# Kaon semileptonic form factors as functions of the momentum transfer with $N_f$=2+1+1 Twisted Mass fermions

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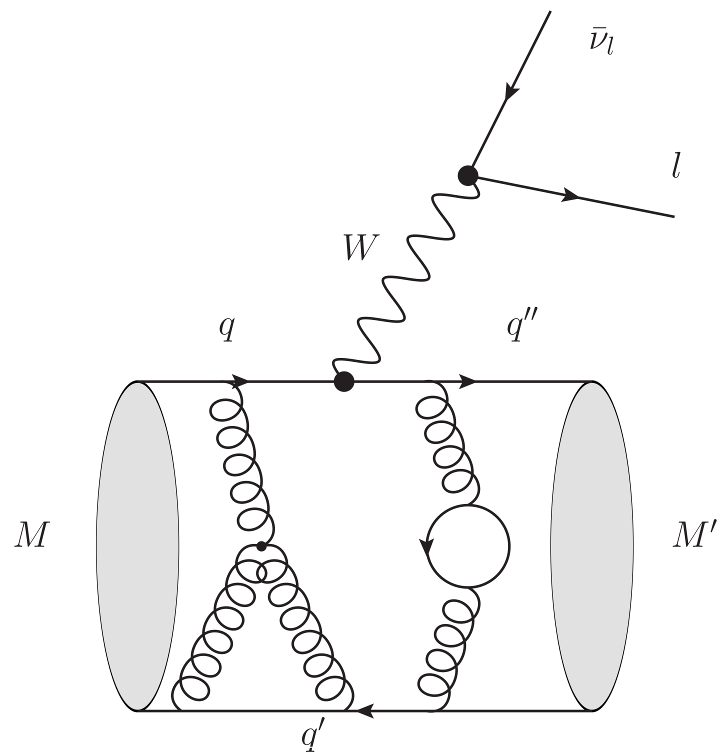
# Motivation

Kaon semileptonic decay rate is regulated by the vector and scalar form factors $f_+$ and $f_0$ which are functions of the square momentum transfer




Semileptonic decay rate :

$$\Gamma_{K_{\ell 3}} = \frac{G_F^2 m_K^5}{192\pi^3} C_K^2 S_{\mathrm{EW}} \left(|V_{us}| f_+^{K^0\pi^-}(0)\right)^2 I_{K\ell} \times \left(1 + \delta_{\mathrm{EM}}^{K\ell} + \delta_{\mathrm{SU(2)}}^{K\pi}\right)^2 ,$$

these form factors are measured with increasing precision by experiments.

Typical lattice calculations focus on the value of the form factors at $q^2=0$, it is however interesting to compute these form factors on the lattice in all the physical $q^2$ range and to compare them with experimental measurements

# Outline

- Simulation Details
- General strategy
- Plateaux
- Simultaneous fit of the $q^2$, $m_l$ and $a^2$ dependence
- FSE
- Dispersive parametrization and comparison with experiments
- Results and CKM unitarity tests
- Summary and Conclusions

# Simulation Details

Something on the action:

- Wilson Twisted Mass action at maximal twist with Nf=2+1+1 sea quarks
- Osterwalder-Seiler valence quark action
- Iwasaki gluon action

# Simulation Details

Details of the ensembles used in this $N_f$ =2+1+1 analysis

The valence light quark mass is put equal to the sea quark mass

| ensemble | $\beta$ | $V/a^4$ | $a\mu_{sea} = a\mu_l$ | $a\mu_\sigma$ | $a\mu_\delta$ | $N_{cfg}$ | $a\mu_s$ | $a\mu_c$ |
|---|---|---|---|---|---|---|---|---|
| $A30.32$ | 1.90 | $32^3 \times 64$ | 0.0030 | 0.15 | 0.19 | 150 | 0.0145, | 0.1800, 0.2200, |
| $A40.32$ | | | 0.0040 | | | 90 | 0.0185, | 0.2600, 0.3000, |
| $A50.32$ | | | 0.0050 | | | 150 | 0.0225 | 0.3600, 0.4400 |
| $A40.24$ | 1.90 | $24^3 \times 48$ | 0.0040 | 0.15 | 0.19 | 150 | | |
| $A60.24$ | | | 0.0060 | | | 150 | | |
| $A80.24$ | | | 0.0080 | | | 150 | | |
| $A100.24$ | | | 0.0100 | | | 150 | | |
| $B25.32$ | 1.95 | $32^3 \times 64$ | 0.0025 | 0.135 | 0.170 | 150 | 0.0141, | 0.1750, 0.2140, |
| $B35.32$ | | | 0.0035 | | | 150 | 0.0180, | 0.2530, 0.2920, |
| $B55.32$ | | | 0.0055 | | | 150 | 0.0219 | 0.3510, 0.4290 |
| $B75.32$ | | | 0.0075 | | | 75 | | |
| $B85.24$ | 1.95 | $24^3 \times 48$ | 0.0085 | 0.135 | 0.170 | 150 | | |
| $D15.48$ | 2.10 | $48^3 \times 96$ | 0.0015 | 0.12 | 0.1385 | 60 | 0.0118, | 0.1470, 0.1795, |
| $D20.48$ | | | 0.0020 | | | 90 | 0.0151, | 0.2120, 0.2450, |
| $D30.48$ | | | 0.0030 | | | 90 | 0.0184 | 0.2945, 0.3595 |

Range of the simulated pion masses

| $\beta$ | $L(fm)$ | $M_\pi$(MeV) | $M_\pi L$ |
|---|---|---|---|
| 1.90 | 2.84 | 245.41 | 3.53 |
| | | 282.13 | 4.06 |
| | | 314.43 | 4.53 |
| 1.90 | 2.13 | 282.13 | 3.05 |
| | | 343.68 | 3.71 |
| | | 396.04 | 4.27 |
| | | 442.99 | 4.78 |
| 1.95 | 2.61 | 238.67 | 3.16 |
| | | 280.95 | 3.72 |
| | | 350.12 | 4.64 |
| | | 408.13 | 5.41 |
| 1.95 | 1.96 | 434.63 | 4.32 |
| 2.10 | 2.97 | 211.18 | 3.19 |
| | | 242.80 | 3.66 |
| | | 295.55 | 4.46 |

| Lattice Spacings | |
|---|---|
| a($\beta = 1.90$) | 0.0885(36)fm |
| a($\beta = 1.95$) | 0.0815(30)fm |
| a($\beta = 2.10$) | 0.0619(18)fm |

To inject momenta we used non-periodic boundary conditions

| $\beta$ | $V/a^4$ | $\theta$ |
|---|---|---|
| 1.90 | $32^3 \times 64$ | 0.0, $\pm 0.400$, $\pm 0.933$, $\pm 1.733$ |
| | $24^3 \times 48$ | 0.0, $\pm 0.300$, $\pm 0.700$, $\pm 1.300$ |
| 1.95 | $32^3 \times 64$ | 0.0, $\pm 0.366$, $\pm 0.854$, $\pm 1.588$ |
| | $24^3 \times 48$ | 0.0, $\pm 0.275$, $\pm 0.641$, $\pm 1.191$ |
| 2.10 | $48^3 \times 96$ | 0.0, $\pm 0.424$, $\pm 0.986$, $\pm 1.832$ |

Three different values of the lattice spacing: 0.06 *fm* ÷ 0.09 *fm*

Different volumes: 2 *fm* ÷ 3 *fm*

Pion masses in range 210 ÷ 440 MeV

Momentum range up to 350 MeV

# General strategy

Over-constrain the scalar and vector form factors using a ratio of 3-points correlation function and the scalar density and extract with a combined fit our best determination of $f_0$ and $f_+$ on each ensemble for different values of $q^2$

