Stochastic Perturbation Theory and Gradient Flow in ϕ^4 Theory

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Abstract

We investigate different methods for solving Lattice Perturbation Theory in ϕ^4 theory in a finite periodic box in four dimensions, defined by the action

$$S(\varphi) = \int d^4x \left\{ \frac{1}{2}\partial_\mu\varphi(x)\partial_\mu\varphi(x) + \frac{1}{2}m^2\varphi(x)^2 + \frac{g}{4!}\varphi(x)^4 \right\}. \quad (1)$$

We studied both Instantaneous Stochastic Perturbation Theory (ISPT) [1] and numerical perturbation theory methods based on the Langevin (LSPT) [2] and Hybrid Molecular Dynamics equations (HSPT).

ISPT

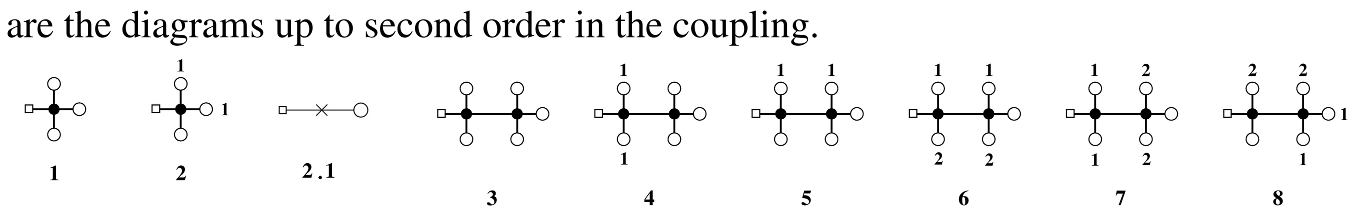
ISPT is based on the trivializing map that transforms Gaussian distributed random fields η into stochastic fields ϕ such that

$$\langle \phi(x_1)...\phi(x_n)\rangle_\eta = \langle \varphi(x_1)...\varphi(x_n)\rangle,$$

where φ is the quantum field of the action (1). The stochastic field can be expressed to any finite order in perturbation theory as series of rooted tree diagrams $v(x, \mathcal{R}_i)$

$$\phi(x) = \sum_n \phi_n g_0^n = \sum_i c_i v(x, \mathcal{R}_i).$$

These are the diagrams up to second order in the coupling.



We wrote a code for the automatic computation of these diagrams. Our routine starts from the root vertex and explores all the vertices recursively to generate the field. As a test we computed the coupling defined in [3]

$$g = -\frac{\chi_4}{\chi_2}m^4 = g_0 + c_1 g_0^2 + c_2 g_0^3 + O(g_0^4),$$

where χ_2 and χ_4 are the connected two- and four-point functions at zero momenta. We computed it in both bare and renormalized theory and the results are in agreement with the analytic value.

$L = 4\ z = 6$	c_1	c_2
analytic m_R	$-1.98 \cdot 10^{-2}$	$6.0 \cdot 10^{-4}$
ISPT m_R	$-2.00(6) \cdot 10^{-2}$	$5.9(2) \cdot 10^{-4}$
analytic m_0	$-2.15 \cdot 10^{-2}$	$1.09 \cdot 10^{-3}$
ISPT m_0	$-2.24(8) \cdot 10^{-2}$	$1.11(2) \cdot 10^{-3}$

Table 1: ISPT results for g obtained with 10^7 configurations

We considered it since three loop results are available. However in the computation of this coupling there are big cancellation of the disconnected part that lead to large statistical errors, so we considered other observables in the following.

LSPT

Another way to generate stochastic fields that represent the theory is via a Markov process generated according to the Langevin equations

$$\partial_{t_s}\phi(x,t_s) = \partial^2\phi(x,t_s) - (m^2+\delta m^2)\phi(x,t_s) - \frac{g_0}{3!}\phi(x,t_s)^3 + \eta(x,t_s),$$

where η is a Gaussian distributed field. The equation is discretized in the stochastic time $\varepsilon = \delta t^2$ (δt is the integration step in HSPT) and is integrated order by order in g_0 using a second order Runge-Kutta scheme, thus we expect $O(\varepsilon^2)$ errors in our observables. It is necessary to extrapolate to $\varepsilon = 0$.

