Determination of f_K/f_π from 2+1+1 flavor 4-stout staggered lattices

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Lattice 2015
Kobe – 15.7.2015

Overview (1): Leptonic decays and CKM matrix elements

Leptonic decays of pseudoscalar mesons:

$$\Gamma(P \to \ell\bar{\nu}_\ell) = \frac{G_F^2}{8\pi}|V_{ij}|^2 f_P^2 m_\ell^2 M_P (1 - m_\ell^2/M_P^2)^2 \tag{1}$$

$\Gamma(P \to \ell\bar{\nu}_\ell)$ can be obtained from total width/lifetime and branching fraction in PDG. In other words, experiment determines product $|V_{ij}|\, f_P$.

Pion: $|V_{ud}| = 0.97425(22)$ is know with 0.23 permille precision [β^-, HardyTowner 08]
$\longrightarrow$ use it to get $f_\pi^{\mathrm{phys}} = 130.41(20)\,\mathrm{MeV}$ [PDG]

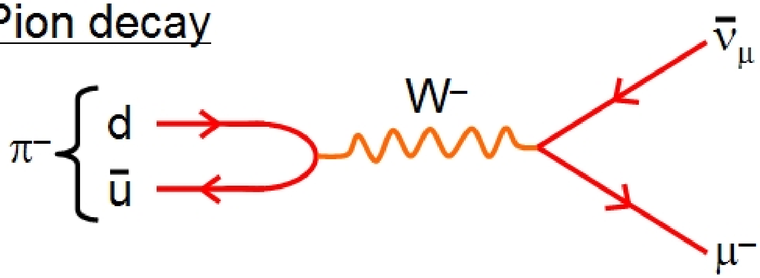
Kaon: $|V_{us}|$ is not known with high precision from other processes
$\longrightarrow$ calculate f_K on the lattice and determine $|V_{us}|$

To further enhance precision form ratio and calculate only f_K/f_π [Marciano 04]:

$$\frac{\Gamma(K \to \ell\nu_\ell)}{\Gamma(\pi \to \ell\nu_\ell)} = \frac{|V_{us}|^2}{|V_{ud}|^2} \frac{f_K^2}{f_\pi^2} \frac{M_K}{M_\pi} \frac{(1 - m_\ell^2/M_K^2)^2}{(1 - m_\ell^2/M_\pi^2)^2} (1 + \delta_{\mathrm{em}}) \tag{2}$$

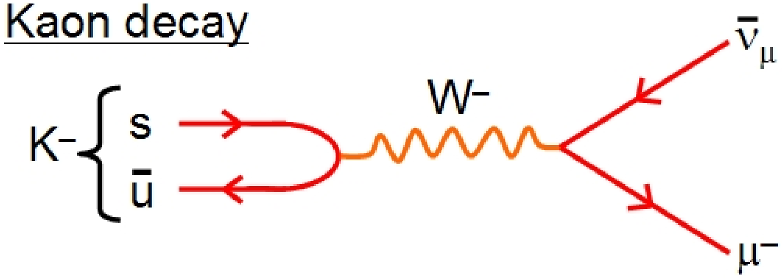
Overview (2): Leptonic decays on the lattice

Pion decay



$\pi^-(d\bar{u}) \to \mu^- + \bar{\nu}_\mu$

Kaon decay

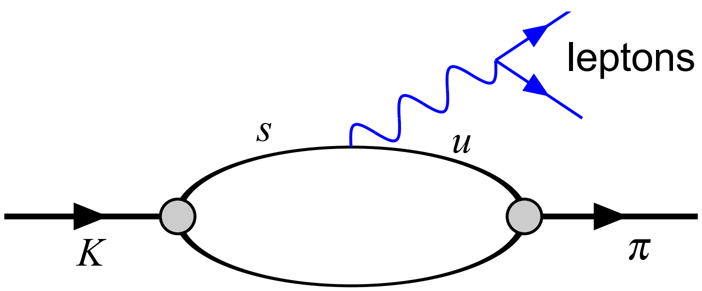


$K^-(s\bar{u}) \to \mu^- + \bar{\nu}_\mu$

$$J_\mu^{\mathrm{CC}} = (\bar{u}, \bar{c}, \bar{t})\gamma_\mu \frac{1}{2}[1-\gamma_5]\begin{pmatrix} d' \\ s' \\ b' \end{pmatrix}$$

$$\begin{pmatrix} d' \\ s' \\ b' \end{pmatrix} = \underbrace{\begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix}}_{V_{\mathrm{CKM}}} \begin{pmatrix} d \\ s \\ b \end{pmatrix}$$

$$\begin{aligned} \langle 0|(\bar{u}\gamma_\mu\gamma_5 d)(x)|\pi^-(p)\rangle &= \mathrm{i} f_\pi p_\mu e^{\mathrm{i}px} \\ \langle 0|(\bar{u}\gamma_\mu\gamma_5 s)(x)|K^-(p)\rangle &= \mathrm{i} f_K p_\mu e^{\mathrm{i}px} \end{aligned}$$



$\Rightarrow \quad V_{us}$

$\Longrightarrow$ strong dynamics restricted to matrix elements $\langle 0|A_\mu|\pi\rangle$, $\langle 0|A_\mu|K\rangle$ and form factors $\langle\pi|V_\mu|K\rangle$ etc.

Setup (1): Analysis and scale setting strategy

Action: tree-level Symanzik gluons

Action: staggered fermions with 4 steps of $\rho = 0.125$ stout smearing

$N_f = 2+1+1$: two flavors with a joint quark mass in the vicinity of physical
$m_{ud} = (m_u + m_d)/2$ via HMC (square root of det)
one flavor in the vicinity of physical m_s via RHMC (quarter root)
one flavor in the vicinity of physical m_c via RHMC (quarter root)

Goal: put simulation points below/above m_{ud} [physical quark mass by interpolation]
put simulation points below/above m_s [physical quark mass by interpolation]
fix $(am_c)/(am_s) = 11.85$ [HPQCD 09] so that charm is adjusted automatically
avoid big finite-volume artefacts by keeping $M_\pi L > 4$
simulate at several β-values to allow for $a \to 0$ extrapolation

Note: rudimentary chiral symmetry allows to calculate af_P without renormalization

Ergo: use af_π to set the scale
use $a^2 M_\pi^2$ to adjust up/down-quark mass
use $2a^2 M_K^2 - a^2 M_\pi^2$ to adjust strange-quark mass
treat af_K or f_K/f_π as the observable of interest