Perform a combined fit of the scalar and vector form factors $f_0$ and $f_+$ studying simultaneously the $q^2$, $m_l$ and $a^2$ dependence using different assumptions

- SU(2) inspired formula,
- z expansion
- polar or polynomial behavior of $f_0$ as a function of $q^2$

Once we obtain the form factors as a function of the square 4-momentum transfer, we compute $f_0$ and $f_+$ in a set of reference $q^2$ values and fit them with a dispersive parametrization to compare our determination of $\Lambda_+$ and logC with the experimental one

# Extraction of the form factors

The two kaon semileptonic form factors $f_0$ and $f_+$ can be determined from the matrix element of the vector current

$$\langle \pi(p') |\ V_\mu\ | K(p) \rangle = (p_\mu + p'_\mu) f_+(q^2) + (p_\mu - p'_\mu) f_-(q^2)$$

$$f_+(q^2) = \frac{(E - E') V_i - (p_i - p'_i) V_0}{2Ep'_i - 2E'p_i}$$

$$f_-(q^2) = \frac{(p_i + p'_i) V_0 - (E + E') V_i}{2Ep'_i - 2E'p_i}$$

$$f_0(q^2) = f_+(q^2) + \frac{q^2}{m_K^2 - m_\pi^2} f_-(q^2)$$

an alternative way to determine $f_0$ is to use the scalar density

$$\langle \pi(p') | S | K(p) \rangle = f_0(q^2) \frac{m_s - m_l}{M_K^2 - M_\pi^2}$$

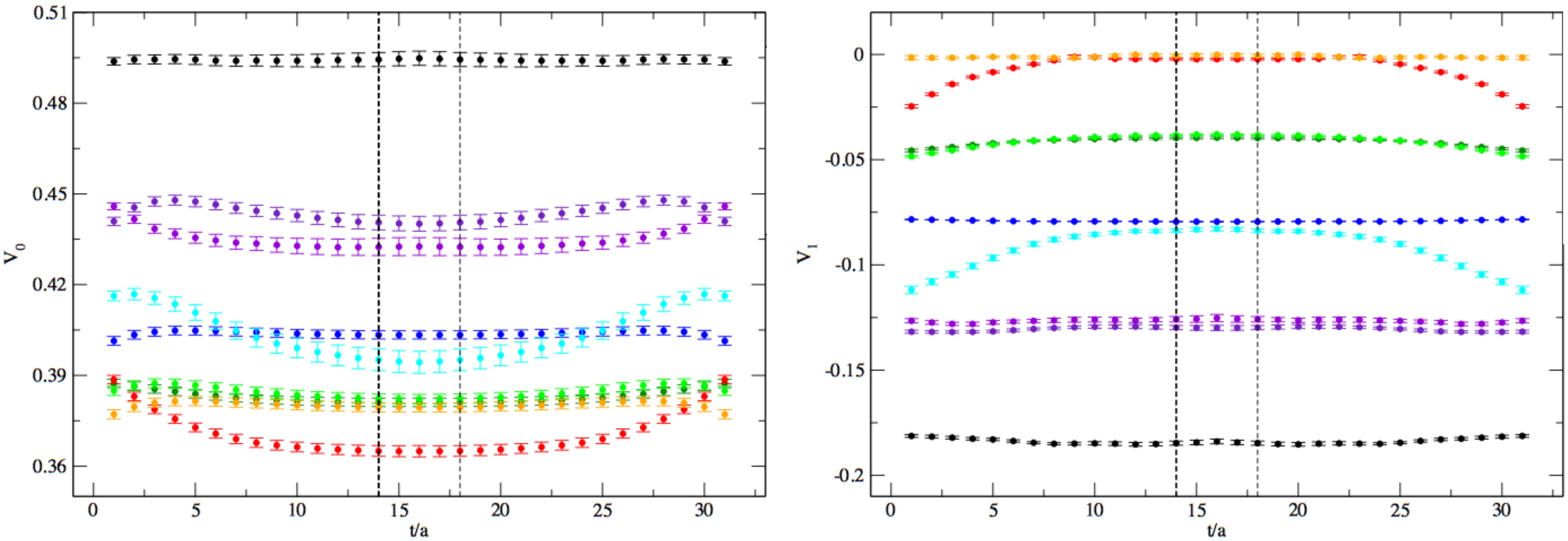
therefore using both matrix elements we are over-constraining $f_0$ and $f_+$ and we get our determination of the form factors on each ensemble for different values of $q^2$ from a combined fit

in practice we get these matrix elements fitting the time dependence of appropriate combinations of 3-points correlation functions

# Plateaux

example of plateaux of $V_0$ and $V_1$ for all the selected kinematics

$\beta = 1.90$
$\mu_l^{(sea)} = 0.0050$
$\mu_s = 0.0145$



# SU(2) ChPT

## SU(2) ChPT

The first formula used to fit the $q^2$, $m_l$ and $a^2$ dependence of the vector and scalar form factor simultaneously is obtained(1) expanding in x the SU(3) NLO expressions found by G&L(2)

$$f_+(s) = F_+(s)\left\{1 + C_+(s)x + \frac{M_K^2}{(4\pi f)^2}\left[-\frac{3}{4}x\log x - xT_1^+(s) - T_2^+(s)\right]\right\}$$

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(1) [*Lubicz et al. arXiv:1012.3573*]

(2) [*Gasser Leutwyler NPB 1985*]

$$s = q^2 / M_K^2 \qquad x = M_\pi^2 / M_K^2$$

we adopt a pole behavior for $F_{+,0}$ (which would only arise at higher order) and a polynomial behavior for $C_{+,0}$

$$F_{+,0}(s) = \frac{F(a^2,s)}{1 - \frac{q^2}{M_{V,S}^2}} \qquad\qquad C_{+,0}(s) = C + C_{+,0}^{(1)}\, s + C_{+,0}^{(2)}\, s^2$$

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$$M_V = M_{PS} + \Delta_{PS,V}$$

$$M_S = M_{PS} + \Delta_{PS,S}$$

$M_{PS}$ has been calculated on the lattice
$\Delta_{PS,V}$ and $\Delta_{PS,S}$ are taken from the PDG

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$$s = q^2 / M_K^2 \qquad x = M_\pi^2 / M_K^2$$

While for the vector form factor, a pole parameterization with the dominance of the $K^*(892)$ ($M_V \sim 892$ MeV) is in good agreement with the data, for the scalar form factor, such dominance is less clear

For this reason we also tried a global fit in which we remove the pole parametrization for $f_0$ replacing it with a polynomial dependence in $q^2$

# (modified) z-expansion

## (modified) z-expansion

We also tried to fit the $q^2$, $m_l$ and $a^2$ dependence of the vector and scalar form factor simultaneously using the z-expansion[(1)] in which however the parameters *a* and *b* depend on the light quark mass and on the lattice spacing squared

(1) *Bourrely Caprini and Lellouch [Phys.Rev. D79 (2009) 013008]*

$$f_+(q^2) = \frac{a_0(m_l, a^2) + a_1(m_l, a^2)\left[z + \frac{1}{2}z^2\right]}{1 - \frac{q^2}{M_V^2}}$$

$$f_0(q^2) = \frac{b_0(m_l, a^2) + b_1(m_l, a^2)\left[z + \frac{1}{2}z^2\right]}{1 - \frac{q^2}{M_S^2}}$$

$$z = \frac{\sqrt{t_+ - q^2} - \sqrt{t_+ - t_0}}{\sqrt{t_+ - q^2} + \sqrt{t_+ - t_0}}$$

