Heavy-heavy current improvement for calculating $\bar{B} \to D^{(*)}\ell\bar{\nu}$ semi-leptonic form factors from Oktay-Kronfeld quarks

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Abstract

We are improving heavy-heavy currents for calculating $\bar{B} \to D^{(*)}\ell\bar{\nu}$ semi- leptonic form factors with Oktay-Kronfeld(OK) heavy quarks. The OK action, which has dimension 6 and 7 interaction terms, can control the discretization errors of heavy quarks (b and c quarks). The OK action is improved through third order in HQET power counting. We report work on heavy-heavy currents to get the systematic improvement of the hadronic matrix elements for $\bar{B} \to D^{(*)}\ell\bar{\nu}$ processes with the OK action.

Physical Motivation

- Decay rate for $\bar{B} \to D^{(*)}\ell\bar{\nu}$ are proportional to $|V_{cb}|^2$.

$$\frac{d\Gamma}{dw}(\bar{B} \to D^{(*)}\ell\bar{\nu}) = |V_{cb}|^2 F_{D^{(*)}}(w)^2 \times \text{Const.}$$

$$\times \text{ electroweak.} \times \text{ kinematic term.}$$

- By calculating semi-leptonic form factors $h_\pm(w), h_{A_i}(w), h_V(w)$ from lattice, we can determine $|V_{cb}|$ using experimental measurements of decay rate. $F_{D^{(*)}}(w)^2$ is expressed in terms of $h_\pm(w), h_{A_i}(w), h_V(w)$.

$$\frac{\langle D(p_D)|V^\mu|B(p_B)\rangle}{\sqrt{M_D M_B}} = (v_B + v_D)^\mu h_+(w) + (v_B - v_D)^\mu h_-(w)$$

$$\frac{\langle D^*(p_{D^*},\epsilon)|A^\mu|B(p_B)\rangle}{\sqrt{M_{D^*} M_B}} = i[\epsilon^{*\mu} h_{A_1}(w)(1+w) - (\epsilon^* \cdot v_B)(v_B^\mu h_{A_2}(w) + v_{D^*}^\mu h_{A_3}(w))]$$

$$\frac{\langle D^*(p_{D^*},\epsilon)|V^\mu|B(p_B)\rangle}{\sqrt{M_{D^*} M_B}} = \varepsilon^{\mu\nu\rho\sigma}\epsilon^*_\nu (v_B)_\rho (v_{D^*})_\sigma h_V(w),$$

where v_B and v_D is hadron velocity and $w = v_B \cdot v_D$.

Improvement of Lattice Gauge Theory

- We can reduce discretization error systematically in arbitrary $m_q a$ with the Wilson fermion by allowing an asymmetry between temporal & spatial interaction, and retaining Wilson's time derivative. [1]
- For heavy quark ($m_q a \simeq 1$), we may interpret Wilson fermion non-relativistically, and match lattice gauge theory to continuum QCD through HQET. ($\delta\mathcal{C}_i = \mathcal{C}_i^{\text{lat}} - \mathcal{C}_i \to 0$) [2, 3, 4]

$$\mathcal{L}_{\text{lat}}^{\text{heavy}} \doteq -\bar{h}(D_4 + m_{1h})h + \sum_i \mathcal{C}_i^{\text{lat}}\mathcal{O}_i, \tag{1}$$

$$\mathcal{L}_{\text{cont}}^{\text{heavy}} \doteq -\bar{h}(D_4 + m_h)h + \sum_i \mathcal{C}_i\mathcal{O}_i, \tag{2}$$

where m_{1h} is rest mass of lattice heavy quark, and $\mathcal{O}_i$ is operator in heavy quark effective theory (HQET).

Matching

- We can match lattice gauge theory to continuum QCD by calculating on-shell matrix elements. There are two strategies for that.