Setup (2): Overview of ensembles

β	am_{ud}	am_s	L/a	T/a	confs	binl
3.55	0.003113	0.0872	32	64	334	16
	0.003113	0.09124	32	64	340	17
	0.003241	0.0872	32	64	339	16
	0.003241	0.09124	32	64	339	16
3.60	0.002673	0.07435	40	64	307	15
	0.002673	0.07742	40	64	309	15
	0.002755	0.07435	40	64	315	15
	0.002755	0.07742	40	64	319	15
3.65	0.002318	0.0643	40	64	591	29
	0.002318	0.06695	40	64	593	29
	0.002388	0.0643	40	64	599	29
	0.002388	0.06695	40	64	598	29
	0.00234061	0.0660754	40	64	632	31
3.70	0.00205349	0.0572911	48	64	1800	90
	0.00205349	0.0590098	48	64	381	19
	0.00211509	0.0590098	48	64	362	18
	0.00208767	0.0560828	48	64	424	21
	0.00202596	0.0560828	48	64	437	21
	0.00205349	0.0572911	32	64	1603	80
	0.00205349	0.0572911	64	64	1032	51
	0.00205349	0.0572911	40	64	1670	83
3.75	0.00176877	0.049593	56	96	318	15
	0.00184096	0.049593	56	96	330	16
	0.00176877	0.0516173	56	96	331	16
	0.00184096	0.0516173	56	96	411	20
	0.001798	0.048644	56	96	201	10
3.84	0.00151556	0.0431935	64	96	401	20
	0.00151556	0.040150	64	96	348	17
	0.00143	0.0431935	64	96	358	17
	0.00143	0.040150	64	96	481	24
3.92	0.0012	0.0332856	80	128	202	10
	0.001207	0.032	80	128	132	6
	0.001172	0.03244	80	128	220	11
	0.001245	0.03424	80	128	394	19

Setup (3): Staggered correlators

The 2-point function $C_{ij}(t)$ of the point-like pseudoscalar density (that belongs to $\gamma_5 \otimes \xi_5$ in the spinor$\otimes$taste representation and thus couples to the pseudo-Goldstone state) with non-singlet flavor content $i\bar{j}$ assumes the form

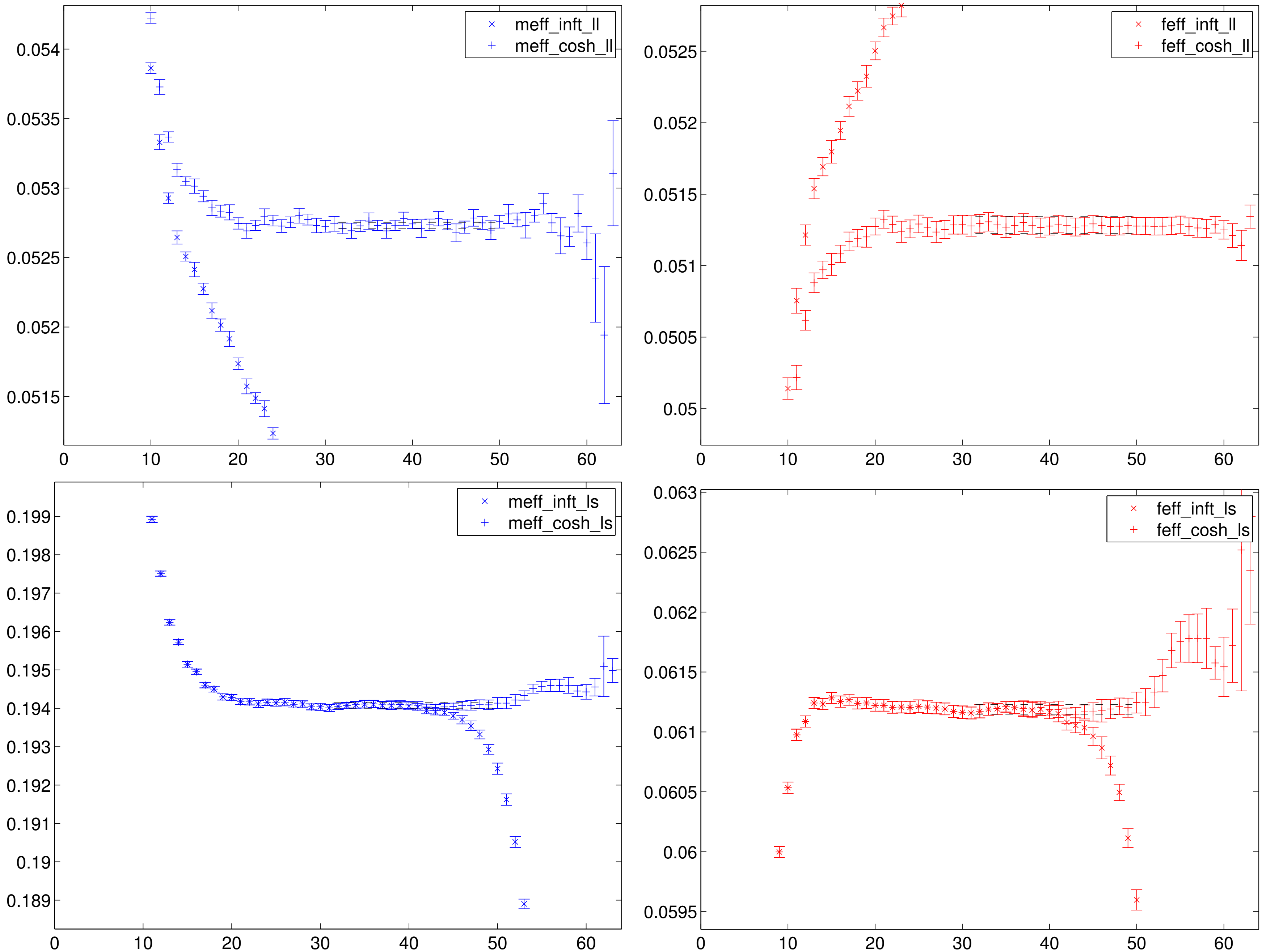
$$C(t) = \sum_{k=0}^{\infty} a_k \left[e^{-E_k t} + e^{-E_k (T-t)} \right] + (-1)^{t/a} \sum_{\ell=0}^{\infty} \tilde{a}_\ell \left[e^{-\tilde{E}_\ell t} + e^{-\tilde{E}_\ell (T-t)} \right] \qquad (3)$$

where the oscillatory term is missing for $m_i = m_j$. Ground state decay constant via

$$f_{ij} = (m_i + m_j) \sqrt{\frac{2a_0}{E_0^3}} \quad \longrightarrow \quad f_{ij} = \sqrt{\frac{a_0}{E_0^3}} \qquad (4)$$

in appropriate range. Check against standard effective mass or effective decay-constant plateaus (plots for finest lattice spacing, i.e. at $\beta = 3.92$) ...

