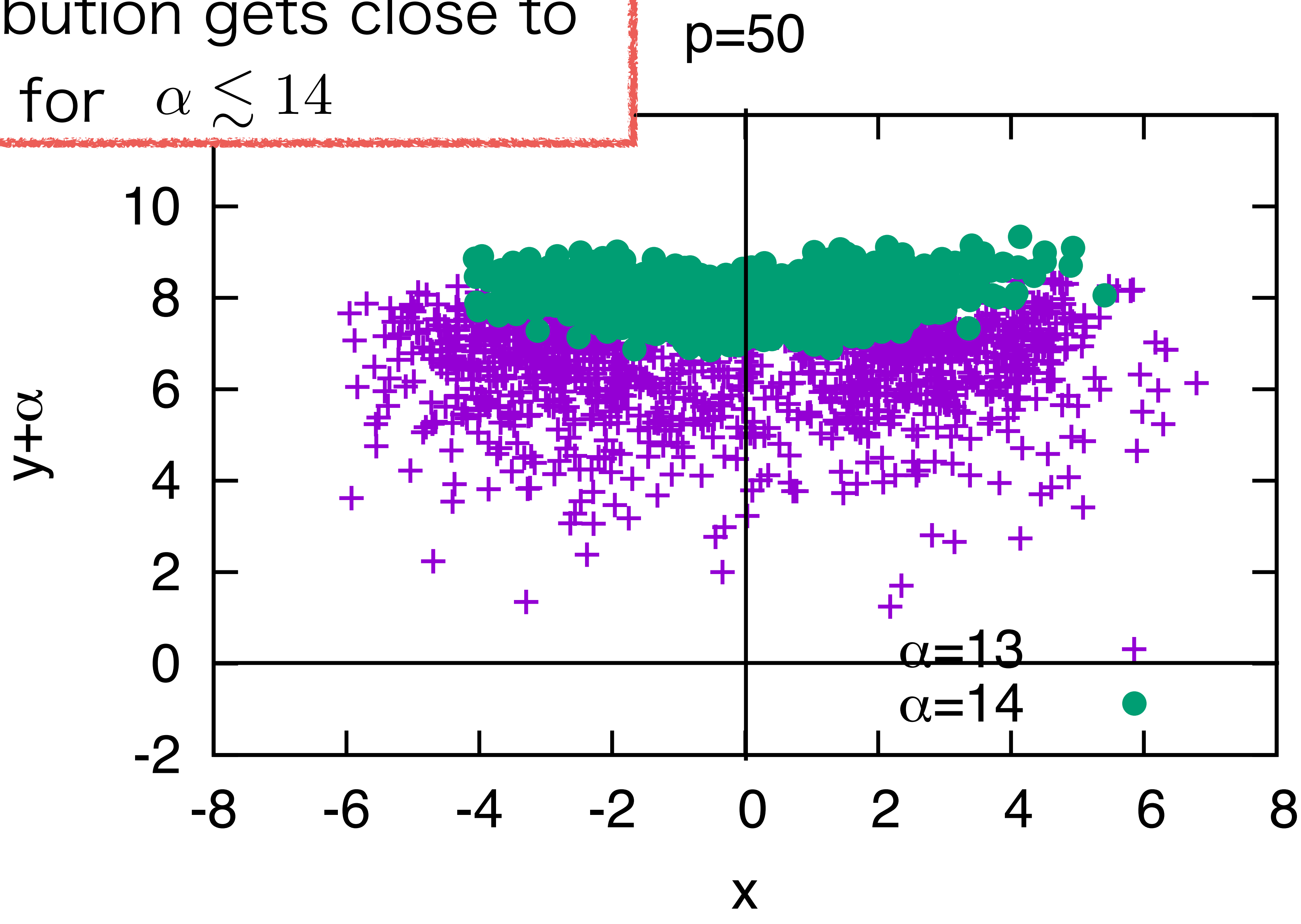
    OK, since the drift term = $-z + \cdots$

  - The drift term should not have singularities

    The drift term is singular at $z = -i\alpha \ (x = 0, \ y = -i\alpha)$
    $\cdots$ OK, if the distribution is zero near the singularity.