HSPT

We also considered whether other stochastic differential equations based on HMD equations may improve LSPT:

$$\partial_{t_s}\phi = \pi, \qquad \partial_{t_s}\pi = \partial^2\phi(x,t_s) - (m^2+\delta m^2)\phi(x,t_s) - \frac{g_0}{3!}\phi(x,t_s)^3.$$

The momentum fields π are also considered as a formal series in the coupling, and they are sampled from a Gaussian distribution at the beginning of each trajectory. At the start of each trajectory the momenta only have a non-zero lowest component, they will acquire higher-order components during the MD evolution.

- In order to integrate these equations order by order in g_0, we employ a 4th order symplectic integrator. We thus expect errors $O(\delta t^4)$ in the integration step-size δt, which is the same as Langevin.
- We randomized the trajectory length τ in order to update all frequency components of the field ϕ at the lowest order in perturbation theory, i.e. the free field ϕ_0 [4].
- We considered adjusting the average trajectory length $\langle\tau\rangle$ to be proportional to the correlation length, hoping to obtain a dynamical critical exponent $z = 1$.

Gradient Flow observable

We considered the gradient flow equation [5] which in the case of ϕ^4 theory can be taken to be:

$$\partial_t\phi(x,t) = \partial^2\phi(x,t).$$

Using the Gradient Flow we can define convenient observables which possess a well defined continuum limit.

$$t^2\langle E\rangle = t^2\langle\phi(t)^4\rangle = E_0 + E_1 g_0 + E_2 g_0^2 + E_3 g_0^3 + O(g_0^4).$$

We studied the continuum limit of E keeping the box size L fixed, as we changed the resolution of our lattice. We must scale all other dimensionful quantities, in particular the mass and the flow time; we therefore introduced the dimensionless constants

$$z = mL, \qquad c = \sqrt{8t}/L.$$

To consider the renormalization of the mass, we expanded the field in both δm and g_0, and we computed the corresponding mass counterterms for the field stochastically,

$$\phi = \sum_{j,k}\phi_{(j,k)}\delta m^j g_0^k.$$

We then imposed the following renormalization condition in order to find the expansion of δm in g_0 so we could express ϕ in a single series in g_0,

$$\frac{\chi_2}{\chi_2^*} = \left(1 + \frac{\hat{p}_*^2}{m^2}\right), \qquad p_* = (2\pi/L, 0, 0, 0),$$

where $\hat{p}$ is the lattice momentum $\hat{p}_\mu = 2\sin(p_\mu/2)$ and χ_2^* is the connected two point function evaluated at p_*. We computed this observable in closed form using position space methods up to g_0 order for comparison with stochastic estimates.

Results

First we tested that all the stochastic methods are consistent with the correct value. Figure 1 shows the expectation value of E_1 at $L/a = 4$. For LSPT we extrapolated to $\varepsilon = 0$, in this case we did not need to do so for HSPT. We examined how the errors scale in the continuum limit keeping the number of configuration generated in ISPT, LSPT and HSPT fixed (Fig: 2).