Setup (4): Plateau examples



Setup (5): Correlation among M_π, f_π, M_K, f_K

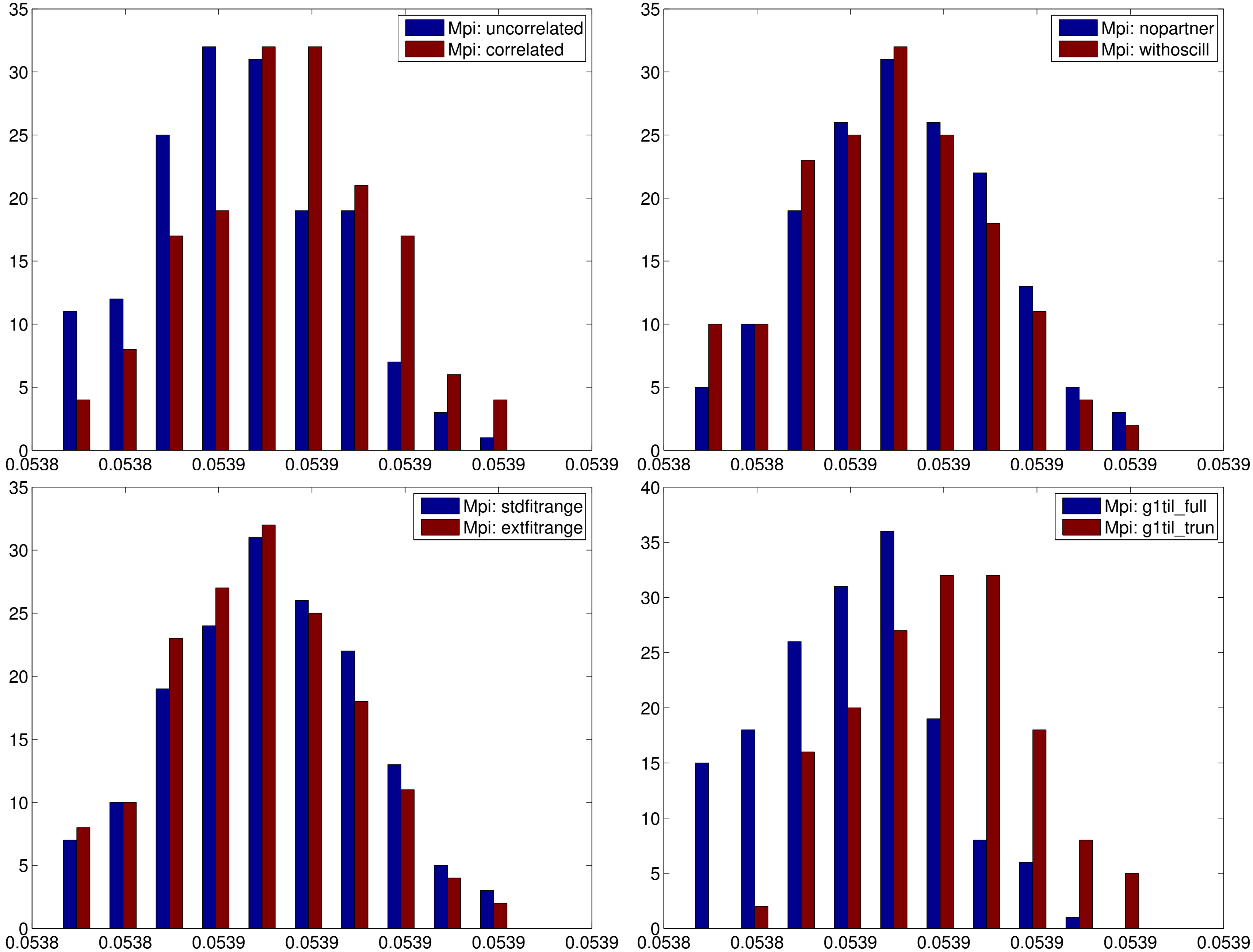
Example from one snapshot fit on one of our $\beta=3.92$ ensembles:

	aM_π	af_π	aM_K	af_K
cen	5.2740e-02	5.1278e-02	1.9408e-01	6.1185e-02
aM_π	8.4102e-10	-9.4434e-10	1.6672e-09	-3.9388e-10
af_π	-9.4434e-10	3.7703e-09	-2.4745e-09	2.1902e-09
aM_K	1.6672e-09	-2.4745e-09	5.8204e-09	-7.0353e-10
af_K	-3.9388e-10	2.1902e-09	-7.0353e-10	2.0703e-09
err	2.9000e-05	6.1403e-05	7.6292e-05	4.5501e-05
aM_π	1.0000e+00	-5.3032e-01	7.5352e-01	-2.9850e-01
af_π	-5.3032e-01	1.0000e+00	-5.2824e-01	7.8391e-01
aM_K	7.5352e-01	-5.2824e-01	1.0000e+00	-2.0267e-01
af_K	-2.9850e-01	7.8391e-01	-2.0267e-01	1.0000e+00
rel	5.4988e-04	1.1974e-03	3.9310e-04	7.4366e-04

Information regarding correlation is relevant to allow for switching to set of "tertiary" observables, e.g. the desired set $(M_\pi^2/f_\pi^2, [2M_K^2-M_\pi^2]/f_\pi^2, f_K/f_\pi)$.

On this ensemble $f_K/f_\pi=1.1932(9)$ is known with better-than-one-permille precision.

Setup (6): single-ensemble systematics



Finite size effects (1): ChPT predictions

ChPT predicts relative finite-volume effects in $M_P(L), f_P(L)$ [GasserLeutwyler 87]

$$M_\pi(L) = M_\pi \left\{1 + \frac{1}{2N_f}\xi\tilde{g}_1(M_\pi L) + O(\xi^2)\right\} \quad (5)$$

$$f_\pi(L) = f_\pi \left\{1 - \frac{N_f}{2}\xi\,\tilde{g}_1(M_\pi L) + O(\xi^2)\right\} \quad (6)$$

with $\xi = M_\pi^2/(4\pi F_\pi)^2 = M_\pi^2/(8\pi^2 f_\pi^2)$ and the shape function $\tilde{g}_1$ given by

$$\tilde{g}_1(z) = \frac{24}{z}K_1(z) + \frac{48}{\sqrt{2}z}K_1(\sqrt{2}z) + \frac{32}{\sqrt{3}z}K_1(\sqrt{3}z) + \frac{24}{2z}K_1(2z) + \ldots \quad (7)$$

