Renormalizability of the Schrödinger Functional

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Quantum Field Theory with Boundaries

- It is of interest to study quantum field theories with boundaries:
    - Schrödinger functional;
    - Casimir effect.
- Boundary conditions may be imposed by surface interactions in the action (K. Symanzik, *Nucl. Phys.* **B190**, 1–44, 1981).
- Our proof is independent of regulator: we shall use continuum notation for simplicity, but results apply equally well on the lattice.
- The proof is in Euclidean space: extension to Minkowski space as distributions presumably follows by partial integration onto test functions (Lowenstein and Speer, *Commun. Math. Phys.* **47**, 43–51, 1976).
- For a scalar field ϕ with the Lagrangian $\mathcal{L} = \frac{1}{2}(\partial\phi)^2 + \frac{1}{2}m^2\phi^2 + \frac{1}{4!}\lambda\phi^4$ we add the surface term $K = \frac{1}{2}c\phi_-\delta'(\sigma)\phi_+$ with $c = \pm 1$.
    - The function σ vanishes on the wall.
        - For a planar wall that is orthogonal to a unit vector w and a distance ℓ from the origin we could take $\sigma(x) = x\cdot w - \ell$.
        - In general we can take σ to be a smooth function corresponding to a wall that is topologically equivalent to a plane.
    - $\phi_\pm$ is the field on either side of the wall.

Quadratic Interactions

- The boundary conditions are imposed by a local interaction that is quadratic in the field ϕ.
- There is no small parameter associated with this wall interaction.
- This is analogous to the mass term $\frac{1}{2}m^2\phi^2$. We can
  - either treat this as part of the propagator, $(k^2 + m^2)^{-1}$;
  - or treat it "perturbatively" as a two-point vertex $-m^2$ with the massless propagator $\Delta = 1/k^2$.
  - In the latter case we can sum the two-point interactions to all orders in m

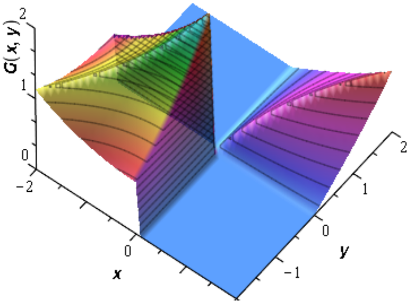
$$\begin{aligned}\Delta_M &= \Delta + \Delta(-m^2)\Delta + \Delta(-m^2)\Delta(-m^2)\Delta + \cdots = \Delta \sum_{n=0}^{\infty} \left[(-m^2)\Delta\right]^n \\ &= \Delta + \Delta(-m^2)\Delta_M = \frac{\Delta}{1 + m^2\Delta} = \frac{1}{k^2 + m^2}.\end{aligned}$$

  - This series only converges for $k^2 \leq m^2$.
  - It has a unique analytic continuation $\forall k^2 \neq m^2$, even though there is no small parameter
  - The mass renormalization is $m^2 \to m^2 + \delta m^2$, where δm^2 is treated as a countervertex order by order in the loop expansion.

Integral Equation

- The Green's function $H(x,y)$ for the quadratic kernel without walls $L(x,y) = \delta(x-y)(-\partial^2 + m^2)$ satisfies $\int dz\, L(x,z)H(z,y) = \delta(x,y)$, which we shall abbreviate as $LH = 1$.
  - Because there are no walls H is translationally invariant and is only a function of $x-y$. We require that $\lim_{|x-y|\to\infty} H(x-y) = 0$.
- The Green's function $G(x,y)$ for the full quadratic kernel $L+K$ where the wall interaction is $K(x,y) = \int dz\, \delta(x-z_-)\delta(y-z_+)\delta'(\sigma(z))$ satisfies $(L+K)G = 1$, where $z_\pm = z \pm \varepsilon\partial\sigma$ with $\varepsilon \to 0$.
- We may thus find G "non-perturbatively" by solving the integral equation $H(L+K)G = H \Rightarrow G = H - HKG$.
  - $G(x,y)$ is not translationally invariant, so it is not just a function of $x-y$.
- We require $G(x_-, x_+) = 0$, so the two sides of the wall are decoupled. The propagator's derivatives must also vanish across the wall.
- Since the propagator G vanishes across the wall so does any connected Green's function that couples points on opposite sides of the wall, as it is a convolution of propagators.