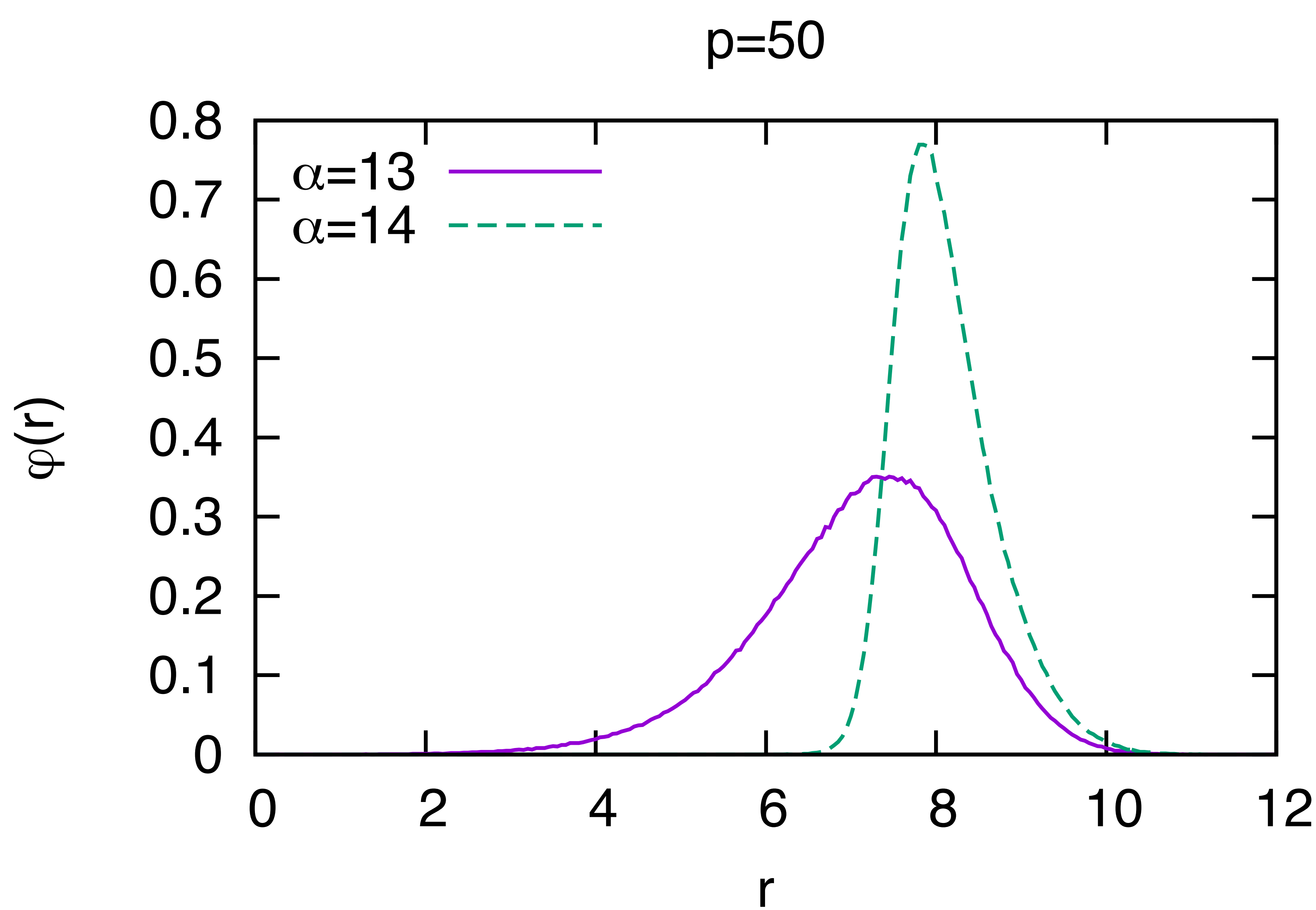
Result (p=50)