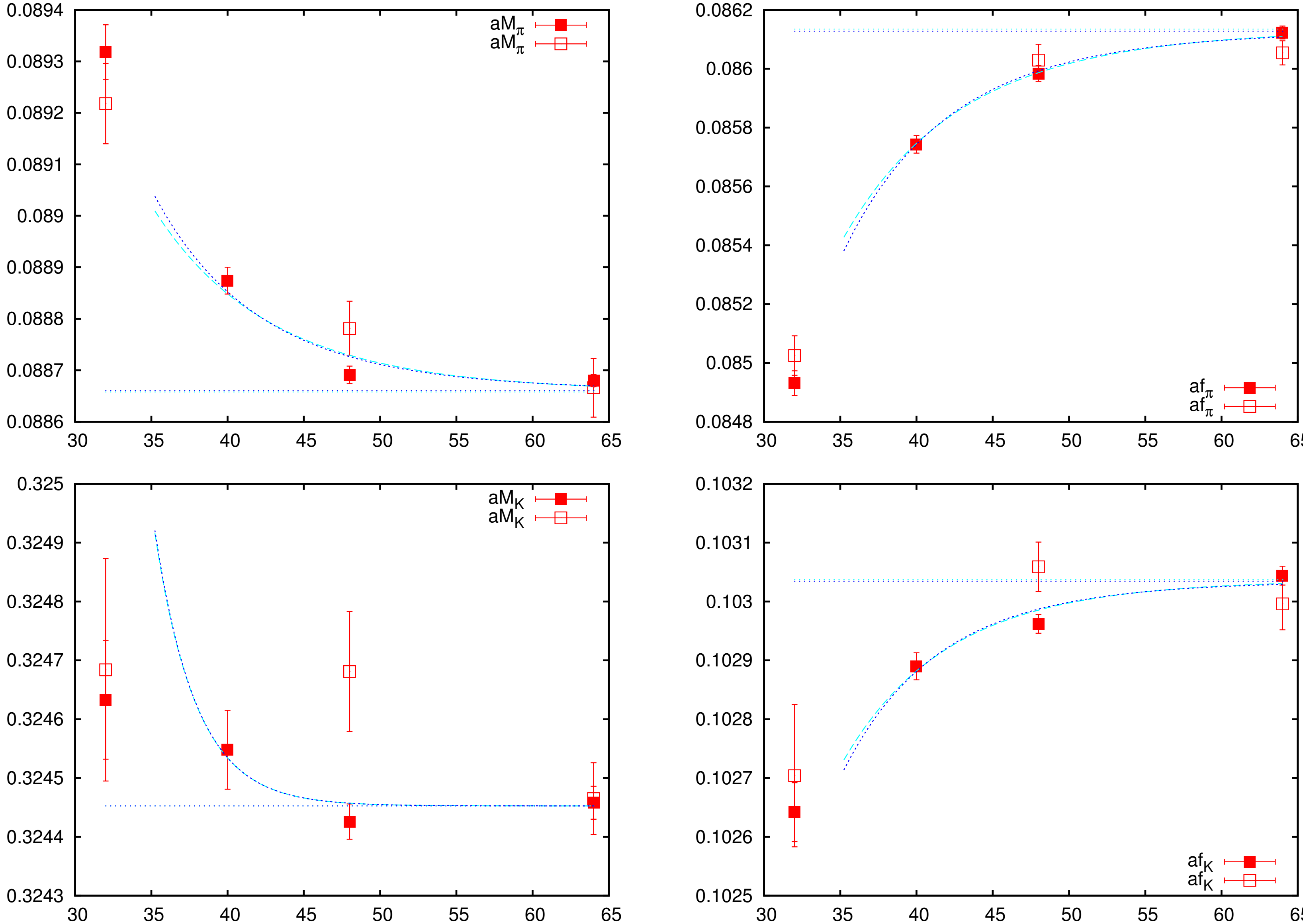
$$K_1(z) = \sqrt{\frac{\pi}{2z}}e^{-z}\left\{1 + \frac{3}{8z} - \frac{3\cdot5}{2(8z)^2} + \frac{3\cdot5\cdot21}{6(8z)^3} - \frac{3\cdot5\cdot21\cdot45}{24(8z)^4} + \ldots\right\} \quad (8)$$

but for realistic masses/volumes this expansion converges slowly [CD 03, CDH 05].

We ignore all analytical knowledge about higher-order terms and instead use (5,6) with re-fitted coefficients [to be compared to the NLO-predictions $1/(2N_f)$ and $-N_f/2$].

For an alternative with correct asymptotics we choose $\tilde{g}_1^{\text{trun}}(z) \equiv 12\sqrt{2\pi}e^{-z}z^{-3/2}$.

Finite size effects (2): Extra ensembles at $\beta = 3.7$



Normal ensemble at $L/a = 48$, extra ensembles at $L/a = 32, 40, 64$ (some run twice). Fitted coefficient in front of $\tilde{g}_1$ exceeds NLO prediction by factor 1.5 to 3.4 ("resum").

Autocorrelation (1): Definitions

Reminder: We chose $n_{\text{bins}} = 20$ which implies binlength$=O(10)$ on most ensembles.

We consider the usual estimator for the autocorrelation function of an observable X

$$\hat{A}_X(t) = \frac{1}{N-t-1} \sum_{i=1}^{N-t} (x_i - \overline{x_i})(x_{i+t} - \overline{x_{i+t}}) \tag{9}$$

whereupon the autocorrelation function at $t = 0$ is the usual estimator of the variance

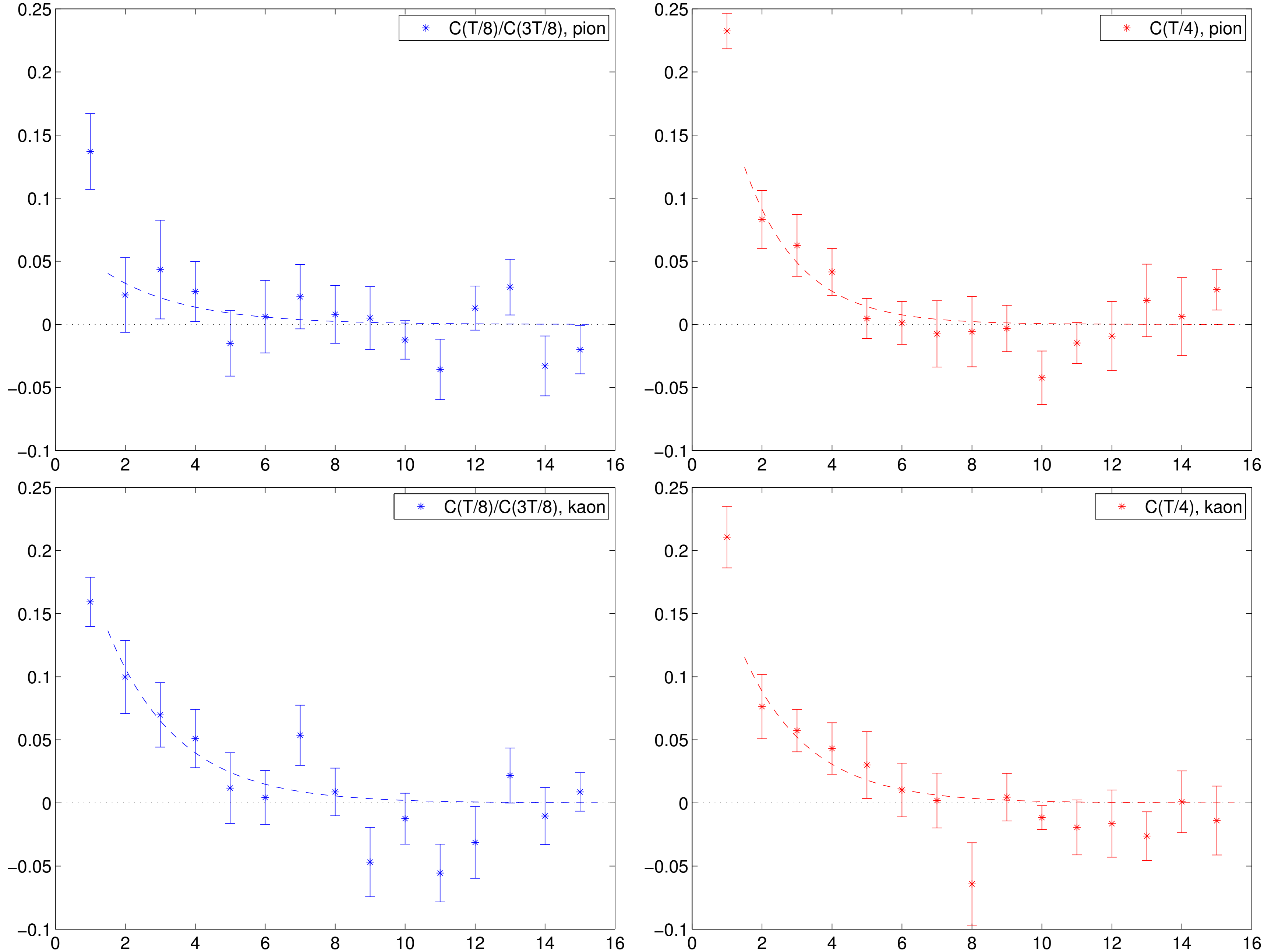
$$\hat{A}_X(0) = \frac{1}{N-1} \sum_{i=1}^{N} (x_i - \overline{x_i})^2 = \hat{V}(x) \tag{10}$$