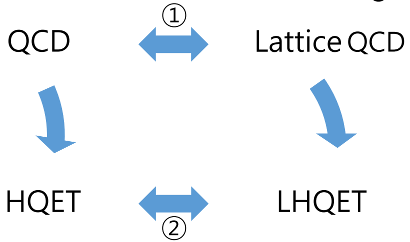

Figure: Matching process. HQET (LHQET) is heavy-quark effective theory matched to underlying continuum (lattice) QCD.

- Compare on-shell matrix elements of lattice and continuum QCD in the desired order of HQET expansion : Case ①
- First, match (lattice) QCD to heavy-quark effective theory to obtain (LHQET) HQET couplings. Then set LHQET couplings to the HQET values : Case ②

Improvement of Lattice Gauge Theory - Action

- Fermilab action.[1]

$$S_0 = m_0 a^4 \sum_x \bar{\psi}(x)\psi(x) + a^4 \sum_x \bar{\psi}(x)\gamma_4 D_{4\text{lat}}\psi(x) + \zeta a^4 \sum_x \bar{\psi}(x)\vec{\gamma}\cdot\vec{D}_{\text{lat}}\psi(x) - \frac{1}{2}a^5 \sum_x \bar{\psi}(x)\Delta_{4\text{lat}}\psi(x) - \frac{1}{2}r_s\zeta a^5 \sum_x \bar{\psi}(x)\Delta^{(3)}_{\text{lat}}\psi(x)$$

- Dimension-five interactions.

$$S_B = -\frac{1}{2}c_B\zeta a^5 \sum_x \bar{\psi}(x) i\vec{\Sigma}\cdot\vec{B}_{\text{lat}}\psi(x)$$

$$S_E = -\frac{1}{2}c_E\zeta a^5 \sum_x \bar{\psi}(x)\vec{\alpha}\cdot\vec{E}_{\text{lat}}\psi(x)$$

- c_B, c_E : fixed by matching one-gluon vertex for the interaction with background field.

Improvement of Lattice Gauge Theory - Action

- Oktay-Kronfeld(OK) action [3] contains six more non-zero couplings at dimension 6 and 7 to improve in the level of $\mathcal{O}(\lambda^3)$. $(\lambda \sim a\Lambda, \Lambda/m_h)$ at tree-level.

$$
\begin{aligned}
S_6 = {} & c_1 a^6 \sum_x \bar{\psi}(x) \sum_i \gamma_i D_{i\text{lat}} \Delta_{i\text{lat}} \psi(x) \\
& + c_2 a^6 \sum_x \bar{\psi}(x) \{\vec{\gamma} \cdot \vec{D}_{\text{lat}}, \Delta^{(3)}_{\text{lat}}\} \psi(x) \\
& + c_3 a^6 \sum_x \bar{\psi}(x) \{\vec{\gamma} \cdot \vec{D}_{\text{lat}}, i\vec{\Sigma} \cdot \vec{B}_{\text{lat}}\} \psi(x) \\
& + c_{EE} a^6 \sum_x \bar{\psi}(x) \{\gamma_4 D_{4\text{lat}}, \vec{\alpha} \cdot \vec{E}_{\text{lat}}\} \psi(x) \\
S_7 = {} & a^7 \sum_x \bar{\psi}(x) \sum_i \left[c_4 \Delta^2_{i\text{lat}} \psi(x) + c_5 \sum_{j \neq i} \{ i\Sigma_i B_{i\text{lat}}, \Delta_{j\text{lat}} \} \right] \psi(x)
\end{aligned}
$$

- Coefficients c_i are fixed by matching dispersion relation, one-gluon vertex, Compton scattering.