The distribution gets close to
$z+i\alpha=0$ for $\alpha \lesssim 14$



Radial distribution

$$\varphi(r) = \frac{1}{2\pi r}\int dxdyP(x,y;\infty)\delta(\sqrt{x^2+(y+\alpha)^2}-r)$$



Non-logarithmic case

$$Z = \int_{-\infty}^{\infty} dx e^{-\frac{1}{(x+i\alpha)^2} - x^2/2}$$

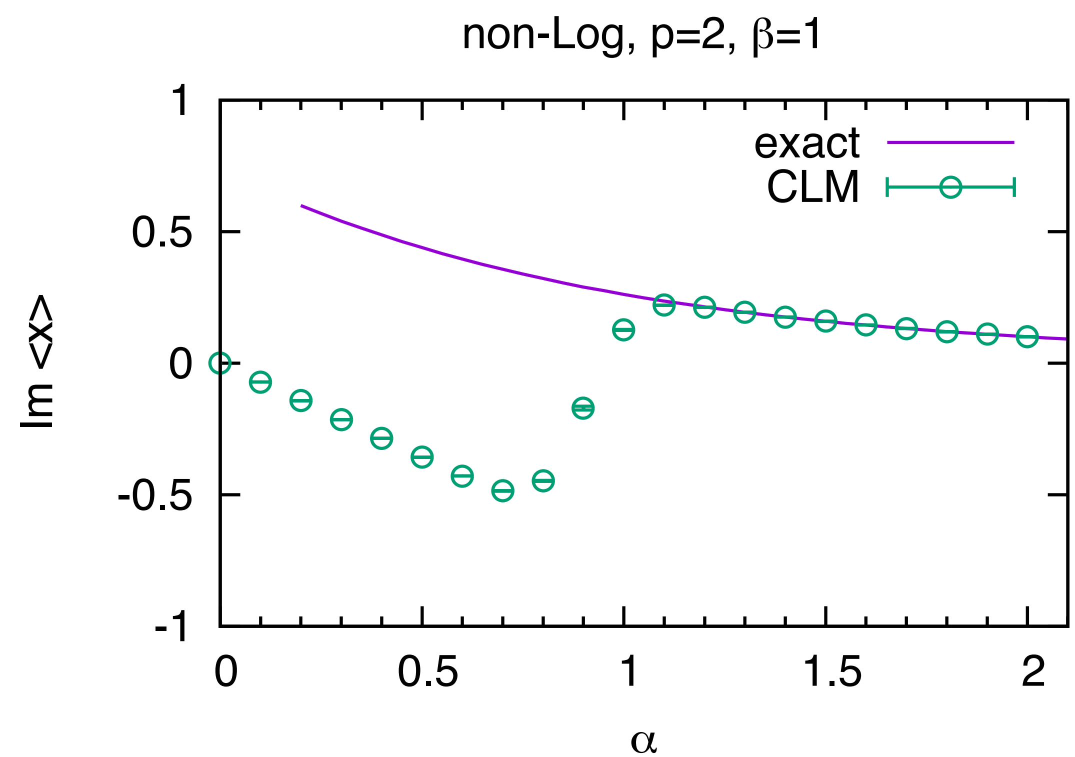
- action $S(z) = \frac{1}{(z+i\alpha)^2} + z^2/2$
- drift = $\frac{2}{(z+i\alpha)^3} - z$