and in the subsequent figure we show the normalized autocorrelation function

$$\hat{\Gamma}_X(t) = \hat{A}_X(t)/\hat{A}_X(0) \tag{11}$$

for $X = C(T/8)/C(3T/8)$ [our proxy for the meson mass, left], and $X = C(T/4)$ [our proxy for the meson decay constant, right], for the pion (top) and the kaon (bottom).

Autocorrelation (2): Justification of binlength$=O(10)$



Mass interpolation (1): choice of coordinates and input values

Reminder: x-coordinate is M_π^2/f_π^2, which value reflects $m_{ud}^\text{phys} \equiv m_l^\text{phys}$?
Reminder: y-coordinate is $(2M_K^2 - M_\pi^2)/f_\pi^2$, which value reflects m_s^phys ?

FLAG (I and II) discusses isospin-corrections in quite some detail. Our simulations do not contain (electromagnetic or strong) isospin breaking effects, therefore physical input values must be corrected for this artefact. Overall recommendation is to use

$$M_\pi^\text{symm} = 134.8(3)\,\text{MeV}\;, \quad M_K^\text{symm} = 494.2(4)\,\text{MeV}\;, \quad f_\pi^\text{phys} = 130.41(20)\,\text{MeV}$$

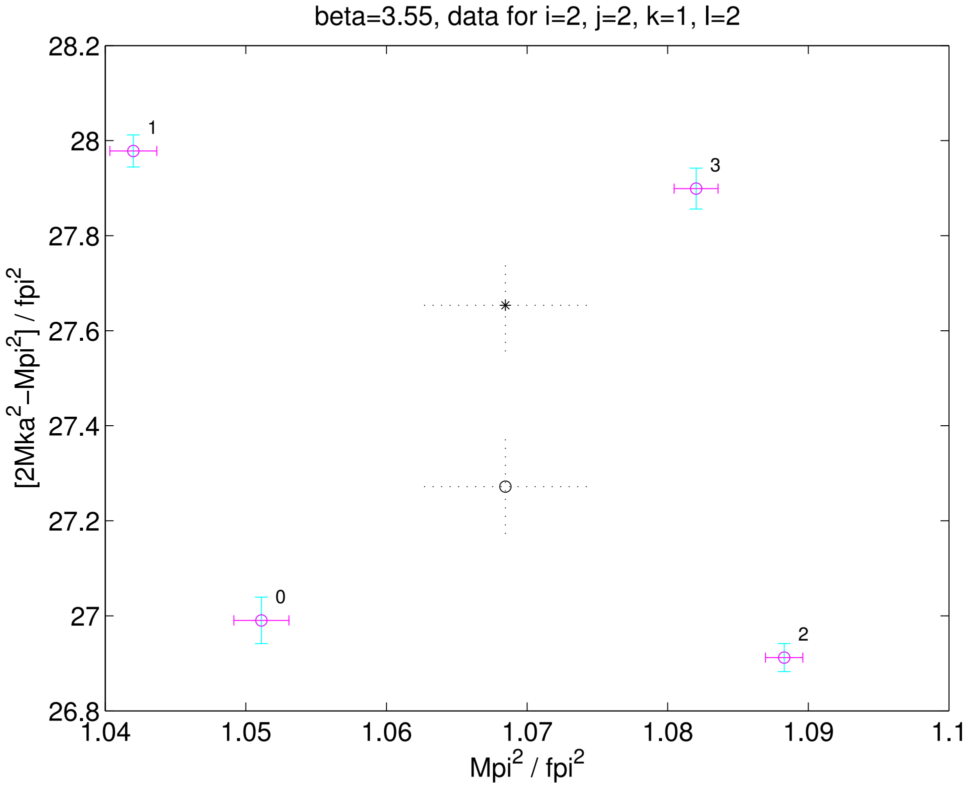
where "symm" stands for "isospin-symmetric" and the value of f_π is from PDG.