Improvement of Lattice Gauge Theory - Current

- The improved lattice operator can be constructed by linear combination of lattice operators with the same quantum number as continuum operator.
- In the case of flavor-changing bilinear,

$$\bar{c}\Gamma b = Z_\Gamma \left[\bar{\psi}_c \Gamma \psi_b + \sum_{d \geq 4} \sum_i C^d_{\Gamma i} O^d_{\Gamma i} \right],$$

  Z_Γ : matching factor.
  $\bar{c}\Gamma b$: continuum bilinear.
  $\bar{\psi}_c \Gamma \bar{\psi}_b$: leading lattice bilinear.
  $O^d_{\Gamma i}$: higher dimensional lattice operator with dimension d.
- By matching on-shell matrix elements, we can obtain coefficients Z_Γ and $C^d_{\Gamma i}$.
- We need improved current in the level of $\mathcal{O}(\lambda^3)$ in HQET power counting.

Improvement of Lattice Current - $\mathcal{O}(\lambda)$ case

- Consider following continuum matrix element at tree-level,

$$\langle c(\eta_c, \vec{p}_c)|\ \bar{c}\Gamma b\ |b(\eta_b, \vec{p}_b)\rangle \to \sqrt{\frac{m_c}{E_c}}\sqrt{\frac{m_b}{E_b}}\ \bar{u}_c(\eta_c, \vec{p}_c)\Gamma u_b(\eta_b, \vec{p}_b),$$

- For the lattice [1],

$$\langle c(\eta_c, \vec{p}_c)|\ \bar{\psi}_c\Gamma\psi_b\ |b(\eta_b, \vec{p}_b)\rangle \to \mathcal{N}_c(\vec{p}_c)\mathcal{N}_b(\vec{p}_b)\ \bar{u}_c^{\mathsf{lat}}(\eta_c, \vec{p}_c)\Gamma u_b^{\mathsf{lat}}(\eta_b, \vec{p}_b),$$

where $\mathcal{N}(\vec{p})$ is lattice normalization factor for external state.

- Expand continuum and lattice spinor for small spatial momentum (compared to $1/a$ or m) gives [1],

$$\sqrt{\frac{m}{E}}u(\eta, \vec{p}) = \left[1 - \frac{i\vec{\gamma}\cdot\vec{p}}{2m}\right]u(\eta, 0) + \mathcal{O}(\frac{\vec{p}^{\ 2}}{m^2}),$$

$$\mathcal{N}(\vec{p})u^{\mathsf{lat}}(\eta, \vec{p}) = e^{-M_1 a/2}\left[1 - \frac{\zeta i\vec{\gamma}\cdot\vec{p}a}{2\sinh M_1 a}\right]u(\eta, 0) + \mathcal{O}(\vec{p}^{\ 2}a^2),$$

where the tree-level rest mass $M_1 a = \log(1 + m_0 a)$.

Improvement of Lattice Current - $\mathcal{O}(\lambda)$ case

- At the zeroth order in momentum $\vec{p}_b$ and $\vec{p}_c$, continuum and lattice matrix elements are different by tree-level matching factor,

$$Z_\Gamma^{[0]} = e^{M_{1b}a/2 + M_{1c}a/2}.$$

- At the first order in momentum $\vec{p}_b$ and $\vec{p}_c$, there is mismatch between lattice and continuum matrix elements. There need dimension four lattice bilinears to match the matrix elements.

$$O^4_{\Gamma 1} = \bar{\psi}_c \Gamma \vec{\gamma} \cdot \vec{D} \psi_b, \qquad O^4_{\Gamma 2} = \bar{\psi}_c \vec{\gamma} \cdot \overleftarrow{\vec{D}} \Gamma \psi_b.$$

- By defining field rotation [1, 2],

$$Q_h = e^{M_{1h}a/2}[1 + a d_{1h} \vec{\gamma} \cdot \vec{D}_{\mathsf{lat}}]\psi_h,$$

we can obtain improved quark-quark matrix elements by adjusting d_{1h},

$$\bar{c}\Gamma b = \bar{Q}_c \Gamma Q_b \Longleftrightarrow d_{1h} = \frac{\zeta}{2\sinh M_{1h}a} - \frac{1}{2m_h a}. \quad (h = b, c)$$