Solution of Integral Equation



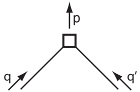
- This solution satisifies Dirichlet boundary conditions on one side and Neumann boundary conditions on the other. Changing the sign of the wall interaction interchanges these.

Feynman Rules

- As well as the usual bulk divergences we have new divergences associated with wall vertices.
- For simplicity we consider the wall $\sigma(x) = x_0 - \ell$.
- The wall vertex is $K(x, y) = \int dz\, \delta(x - z_-)\delta(y - z_+)\delta'(z_0 - \ell)$.
- In momentum space this is

$$\begin{aligned}
\tilde{K}(q, q') &= \int \frac{dx\, dy}{(2\pi)^D} K(x, y) e^{-i(q\cdot x + q'\cdot y)} = \int \frac{dz}{(2\pi)^D} e^{-i(q\cdot z_- + q'\cdot z_+)} \delta'(z_0 - \ell) \\
&= \frac{i}{2\pi} (q + q')_0 e^{-i\ell(q+q')_0} e^{i\varepsilon(q-q')_0} \delta((q + q')_\perp) \\
&= \frac{i}{2\pi} \int dp\, p_0 e^{-i\ell p_0} \delta(q + q' - p) \delta(p_\perp) e^{i\varepsilon' \operatorname{sgn}(q-q')_0}.
\end{aligned}$$

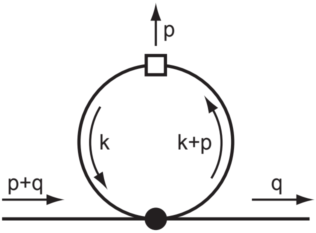
- The location of the wall is specified by the phase $e^{-i\ell p_0}$, and its orientation by the dependence on the sign of $(q - q')_0$.



- We have associated an "external" momentum p with the wall source so that momentum is conserved at the wall vertex.

Single Wall Vertex

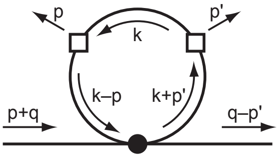
- Consider the one-loop diagram contributing to the two-point function that includes a single wall vertex.
- This is logarithmically divergent in four dimensions, and therefore its divergent part is independent of q and is proportional to the wall vertex $\tilde{K}(p+q, q)$.
- This divergence may be absorbed into a renormalization of the coefficient of the wall vertex, $c \to c + \delta c$.



- We may impose the renormalization condition that the finite part of $\delta c = 0$ so as to maintain the decoupling of the two sides of the wall.

Multiple Wall Vertices

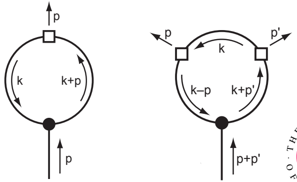
- Consider the one-loop diagram contributing to the two-point function that includes two wall vertices.
- This is logarithmically divergent in six dimensions, and therefore its divergent part is independent of q, but it is *not* proportional to a single wall vertex.
- This divergence is *not* localized on the wall, and cannot be absorbed into a renormalization of the coefficient of the wall vertex.



- In general, if more than one wall vertex appears in an overall divergent graph then the divergence is not localized on the wall.
- Do not be distracted by the fact that ϕ^4 theory without walls is not renormalizable in six dimensions.

Power Counting

- We can easily apply Dyson's power-counting threorem to wall vertices.
- In our example the wall interaction monomial in the action in D-dimensions has dimension $D-2$, just like a mass term.
- Therefore for $D=4$ the only overall divergent n-point functions with one wall vertex must have $n \leq 2$.
- If there are two or more wall vertices then $n \leq 0$.
- $n=1$ is forbidden by $\phi \to -\phi$ symmetry, and $n=0$ is uninteresting, so the only new counterterm required is proportional to the wall vertex and is therefore localized on the wall.
- In ϕ^3 theory we also need to consider divergent tadpoles. These contribute to a non-uniform background source $J(x)$ for the field ϕ, but not to any coupling of the opposite sides of the wall.



Conclusions

- Momentum is not conserved at a wall vertex: this is not suprising, as the wall violates translational invariance. This corresponds to an "external" momentum p flowing into the wall.
- The wall interaction monomial in the action always has the same power-counting dimension as the mass term.
- Imposition of boundary conditions on the field by a local wall interaction induces counterterms that remain localized on the wall to all orders in perturbation theory provided that no more than one wall vertex can appear in any overall divergent two-point function.