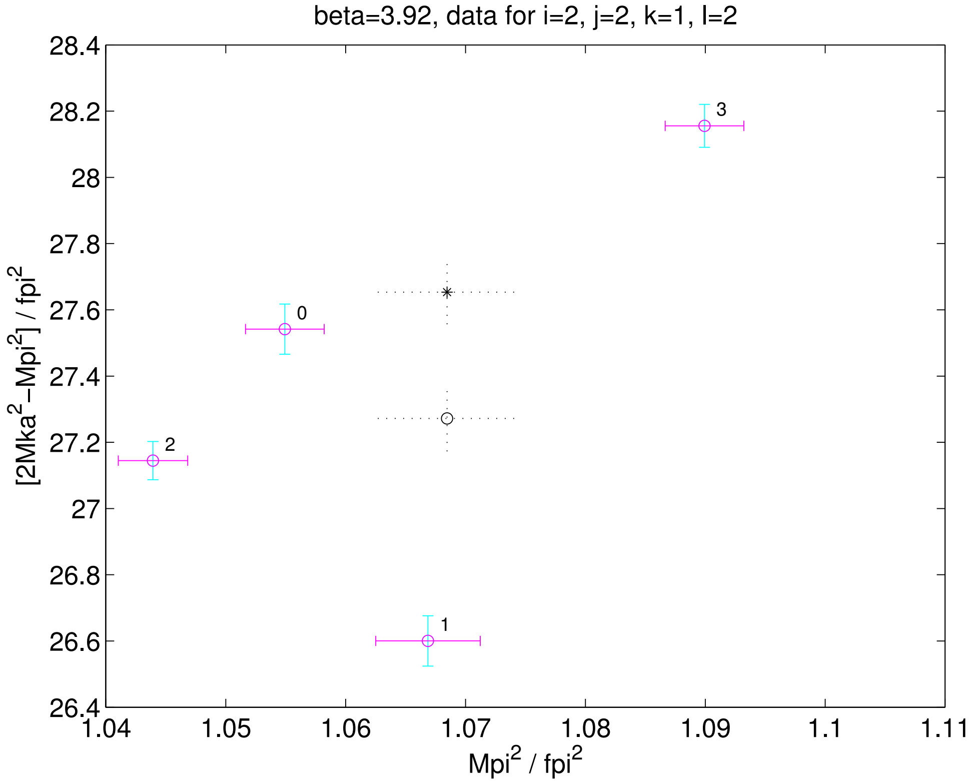
Note: $f_{ll} = f_{ud}$ (one too heavy, other too light) means that one can identify $f_\pi = f_{\pi^\pm}$
Note: $f_{ls} \neq f_{us}$ indicates that one cannot identify f_K with physical $f_{K^\pm}$.

$*$ "target point": pion mass and decay constant and kaon mass as suggested
$\circ$ "compensation point": artificially reduce m_s such that $m_u^\text{phys}+m_s^\text{phys}=m_{ud}^\text{phys}+m_s^\text{comp}$

$$\frac{m_s^\text{comp}}{m_s^\text{phys}} = 1 - \frac{m_d^\text{phys} - m_u^\text{phys}}{m_d^\text{phys} + m_u^\text{phys}} \cdot \frac{m_{ud}^\text{phys}}{m_s^\text{phys}} = 1 - 0.381(27)\cdot 0.03632(28) = 0.9862(10)$$

Mass interpolation (2): target point and compensation point

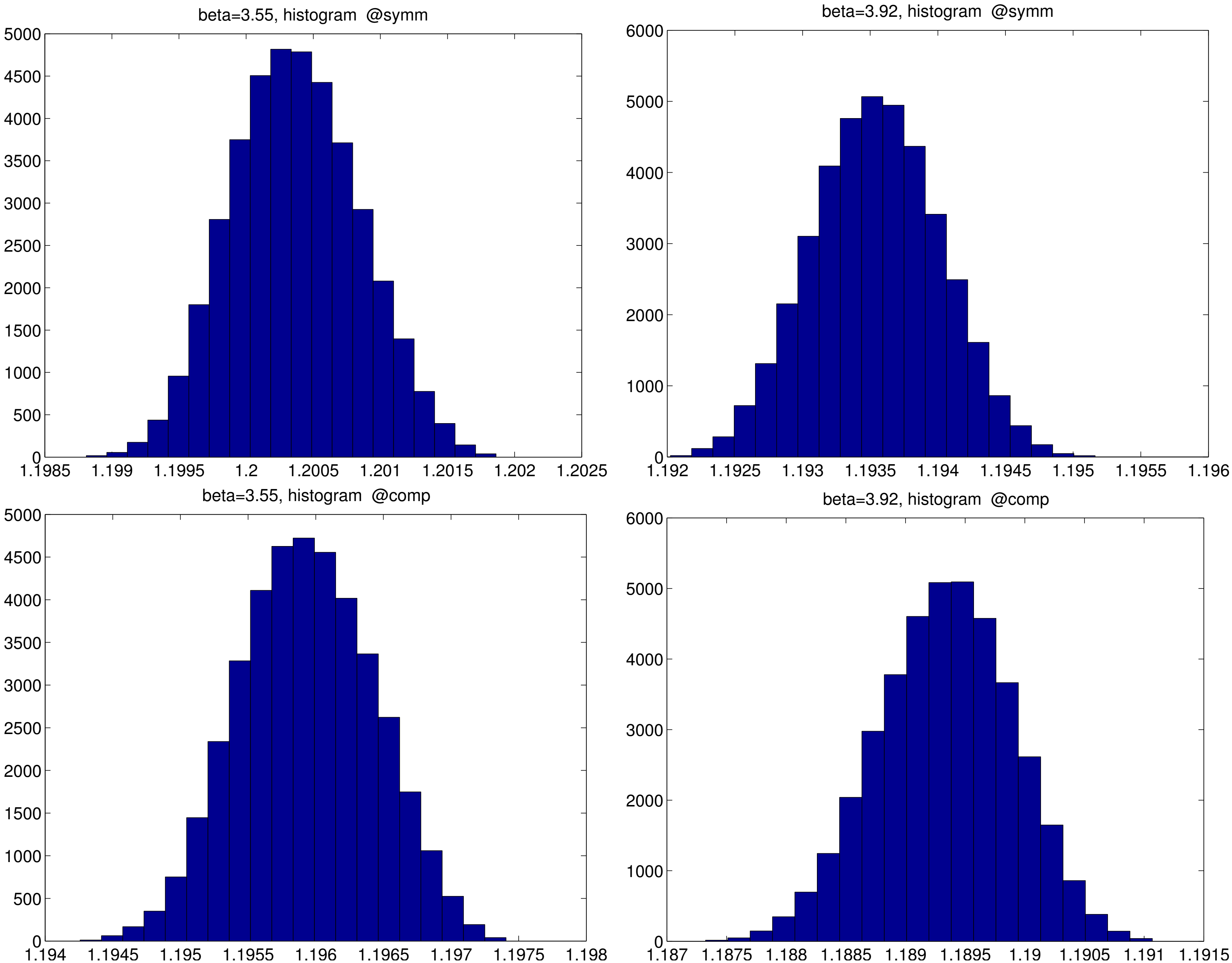




Error bars in ensemble points reflect statistical error of M_π^2/f_π^2 and $(2M_K^2-M_\pi^2)/f_\pi^2$ for one snapshot choice ($\ell=2$ means correlated fit, $k=1$ means without oscillatory term, $j=2$ means extended fitrange, $i=2$ means $\tilde{g}_1^{\mathrm{trun}}$ finite-volume corrections). Relative weight of the four ensembles varies from one jackknife bin to the next.

Systematically loop over all 2^4 options of single-ensemble analysis.
Systematic uncertainty of target/compensation point taken care of by another 3 loops.