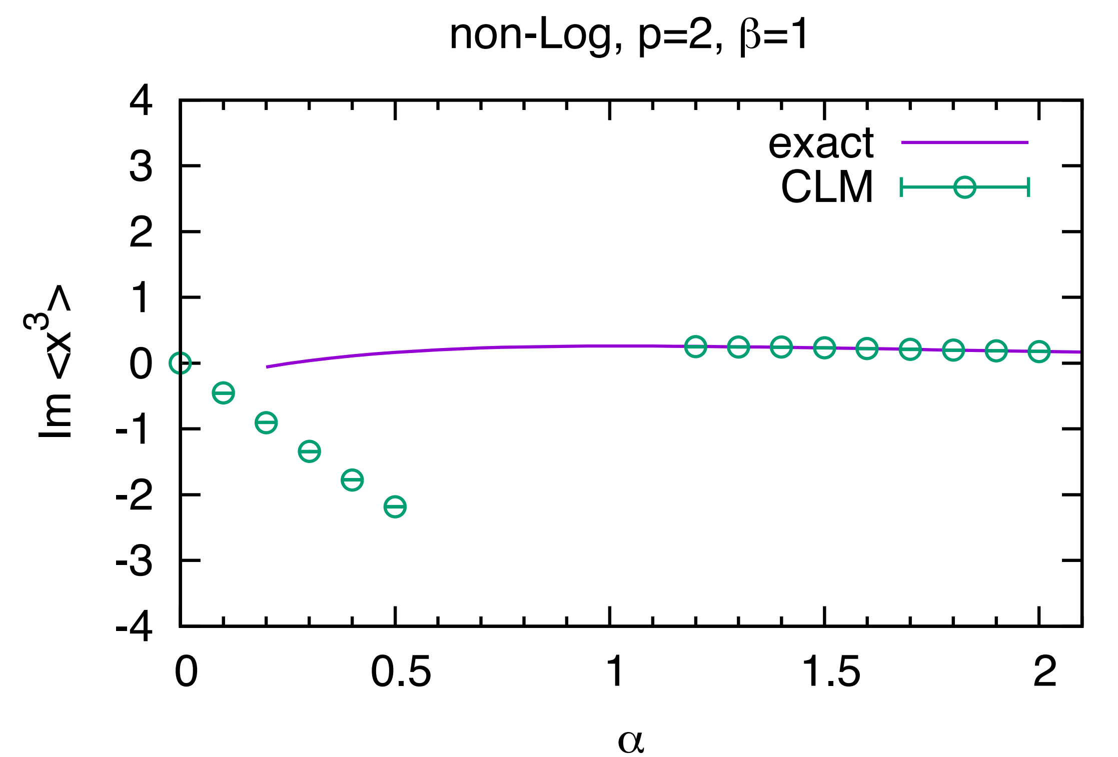
no log

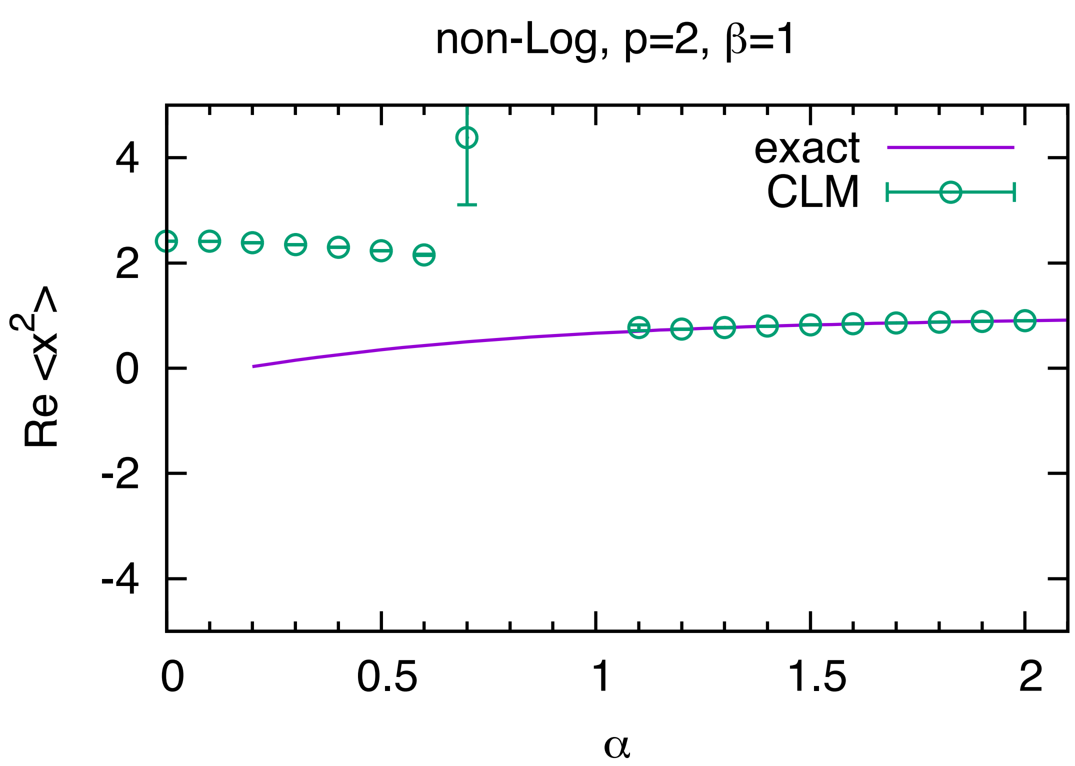