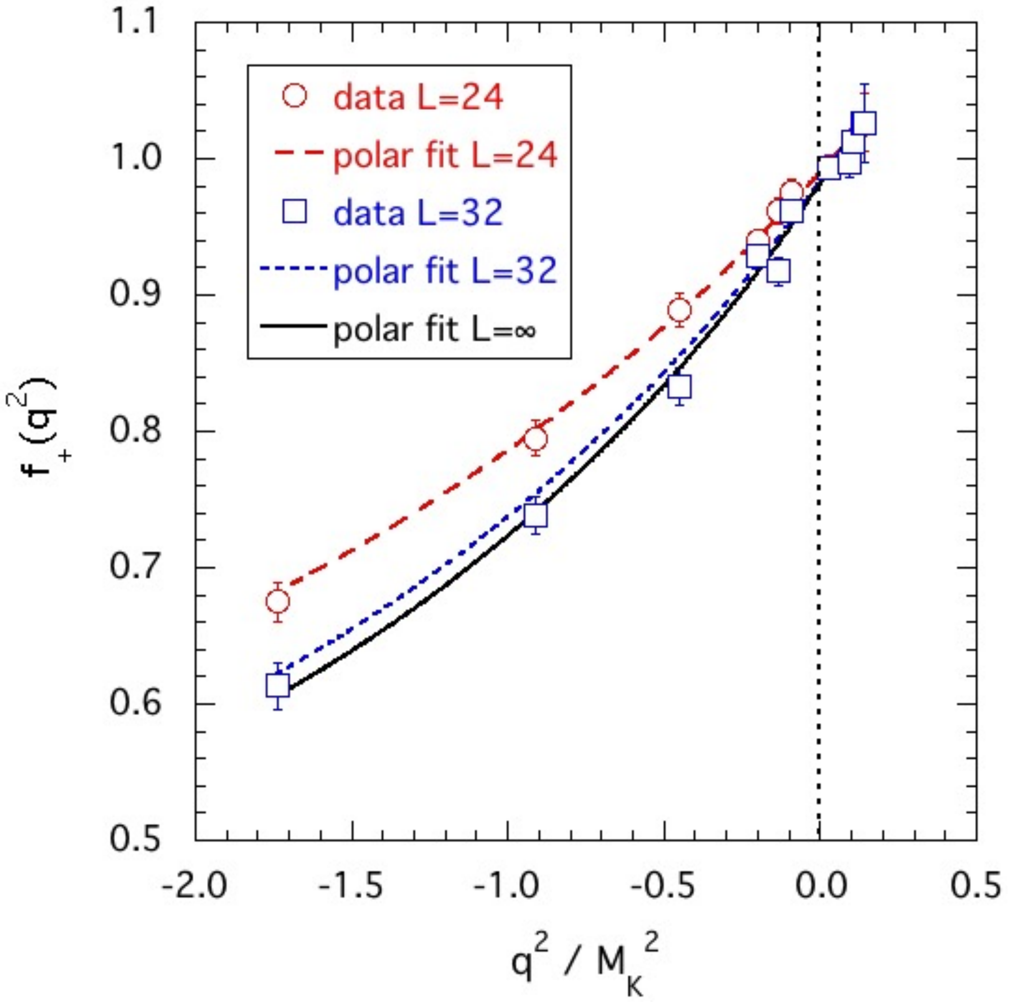
$$t_+ = (M_K + M_\pi)^2$$

$$t_0 = (M_K + M_\pi)\left(\sqrt{M_K} - \sqrt{M_\pi}\right)^2$$

where we adopt for the parameters *a* and *b* a polynomial dependence on the lattice spacing squared and on the light quark mass

# FSE

An investigation of ensemble A40.24 and A40.32 which share the same pion mass at different volumes shows the presence of non negligible FSE in the slope of the form factors. This can be seen in the plot below for $f_+(q^2)$




For this reason we decided to add a FSE correction term in the slope and to fit its magnitude from our data

following what was done in [R. Frezzotti et al. PRD 79(2009)] for FSE correction in the pion electromagnetic form factor we adopt the following phenemenological correction for the slope

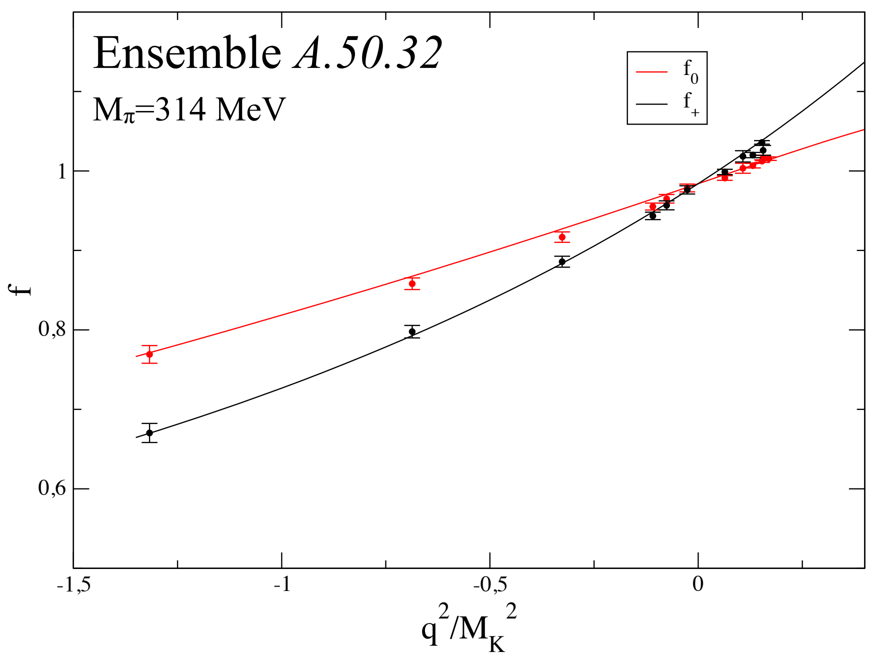
$$1 + P M_\pi^2 \frac{e^{-M_\pi L}}{(M_\pi L)^{\alpha_{eff}}}$$

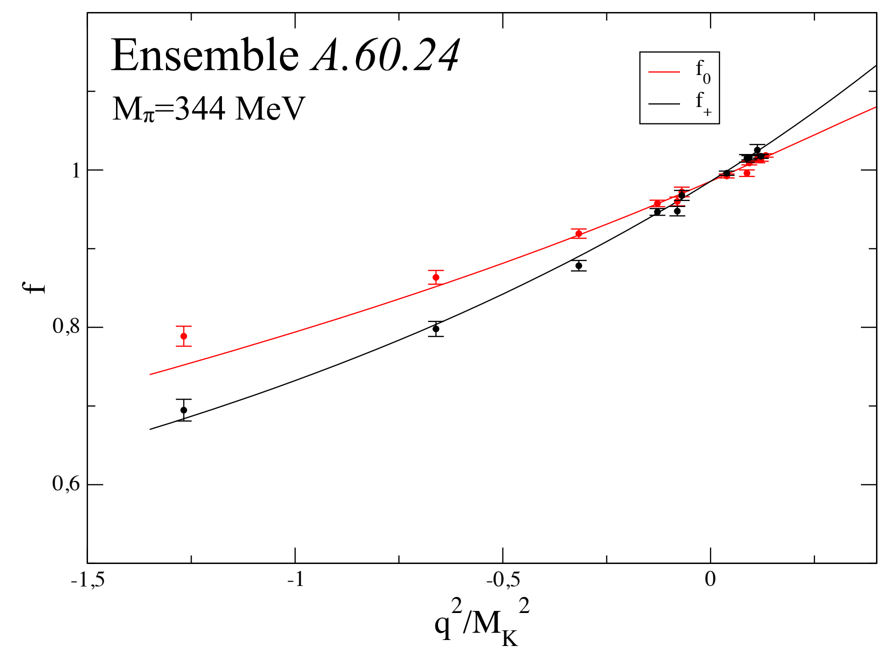
where P is a free parameter determined during the global fit and $\alpha_{eff}$ is an effective fractional power