Mass interpolation (3): example histograms



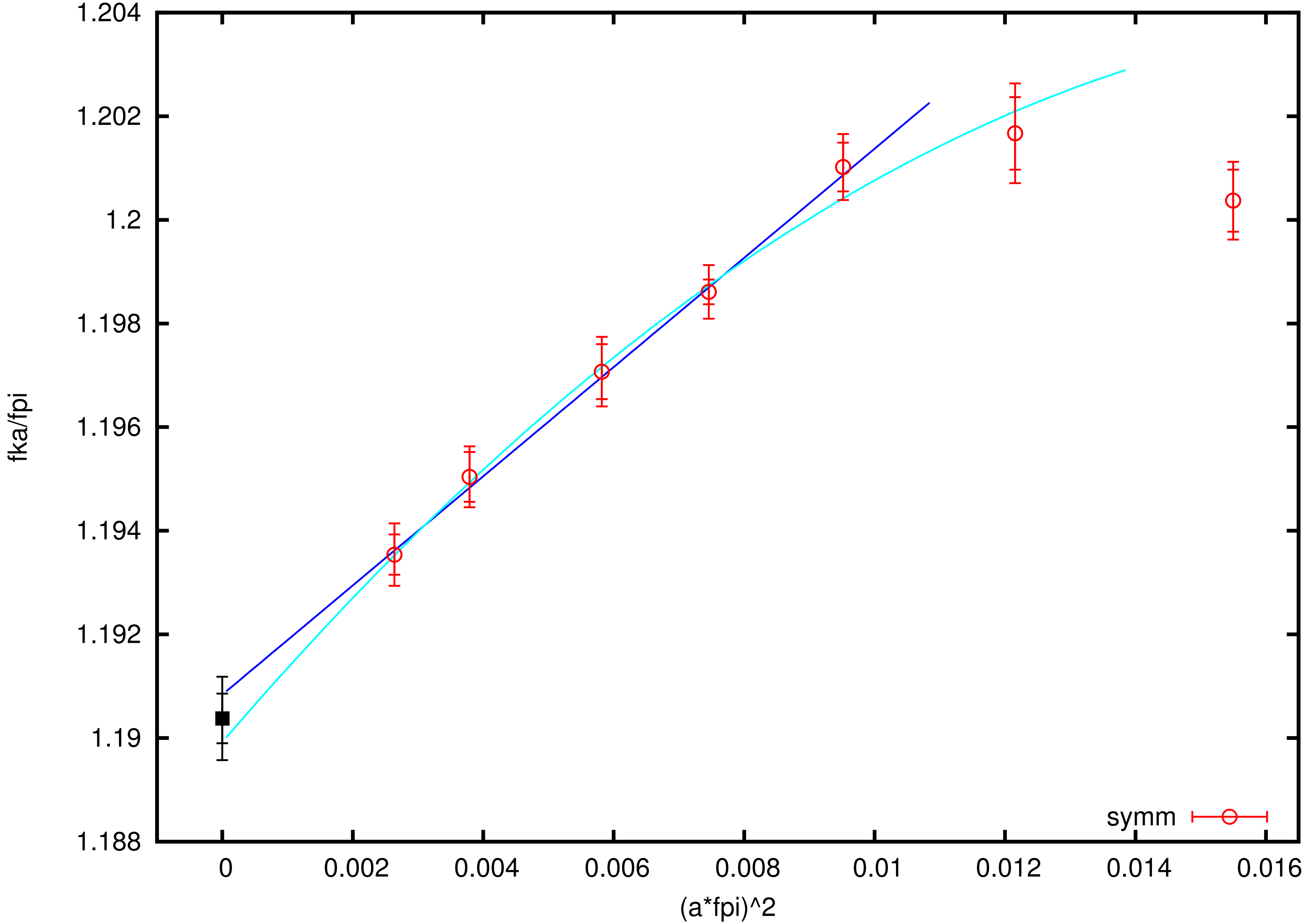
Mass interpolation (4): table of results

Alternative: compute the correction factor $f_{K^\pm}/f_K^{\text{symm}}$ in ChPT [CiriglianoNeufeld 11]

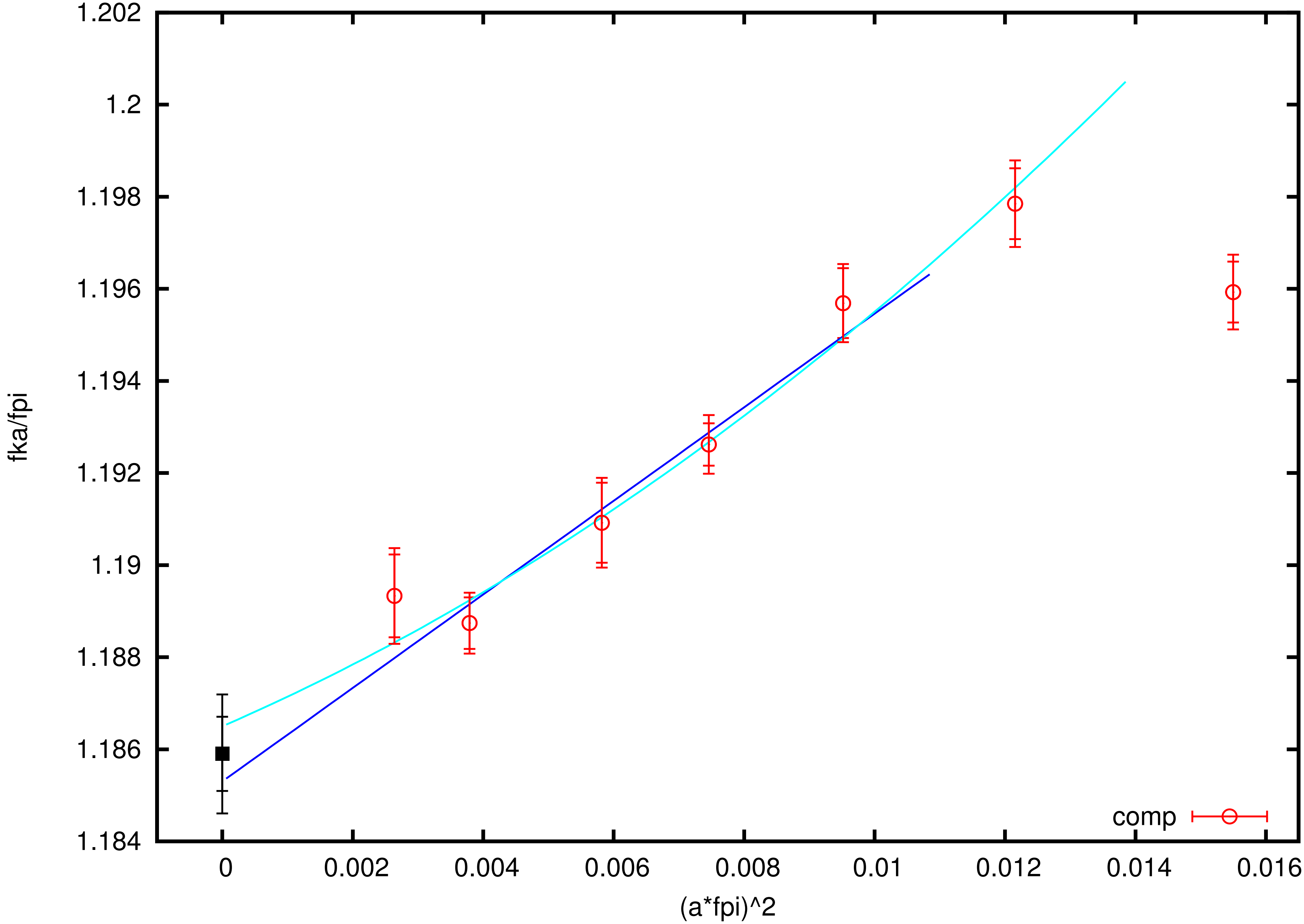
β	af_π^{symm}	$a[\text{fm}]$	$f_K^{\text{symm}}/f_\pi^{\text{symm}}$	$f_K^{\text{corr}}/f_\pi^{\text{symm}}$	$f_K^{\text{comp}}/f_\pi^{\text{symm}}$
3.55	0.12449(6)(9)	0.18837(09)(37)	1.20037(60)(45)	1.19785(60)(66)	1.19593(66)(47)
3.60	0.11025(7)(8)	0.16682(10)(33)	1.20167(70)(66)	1.19914(70)(81)	1.19785(77)(54)
3.65	0.09756(7)(9)	0.14762(10)(32)	1.20102(47)(43)	1.19849(47)(64)	1.19569(76)(38)
3.70	0.08636(4)(9)	0.13068(07)(30)	1.19861(24)(46)	1.19609(24)(66)	1.19262(46)(44)
3.75	0.07628(7)(9)	0.11542(10)(28)	1.19707(53)(41)	1.19456(52)(63)	1.19092(87)(44)
3.84	0.06158(9)(9)	0.09317(13)(25)	1.19504(48)(34)	1.19253(47)(59)	1.18874(56)(35)
3.92	0.05138(5)(5)	0.07774(07)(17)	1.19354(39)(46)	1.19104(39)(67)	1.18933(90)(52)