Improvement of Lattice Current - $\mathcal{O}(\lambda)$ case

- We show how to use HQET matching to demonstrate improvement in the current at $\mathcal{O}(\lambda)$.
- Flavor-changing vector current can be described by, [2]

$$V^\mu \doteq \bar{C}_{V_\parallel} v^\mu \bar{c}_{v'} b_v + \bar{C}_{V_\perp} \bar{c}_{v'} i\gamma^\mu_\perp b_v + \bar{C}_{V_{v'}} {v'}^\mu_\perp \bar{c}_{v'} b_v - \sum_i \bar{B}_{V_i} \bar{\mathcal{Q}}^\mu_{V_i}$$

  where $\ell^\mu_\perp = \ell^\mu + v^\mu v \cdot \ell$, for arbitrary vector ℓ^μ. $(v^2 = -1)$
  $\bar{\mathcal{Q}}^\mu_{V_i}$: dimension 4 operators.
- For the lattice, we can describe bilinear with the right quantum numbers of flavor-changing vector currents in the basis of HQET operators.[2]

$$V^\mu_{\text{lat}} \doteq \bar{C}^{\text{lat}}_{V_\parallel} v^\mu \bar{c}_{v'} b_v + \bar{C}^{\text{lat}}_{V_\perp} \bar{c}_{v'} i\gamma^\mu_\perp b_v + \bar{C}^{\text{lat}}_{V_{v'}} {v'}^\mu_\perp \bar{c}_{v'} b_v - \sum_i \bar{B}^{\text{lat}}_{V_i} \bar{\mathcal{Q}}^\mu_{V_i}$$

- The matching between QCD and lattice vector current is obtained by $\delta\bar{B}_{V_i} = \bar{B}_{V_i} - Z_{V_i}\bar{B}^{\text{lat}}_{V_i} = 0$, where

$$Z_{V_i} = \bar{C}_{V_\perp}/\bar{C}^{\text{lat}}_{V_\perp} \quad \text{when} \quad \bar{\mathcal{Q}}^\mu_{V_i} \perp v^\mu,$$
$$Z_{V_i} = \bar{C}_{V_\parallel}/\bar{C}^{\text{lat}}_{V_\parallel} \quad \text{when} \quad \bar{\mathcal{Q}}^\mu_{V_i} \parallel v^\mu.$$

Improvement of Lattice Current - $\mathcal{O}(\lambda)$ case

- We can define V^{μ}_{lat} in terms of the rotated field Q_h and dimension four lattice operators $O^{\mu}_{V_i}$. [2]

$$V^{\mu}_{\mathsf{lat}} = \bar{Q}_c i\gamma^{\mu} Q_b - \sum_i b_{V_i} O^{\mu}_{V_i}.$$

- By matching V^{μ}_{lat} to the HQET operators, one can obtain $\bar{B}^{\mathsf{lat}}_{V_i}$ in terms of b_{V_i} and d_{1h}.
- We can make $\delta\bar{B}_{V_i} = 0$ while keeping all $b_{V_i} = 0$, if we adjust d_{1h},