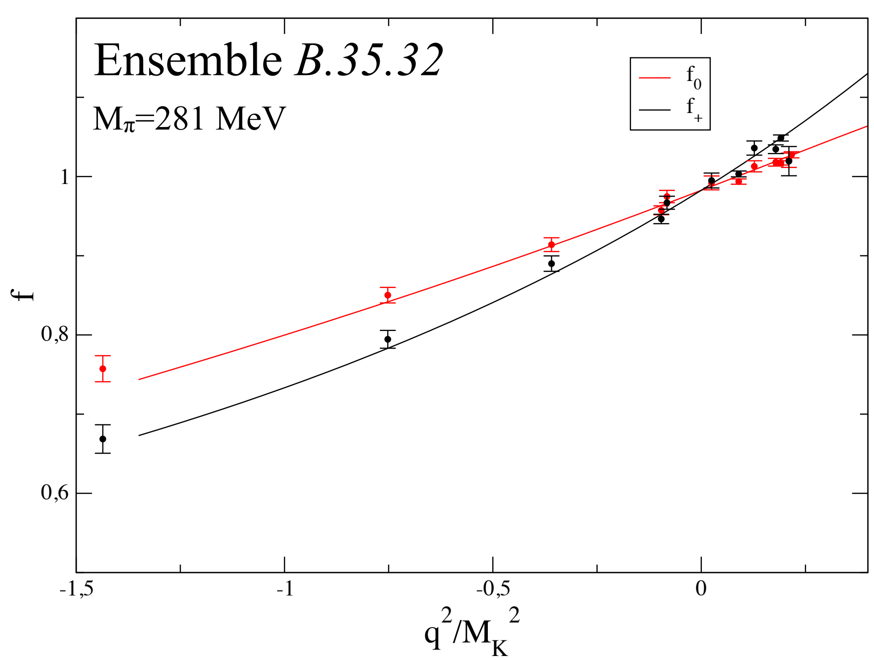
we tested different values of $\alpha_{eff}$ and decided that a good estimate of the systematic uncertainty associated to the FSE is given by the spread on the final results found with $\alpha_{eff}=0$ and $\alpha_{eff}=3/2$

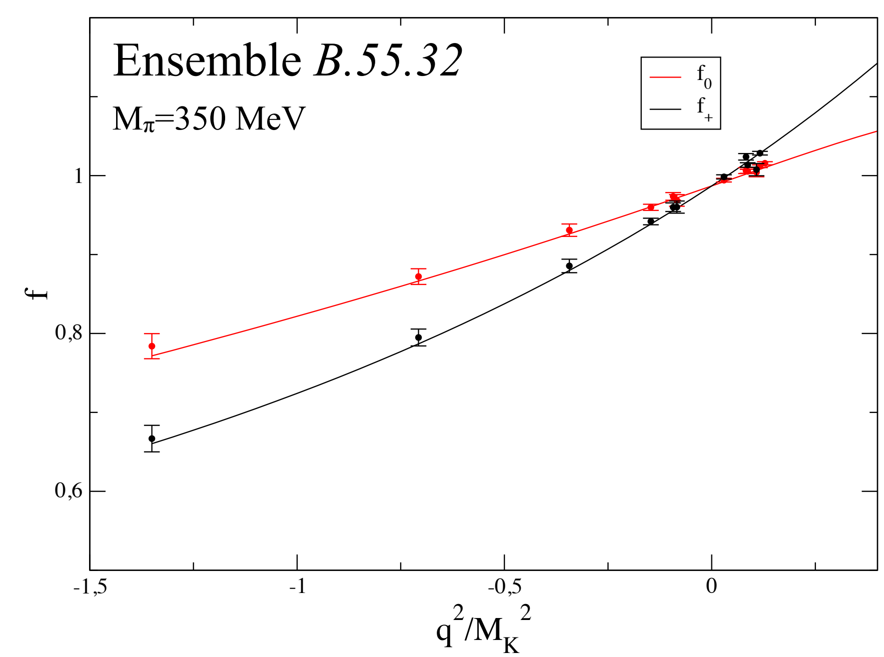
# Comparison between data and global fit on given ensembles

Examples of $q^2$-dependence for some ensembles obtained from the global fit using SU(2) ChPT and assuming a pole behavior for $f_0$