finite for real x

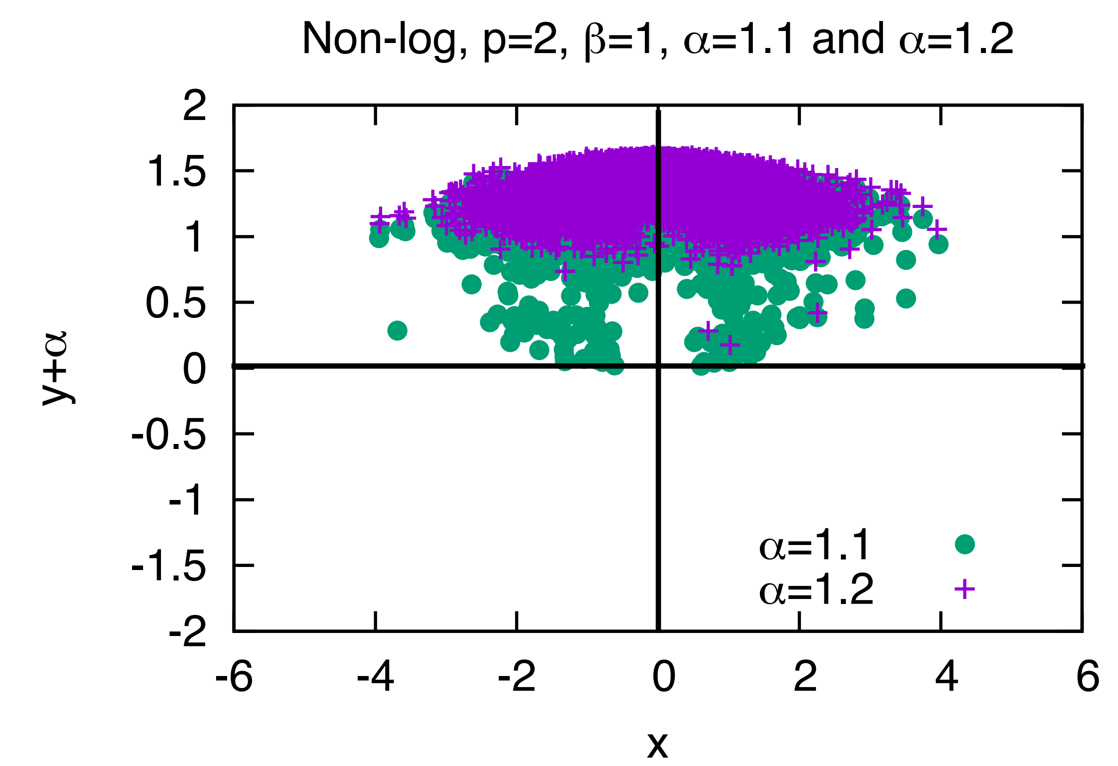
singular at $z = -i\alpha$

Result (non-log)