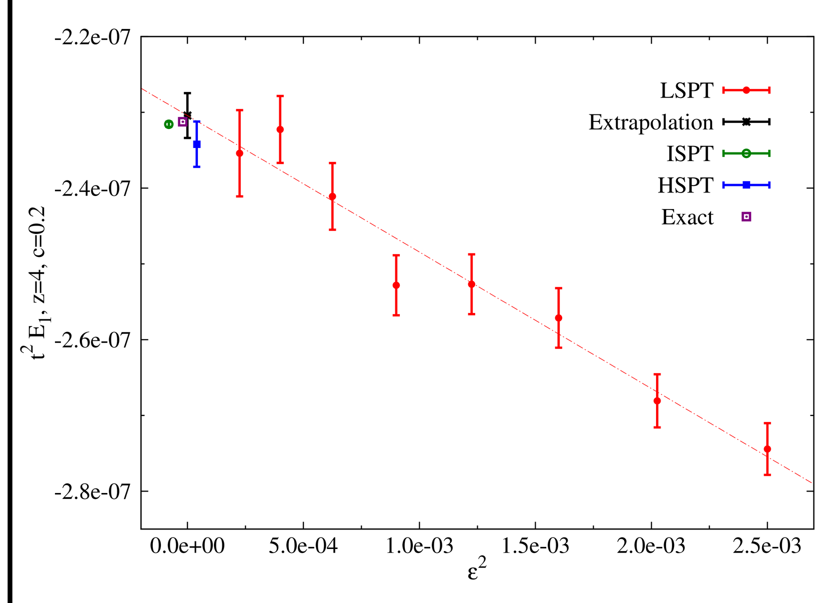


Figure 1: Comparison of different methods in the determination of E_1

In LSPT and HSPT we saw the expected behaviour, consistent with the errors increasing with the integrated autocorrelation (Figures 2.1–2.3). In the case of HSPT, which we kept the average trajectory length $\langle\tau\rangle = 1$, the cost does not only increase for bigger volumes, but also because we reduced the step size $\delta t \propto 1/L$ in order to keep the error in the hamiltonian fixed.

In LSPT we observed that the statistical errors grow like $1/\sqrt{\varepsilon}$ which is consistent with the random walk behaviour with the step size $\delta t = \sqrt{\varepsilon}$.

In the case of ISPT we noted a rapid increase of the errors with L for order g^n with $n > 1$.

In Figure 2.4 we present a plot with increased statistic (10^7 configuration), and we see that the errors scale roughly like L^0, L^2, L^3, L^5 for order $g_0^0, g_0^1, g_0^2, g_0^3$, respectively.