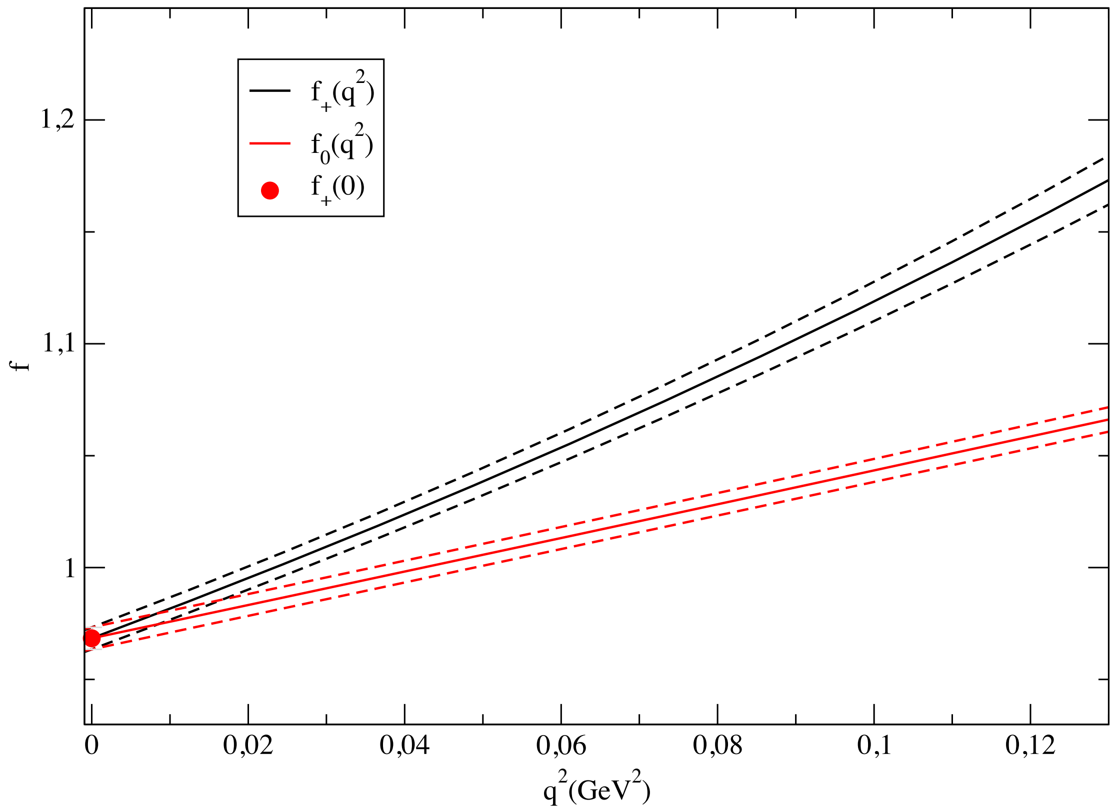



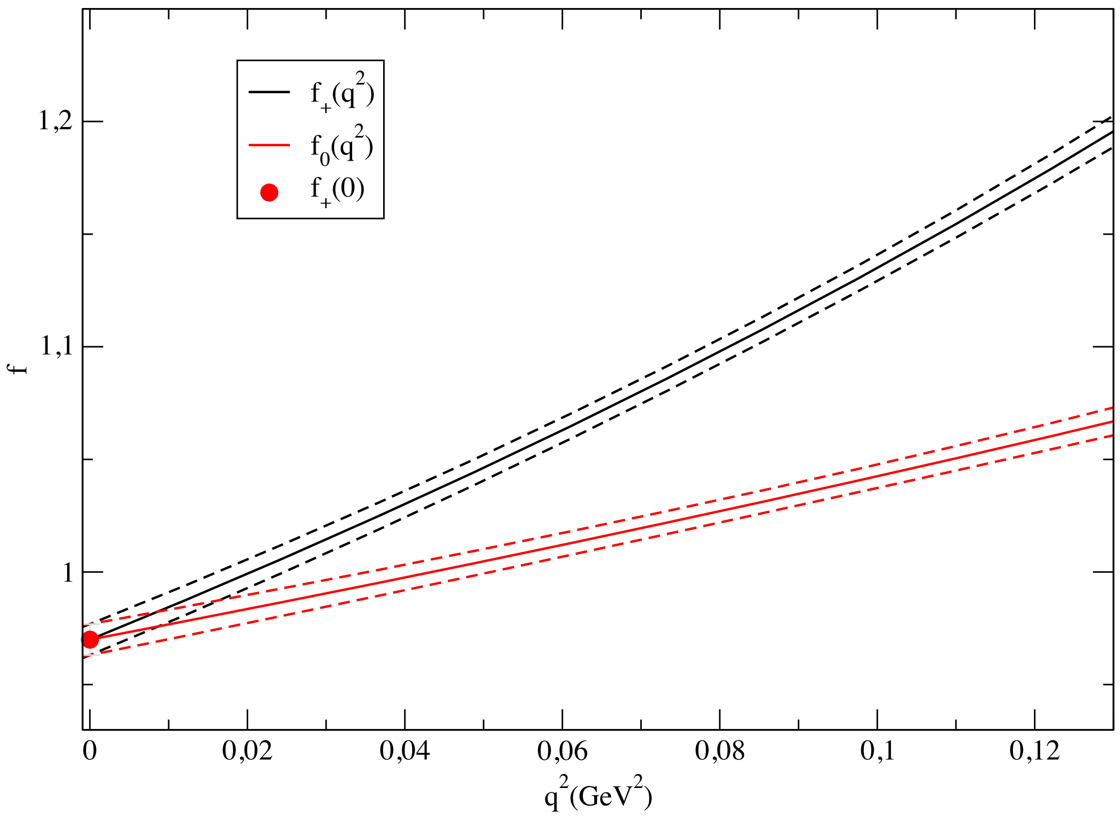
# Global fit

Two example of the form factors obtained with different global fit in the continuum limit and at the physical light and strange quark mass. In the plot is also shown the form factor at zero momentum transfer relevant for the extraction of $|V_{us}|$

SU(2) ChPT




(modified) z-expansion



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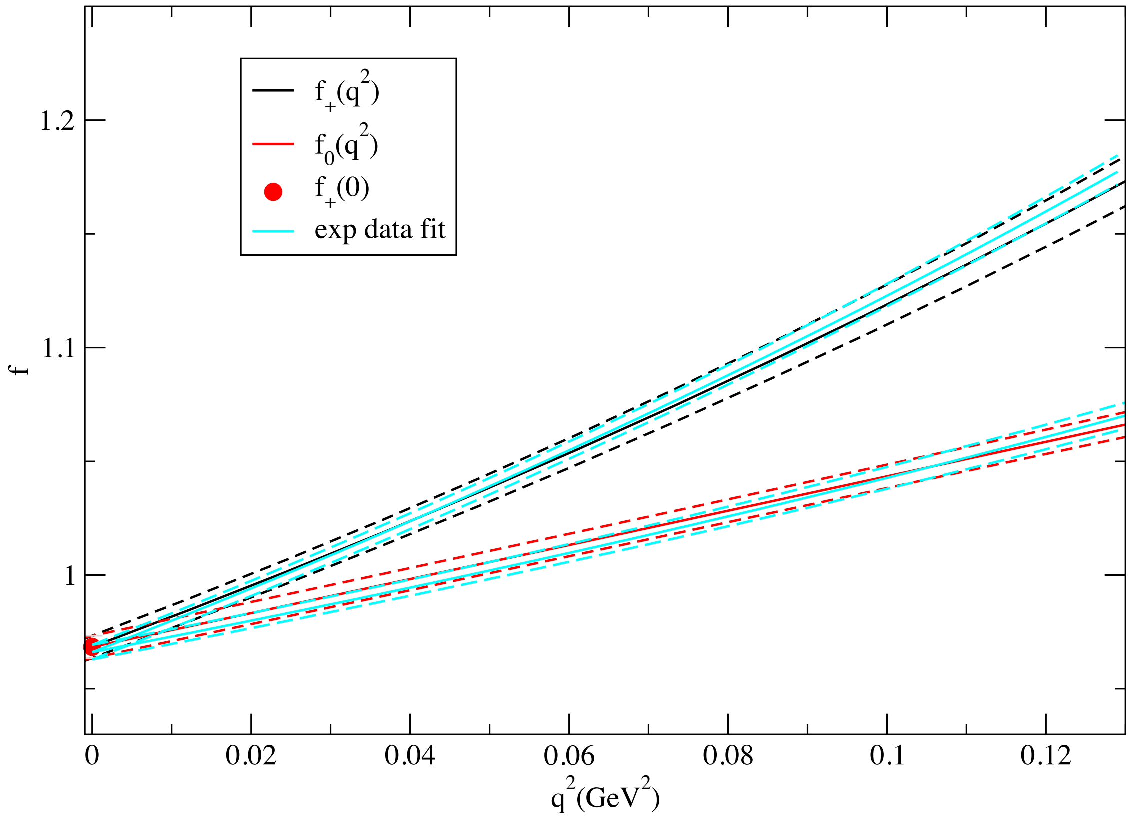