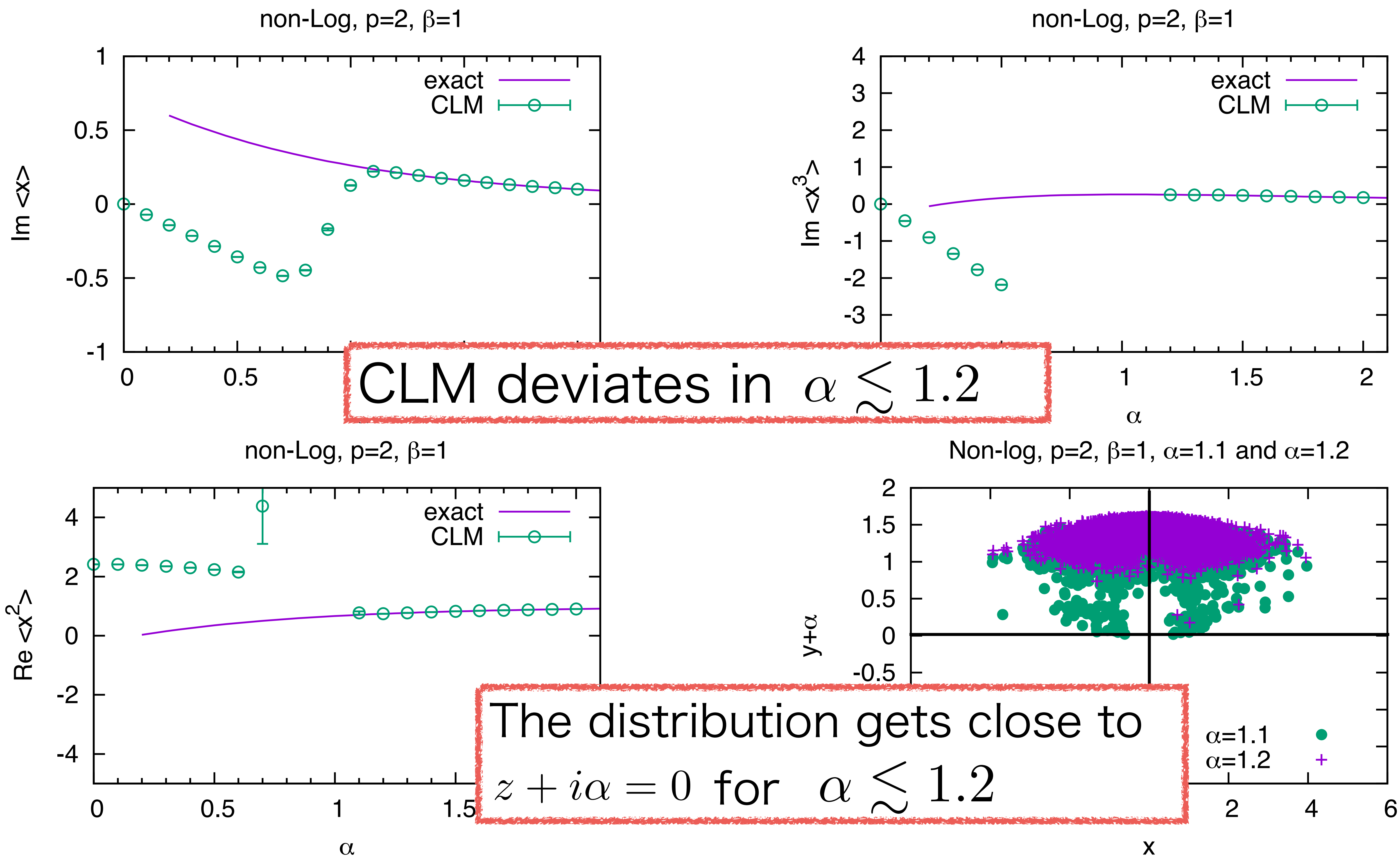








Result (non-log)



CLM deviates in $\alpha \lesssim 1.2$

The distribution gets close to $z + i\alpha = 0$ for $\alpha \lesssim 1.2$

Summary

- Condition for CLM to work
  = probability distribution has a fast fall-off
  and is practically zero near the singularity

- Both are related to the justification of the partial integration

- ”gauge cooling” should be useful in curing the singularity problem, too !!

- The new insights have great impact on applications to finite density QCD at low T with light quarks. (Nagata’s talk)