$$d_{1h} = \frac{\zeta}{2\sinh M_{1h}a} - \frac{1}{2m_h a}. \qquad (h = b, c)$$

Improvement of Lattice Current - $\mathcal{O}(\lambda^3)$ case

- For λ^3 improvement, we define field rotation as follows [5], $(a = 1)$

$$
\begin{aligned}
Q(x) = & e^{M_1/2}\Big[1 + d_1\vec{\gamma}\cdot\vec{D}_{\text{lat}} + \frac{1}{2}d_2\Delta^{(3)}_{\text{lat}} + \frac{1}{2}id_B\vec{\Sigma}\cdot\vec{B}_{\text{lat}} + \frac{1}{2}d_E\vec{\alpha}\cdot\vec{E}_{\text{lat}} \\
& + \frac{1}{4}d_{r_E}\{\vec{\gamma}\cdot\vec{D}_{\text{lat}}, \vec{\alpha}\cdot\vec{E}_{\text{lat}}\} + \frac{1}{4}d_{z_E}\gamma_4(\vec{D}_{\text{lat}}\cdot\vec{E}_{\text{lat}} - \vec{E}_{\text{lat}}\cdot\vec{D}_{\text{lat}}) \\
& + \frac{1}{6}d_3\gamma_i D_{i\text{lat}}\Delta_{i\text{lat}} + \frac{1}{2}d_4\{\vec{\gamma}\cdot\vec{D}_{\text{lat}}, \Delta^{(3)}_{\text{lat}}\} + \frac{1}{4}d_5\{\vec{\gamma}\cdot\vec{D}_{\text{lat}}, i\vec{\Sigma}\cdot\vec{B}_{\text{lat}}\} \\
& + \frac{1}{4}d_{EE}\{\gamma_4 D_{4\text{lat}}, \vec{\alpha}\cdot\vec{E}_{\text{lat}}\} + \frac{1}{4}d_{z_3}\vec{\gamma}\cdot(\vec{D}_{\text{lat}}\times\vec{B}_{\text{lat}} + \vec{B}_{\text{lat}}\times\vec{D}_{\text{lat}})\Big]\psi(x)
\end{aligned}
$$

- And define flavor-changing vector and axial vector current.

$$
\begin{aligned}
V_I^\mu &= \bar{Q}_c i\gamma^\mu Q_b, \\
A_I^\mu &= \bar{Q}_c i\gamma^\mu\gamma_5 Q_b.
\end{aligned}
$$

- Then, obtaining improved flavor-changing currents is equivalent to obtaining parameters d_i which suffice to match to continuum QCD.

Matching Calculation I

- $\langle c(\eta_c, \vec{p}_c)|V_I^\mu|b(\eta_b, \vec{p}_b)\rangle_{\mathsf{lat}} = \langle c(\eta_c, \vec{p}_c)|V^\mu|b(\eta_b, \vec{p}_b)\rangle_{\mathsf{cont}}$
  At tree-level [1],

$$
\begin{aligned}
&\to \mathcal{N}_c(\vec{p}_c)\mathcal{N}_b(\vec{p}_b)\ \bar{u}_c^{\mathsf{lat}}(\eta_c, \vec{p}_c)\bar{R}_c(\vec{p}_c)i\gamma^\mu R_b(\vec{p}_b)u_b^{\mathsf{lat}}(\eta_b, \vec{p}_b) \\
&= \sqrt{\frac{m_c}{E_c}}\sqrt{\frac{m_b}{E_b}}\ \bar{u}_c(\eta_c, \vec{p}_c)i\gamma^\mu u_b(\eta_b, \vec{p}_b),
\end{aligned}
$$

  where $R(\vec{p})$ is the rotation vertex of improved current.
- Expand continuum and lattice spinor in $\mathcal{O}(\vec{p}^{\,3})$ [5], $(a = 1)$

$$
\sqrt{\frac{m_h}{E}}u(\eta, \vec{p}) = \Big[1 - \frac{i\vec{\gamma}\cdot\vec{p}}{2m} - \frac{\vec{p}^{\,2}}{8m^2} + \frac{3i(\vec{\gamma}\cdot\vec{p})\vec{p}^{\,2}}{16m^3}\Big]u(\eta, 0) + \mathcal{O}(\vec{p}^{\,4})
$$

$$
\begin{aligned}
\mathcal{N}(\vec{p})u^{\mathsf{lat}}(\eta, \vec{p}) = &\Big[1 - \frac{i\zeta\vec{\gamma}\cdot\vec{p}}{2\sinh M_1} - \frac{\vec{p}^{\,2}}{8M_X^2} + \frac{1}{6}iw_3\sum_{k=1}^{3}\gamma_k p_k^3 + \frac{3i(\vec{\gamma}\cdot\vec{p})\vec{p}^{\,2}}{16M_Y^3}\Big]u(\eta, 0) \\
&+ \mathcal{O}(\vec{p}^{\,4})
\end{aligned}
$$