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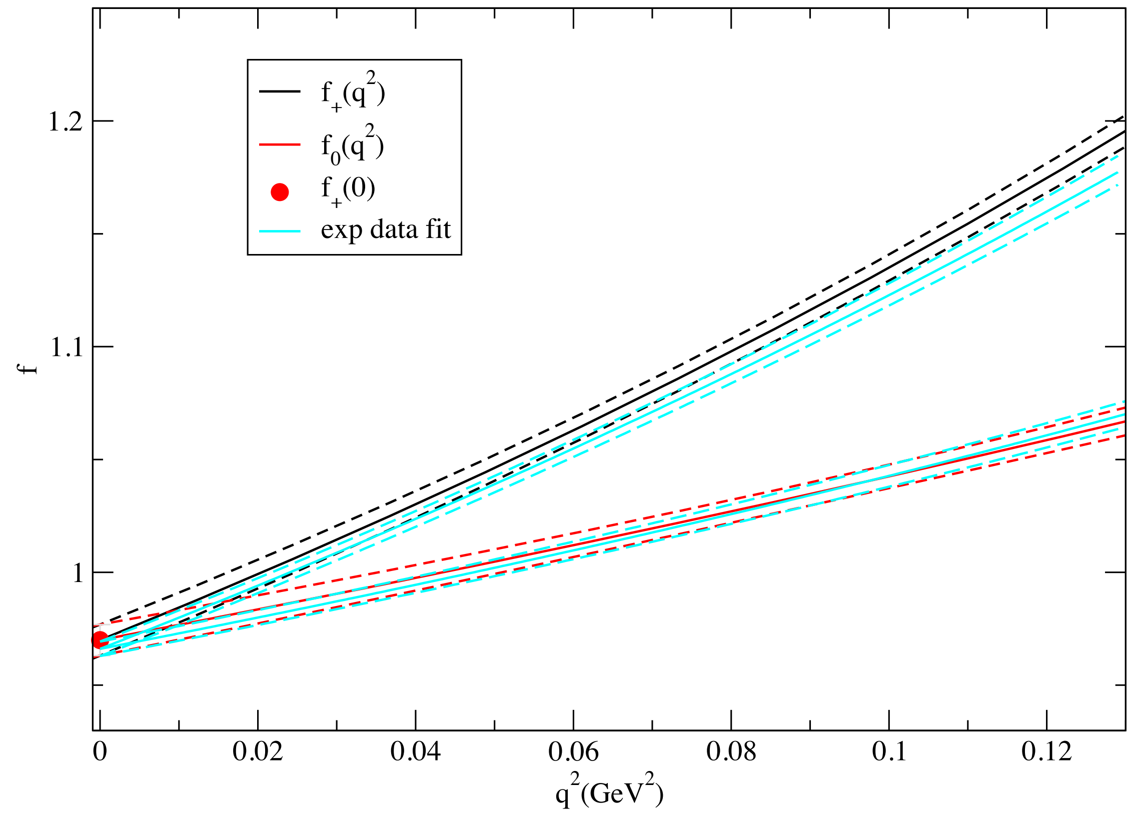



The results can be compared with a dispersive fit of experimental data reported in (1). The most relevant parameters of the dispersive parametrization of the form factors are $\Lambda_+$ and logC.
To get our own determination of $\Lambda_+$ and logC we compute, for each analysis, $f_0$ and $f_+$ in a set of reference $q^2$ values in the physical region and fit them with a dispersive formula.

(1)[*Moulson arXiv:1411.5252(2014)*]

dispersive fit with experimental data from:
KTeV, KLOE, NA48/2, ISTRA+

# Dispersive parametrization

Dispersive parametrization of the form factors is written as follows

$$f_{+,0}(q^2) = f_+(0)\left(1 + \lambda'_{+,0}\frac{q^2}{M_\pi^2} + \frac{1}{2}\lambda''_{+,0}\left(\frac{q^2}{M_\pi^2}\right)^2 + \frac{1}{6}\lambda'''_{+,0}\left(\frac{q^2}{M_\pi^2}\right)^3\right)$$

where the Taylor expansion parameters are related to each other by these equations

$$\lambda'_0 = \frac{m_\pi^2}{\Delta_{K\pi}}\left[\ln C - G(0)\right],$$

$$\lambda''_0 = (\lambda'_0)^2 - 2\frac{m_\pi^4}{\Delta_{K\pi}}G'(0),$$

$$\lambda'''_0 = (\lambda'_0)^3 - 6\frac{m_\pi^4}{\Delta_{K\pi}}G'(0)\lambda'_0 - 3\frac{m_\pi^6}{\Delta_{K\pi}}G''(0)$$

$$\lambda'_+ = \Lambda_+,$$

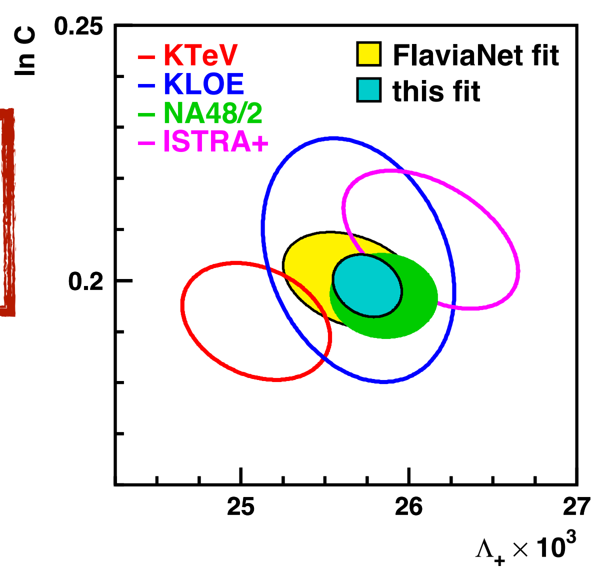
$$\lambda''_+ = (\lambda'_+)^2 + 2m_\pi^2 H'(0),$$

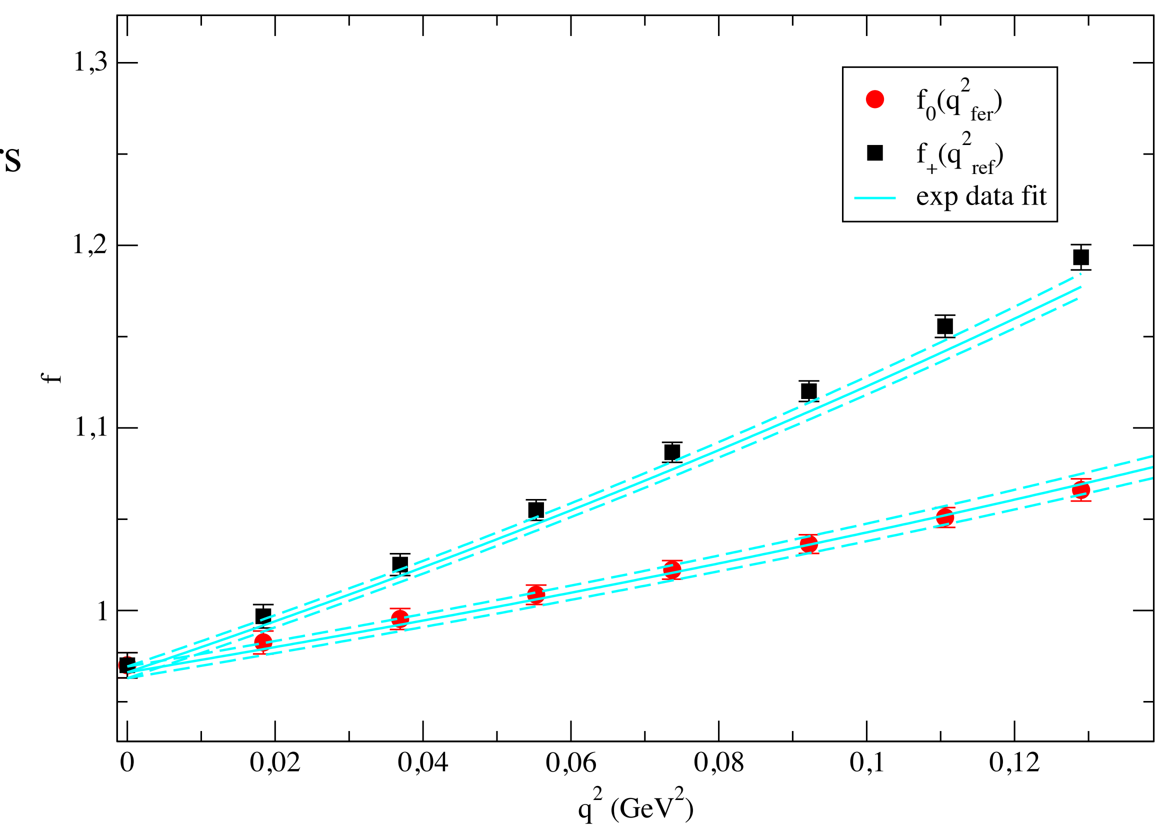
$$\lambda'''_+ = (\lambda'_+)^3 + 6m_\pi^2 H'(0)\lambda'_+ + 3m_\pi^4 H''(0),$$