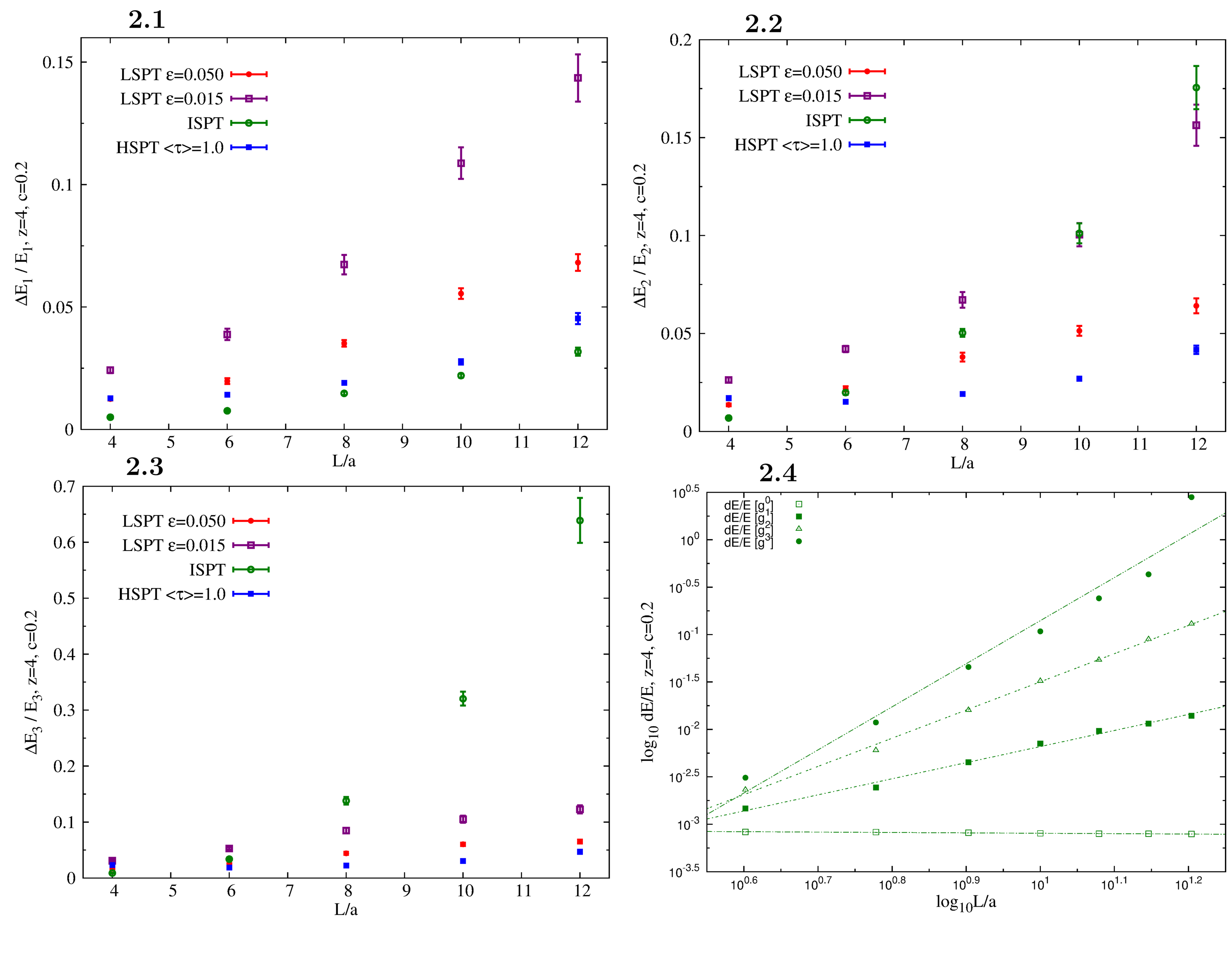


Figure 2: Scaling behaviour of the relative error of E_1 in 2.1, E_2 in 2.2, E_3 in 2.3. **Figure: 2.4:** ISPT only up to g_0^3

Finally, we investigated how the integrated autocorrelation A_i scales with L/a in HSPT: we compared the case $\langle\tau\rangle = 1$ with the case when $\langle\tau\rangle = L/a$ (Figure 3). The results are consistent with expectations. A_i at $\langle\tau\rangle = 1$ grows like a random walk behaviour $(L/a)^2$, whereas for $\langle\tau\rangle = L/a$ it is constant, i.e. the configurations are effectively independent.

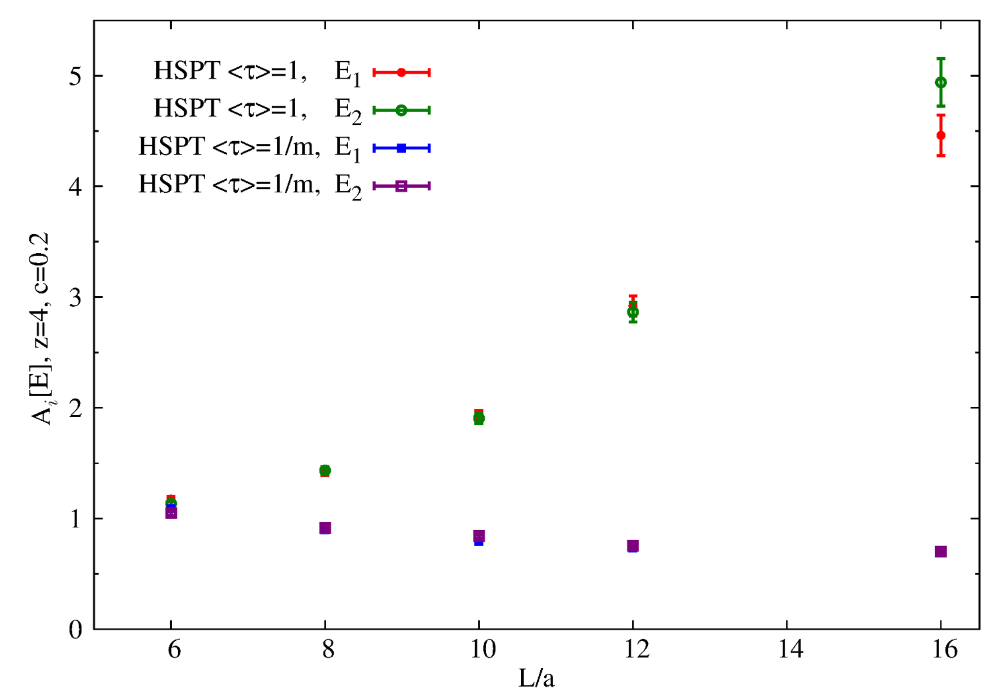


Figure 3: Scaling of A_i with $\langle\tau\rangle = 1$ and $\langle\tau\rangle = L/a$ for E_1 at orders g_0 and g_0^2

Conclusions

- The variance of ISPT grows catastrophically as we increase the order of g_0. It does not appear competitive in its present form.
- Relative merits of LSPT and HSPT, or more generally what the cheapest $\langle\tau\rangle$ is in HSPT for fixed errors is still to be determined.

Acknowledgements

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