Matching Calculation I

- M_X, M_Y, and rotation breaking parameter w_3 are expressed by couplings of action [1, 5],

$$\frac{1}{M_X^2} = \frac{\zeta^2}{\sinh^2 M_1} + \frac{2r_s\zeta}{e^{M_1}}, \qquad w_3 = \frac{3c_1 + \zeta/2}{\sinh M_1},$$
$$\frac{1}{M_Y^3} = \frac{8}{3\sinh M_1}\left\{2c_2 + \frac{1}{4}e^{-M_1}\left[\zeta^2 r_s(2\coth M_1 + 1) + \frac{\zeta^3}{\sinh M_1}\left(\frac{e^{-M_1}}{2\sinh M_1} - 1\right)\right] + \frac{\zeta^3}{4\sinh^2 M_1}\right\}.$$

- The rotation vertex remedies mismatch of continuum and lattice spinor in order $\mathcal{O}(\vec{p}^{\,3})$.
  $\rightarrow$ obtain d_1, d_2, d_3, d_4.

Matching Calculation II

- To obtain rotation parameters other than d_1, d_2, d_3, d_4, we are considering matrix elements.

$$\langle q(\eta_2,\vec{p}_2)c(\eta_c,\vec{p}_c)|V_I^\mu|b(\eta_b,\vec{p}_b)q(\eta_1,\vec{p}_1)\rangle_{\text{lat}}$$

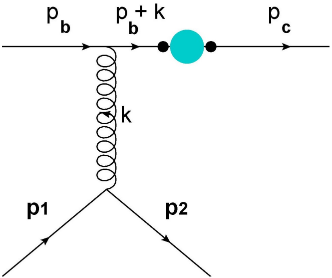

Figure: A diagram for gluon exchange from the OK action one-gluon vertex of b-quark for improved flavor-changing current at tree-level. The large circles represent electroweak current insertions, and the small dots represent field rotation.

- The matching equation for the one-gluon exchange from b-quark line is given by,

$$n_\nu(k)\mathcal{N}(\vec{p}_c)\mathcal{N}(\vec{p}_b)\ \bar{u}^{\text{lat}}(\eta_c,\vec{p}_c)\bar{R}(\vec{p}_c)i\gamma_\mu \Big[R(\vec{p}_b)S^{\text{lat}}(p_b+k)\Lambda_\nu(p_b+k,p_b) + R_\nu(p_b+k)\Big]u^{\text{lat}}(\eta_b,\vec{p}_b) = \sqrt{\frac{m_c}{E_c}}\sqrt{\frac{m_b}{E_b}} \times \bar{u}(\eta_c,\vec{p}_c)i\gamma_\mu S^{\text{cont}}(p_b+k)\gamma_\nu u(\eta_b,\vec{p}_b).$$

$n_\nu(k) = \frac{(2/a)\sin(k_\nu/a)}{k_\nu}$, $\nu = i$ or 4.
R_ν : one-gluon vertex of field rotation.
Λ_ν : one-gluon vertex of OK-action.

Summary

- d_i $(i = 1, 2, 3, 4)$ results at tree-level are given in [5].
- We are caclulating the other rotation parameters to complete $\mathcal{O}(\lambda^3)$ current improvement by calculating four-quark matrix element.

$$\langle q(\eta_2, \vec{p}_2) c(\eta_c, \vec{p}_c) | V_I^{\mu} | b(\eta_b, \vec{p}_b) q(\eta_1, \vec{p}_1) \rangle.$$

- We are enumerating dimension five and six HQET operators to verify the currents constructed with the rotated fields are improved through $\mathcal{O}(\lambda^3)$ in HQET.

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