(modified) z-expansion

P- and S-wave $(K\pi)_{I=1/2}$ elastic scattering can be used to set priors on G and H so that we are left with only three free parameters

experimental results in the lnC, $\Lambda_+$ plane



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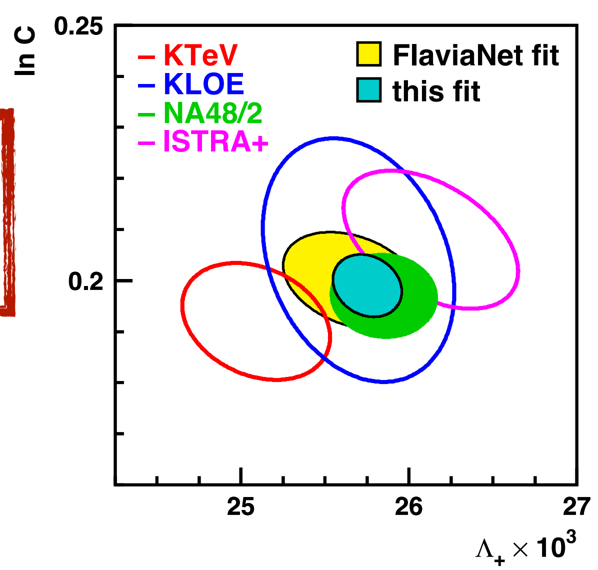
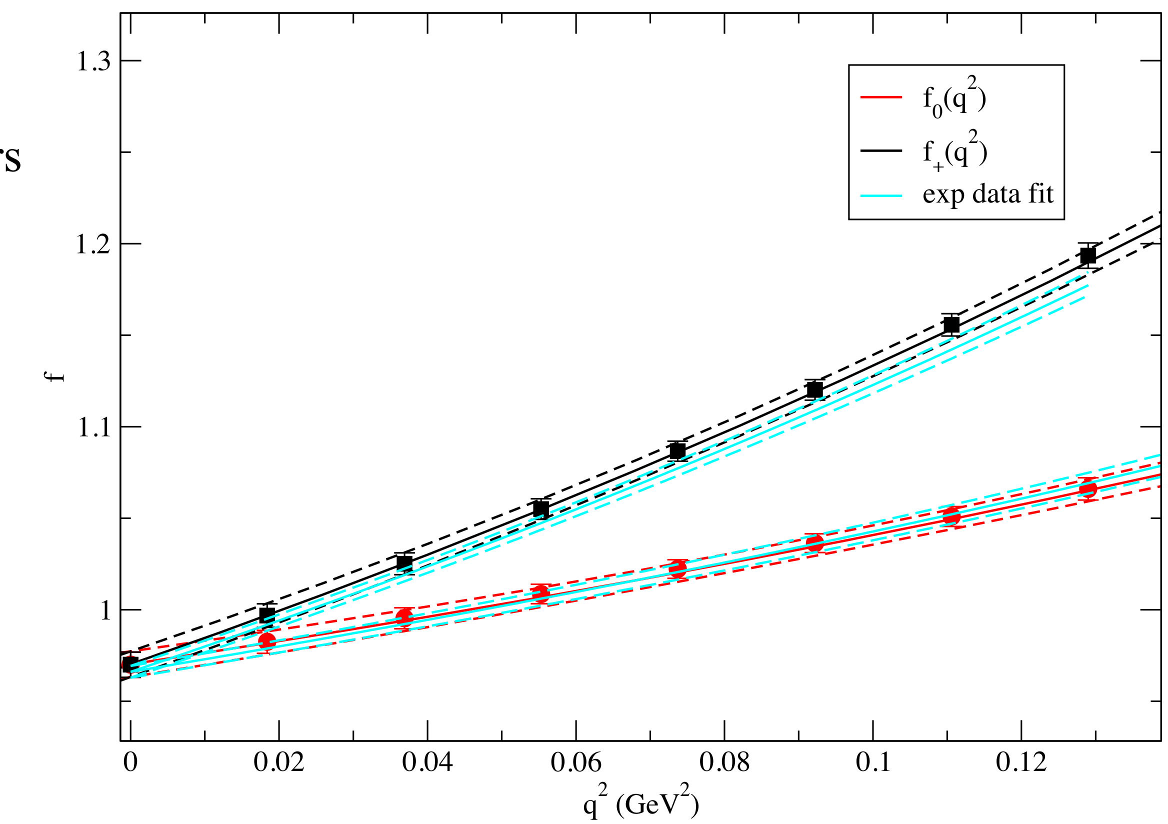


# results and systematics

Combining the results from all these analyses we get:

$$f_+(0) = 0.9684(66)$$

$$\log C = 0.1937(138)$$ NEW

$$\Lambda_+ = 25.2(1.6) \times 10^{-3}$$ NEW

in particular:

$$f_+(0) = 0.9684(59)_{\text{stat+fit}}(29)_{\text{syst}}$$

$$\log C = 0.1937(113)_{\text{stat+fit}}(90)_{\text{syst}}$$

$$\Lambda_+ = 25.2(1.2)_{\text{stat+fit}}(1.1)_{\text{syst}} \times 10^{-3}$$

**stat+fit** is referred to both the statistical uncertainties (including the total error on the light and strange quark mass determination) and the uncertainties due to the fitting procedure

**syst** takes into account: the chiral extrapolation, which have been evaluated comparing the different fit formulas; FSE which are relevant for the slope of the form factors, and discretization effect, which have been evaluated comparing with a fit performed after removing all the data at the coarsest lattice spacing.

# comparison with other results

The results from our analysis:

$$f_+(0) = 0.9684(66)$$

$$\log C = 0.1937(138)$$

$$\Lambda_+ = 25.2(1.6) \times 10^{-3}$$

Recent lattice calculation:

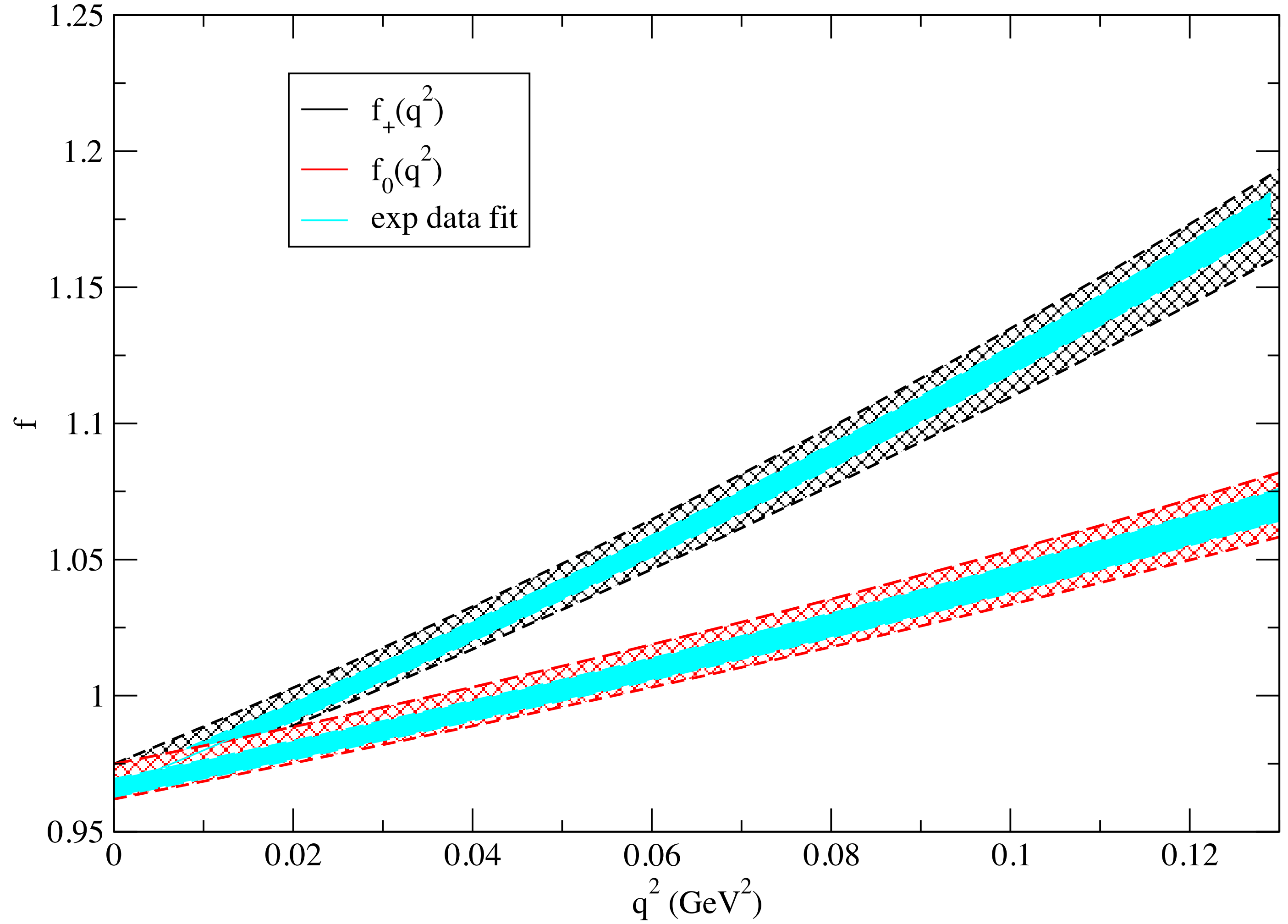
$f_+(0) = 0.9704(33)$ FNAL/MILC $(N_f = 2+1+1)$

$f_+(0) = 0.9685(38)$ RBC/UKQCD $(N_f = 2+1)$

Experimental combined fit:

$$\log C_{EXP} = 0.1985(70)$$

$$\Lambda_{+\,EXP} = 25.75(36) \times 10^{-3}$$




$$|V_{us}| = 0.2234(16)$$

# Testing the CKM unitarity

Testing the first row

$$|V_u|^2 \equiv |V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 = 1$$

$$V_{CKM} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix}$$

| | Experimental input [1] | our result | determination of $\|V_{us}\|$ |
|---|---|---|---|
| $K_{\ell 2}$ | $\left\|\frac{V_{us}}{V_{ud}}\right\| \frac{f_{K^\pm}}{f_{\pi^\pm}} = 0.2758(5)$ | $\frac{f_{K^\pm}}{f_{\pi^\pm}} = 1.184(16)$ [2] | $\|V_{us}\| = 0.2271(33)$ |
| $K_{\ell 3}$ | $\|V_{us}\| f_+(0) = 0.2163(5)$ | $f_+(0) = 0.9684(66)$ | $\|V_{us}\| = 0.2234(16)$ |

$|V_{ud}| = 0.97425(22)$ from β-decay [3]

$K_{\ell 2}$: $$|V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 = 1.0007(16)$$

$K_{\ell 3}$: $$|V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 = 0.9991(8)$$


(1) Eur.Phys.J. C69 (2010) 399-424

(2) PRD 91 (2015 )(Carrasco et al.)

(3) Phys.Rev. C79 (2009) 055502

# Conclusions

- We presented $N_f$=2+1+1 results for the semileptonic form factors $f_+(q^2)$ and $f_0(q^2)$ and in particular their value at zero momentum transfer, $f_+(0)$, which allowed us to determine $|V_{us}|$

$$f_+(0) = 0.9684(66) \qquad |V_{us}| = 0.2234(16)$$

- we quantify the $q^2$-dependence of the form factors in terms of logC and $\Lambda_+$ which are the parameters relevant to the comparison with the experimental results.

$$\log C = 0.1937(138) \qquad \Lambda_+ = 25.2(1.6) \times 10^{-3}$$

Future plans:

- we plan to include also a global fit of the form factors based on hard-pion ChPT

# Backup

# SU(2) ChPT

## SU(2) ChPT

The first formula used to fit the $q^2$, $m_l$ and $a^2$ dependence of the vector and scalar form factor simultaneously is obtained[1] expanding in x the SU(3) NLO expressions found by G&L[2]

$$f_+(s) = F_+(s)\left\{1 + C_+(s)x + \frac{M_K^2}{(4\pi f)^2}\left[-\frac{3}{4}x\log x - xT_1^+(s) - T_2^+(s)\right]\right\}$$

$$f_0(s) = F_0(s)\left\{1 + C_0(s)x + \frac{M_K^2}{(4\pi f)^2}\left[-\frac{3}{4}x\log x - xT_1^0(s) - T_2^0(s)\right]\right\}$$

(1) [*Lubicz et al. arXiv:1012.3573*]

(2) [*Gasser Leutwyler NPB 1985*]

$$s = q^2 / M_K^2 \qquad x = M_\pi^2 / M_K^2$$

$$\begin{aligned}
T_1^+(s) &= [(1-s)\log(1-s) + s(1-s/2)]\,3(1+s)/4s^2 \,, \\
T_2^+(s) &= [(1-s)\log(1-s) + s(1-s/2)]\,(1-s)^2/4s^2 \,, \\
T_1^0(s) &= [\log(1-s) + s(1+s/2)]\,(9+7s^2)/4s^2 \,, \\
T_2^0(s) &= [(1-s)\log(1-s) + s(1-s/2)]\,(1-s)(3+5s)/4s^2
\end{aligned}$$

# Extraction of the form factors

Extracting the vector and scalar current matrix elements in practice

$$\frac{C_{\mu}^{K\pi}\left(t,\vec{p},\vec{p}'\right)\quad C_{\mu}^{\pi K}\left(t,\vec{p}',\vec{p}\right)}{C_{\mu}^{\pi\pi}\left(t,\vec{p}',\vec{p}'\right)\quad C_{\mu}^{KK}\left(t,\vec{p},\vec{p}\right)} \to \frac{\left(\left\langle \pi\left(p'\right)\right| V_{\mu}\left| K\left(p\right)\right\rangle\right)^{2}}{4\,p_{\mu}p'_{\mu}}$$

$$\frac{C_{S}^{K\pi}\left(t,\vec{p},\vec{p}'\right)\quad C_{S}^{\pi K}\left(t,\vec{p}',\vec{p}\right)}{Z_{K}Z_{\pi}\;e^{-(E_{K}-E_{\pi})T/2}} \to \frac{\left(\left\langle \pi\left(p'\right)\right| S\left| K\left(p\right)\right\rangle\right)^{2}}{\left(4E_{K}E_{\pi}\right)^{2}}$$

# Results from the different fits

| | $f_+(0)$ | $\log C$ | $\Lambda_+ \times 10^3$ |
|---|---|---|---|
| SU(2) ChPT FSE type1 | 0.9696(49) | 0.1894(053) | 24.4(1.1) |
| SU(2) ChPT FSE type2 | 0.9693(49) | 0.1901(052) | 24.5(1.1) |
| z-expansion FSE type1 | 0.9702(70) | 0.1864(142) | 26.9(1.1) |
| z-expansion FSE type2 | 0.9704(78) | 0.1868(195) | 26.7(1.5) |
| SU(2) ChPT no $f_0$ pole | 0.9653(49) | 0.2048(082) | 24.4(1.1) |