Results for af_π, $a[\text{fm}]$, $f_K^{\text{symm}}/f_\pi^{\text{symm}}$, $f_K^{\text{corr}}/f_\pi^{\text{symm}}$, $f_K^{\text{comp}}/f_\pi^{\text{symm}}$ after interpolation to iso-symmetrical point, per β. Statistical errors are correlated among observables at one β, systematic errors are correlated both among observables and across β-values.

Continuum extrapolation (1): isospin-symmetric case



Continuum extrapolation (2): compensation-point case



Systematic uncertainty (1): overall strategy

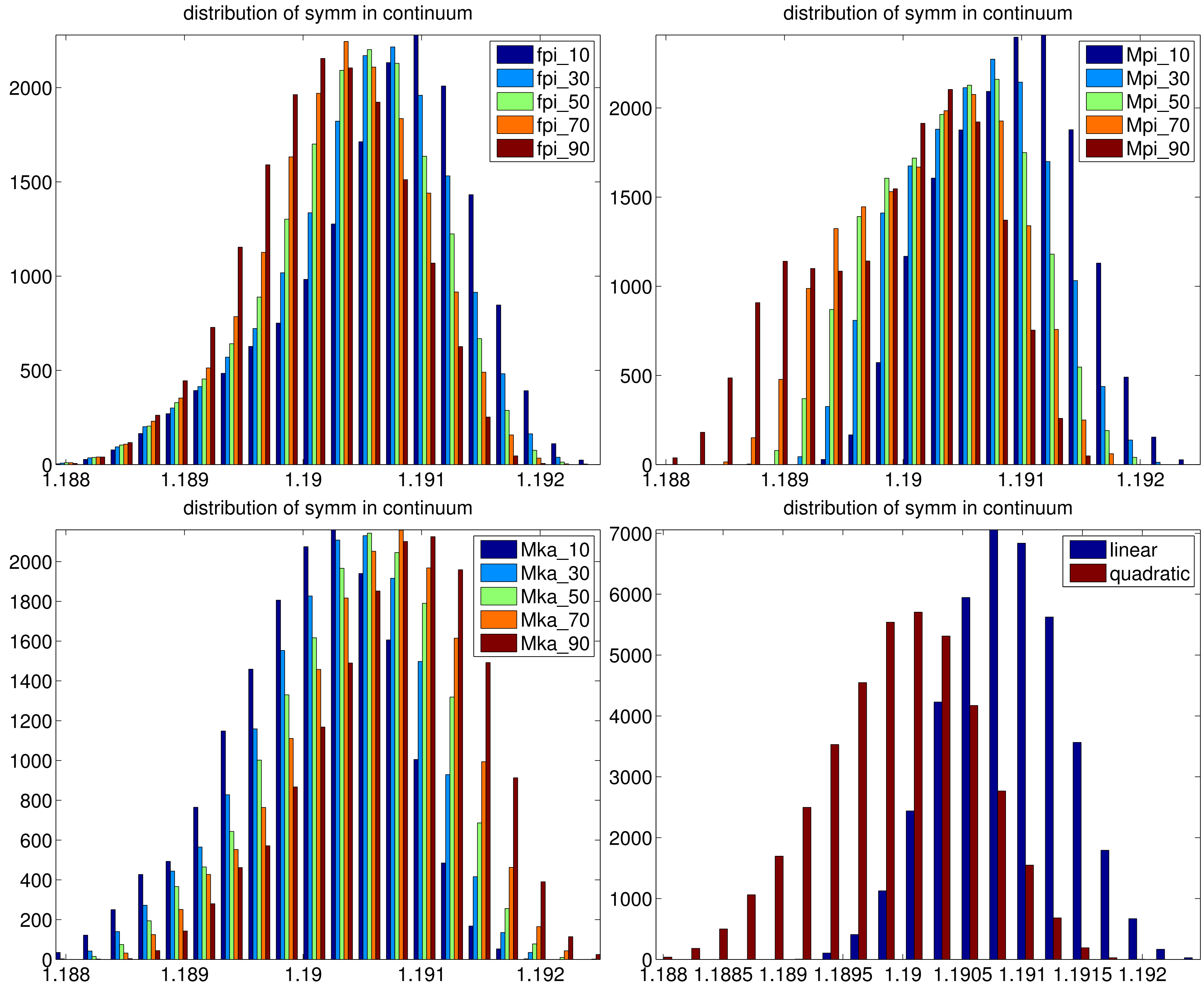
Per ensemble:
Correlated versus uncorrelated (2 choices)
Parity partner effect (with/without oscillations)
Excited states effects (2 choices to tini/tfin)
Finite volume effects (2 choices of function)

Per beta:
Scale setting uncertainty (5 choices: $f_\pi^{\rm phys}$ at 10th, 30th, 50th, 70th, 90th percentile)
Light quark mass uncertainty (5 choices: M_π^2 at 10th, 30th,...,70th ,90th percentile)
Strange quark mass uncertainty (5 choices: $2M_K^2-M_\pi^2$ at 10th,...,90th percentile)

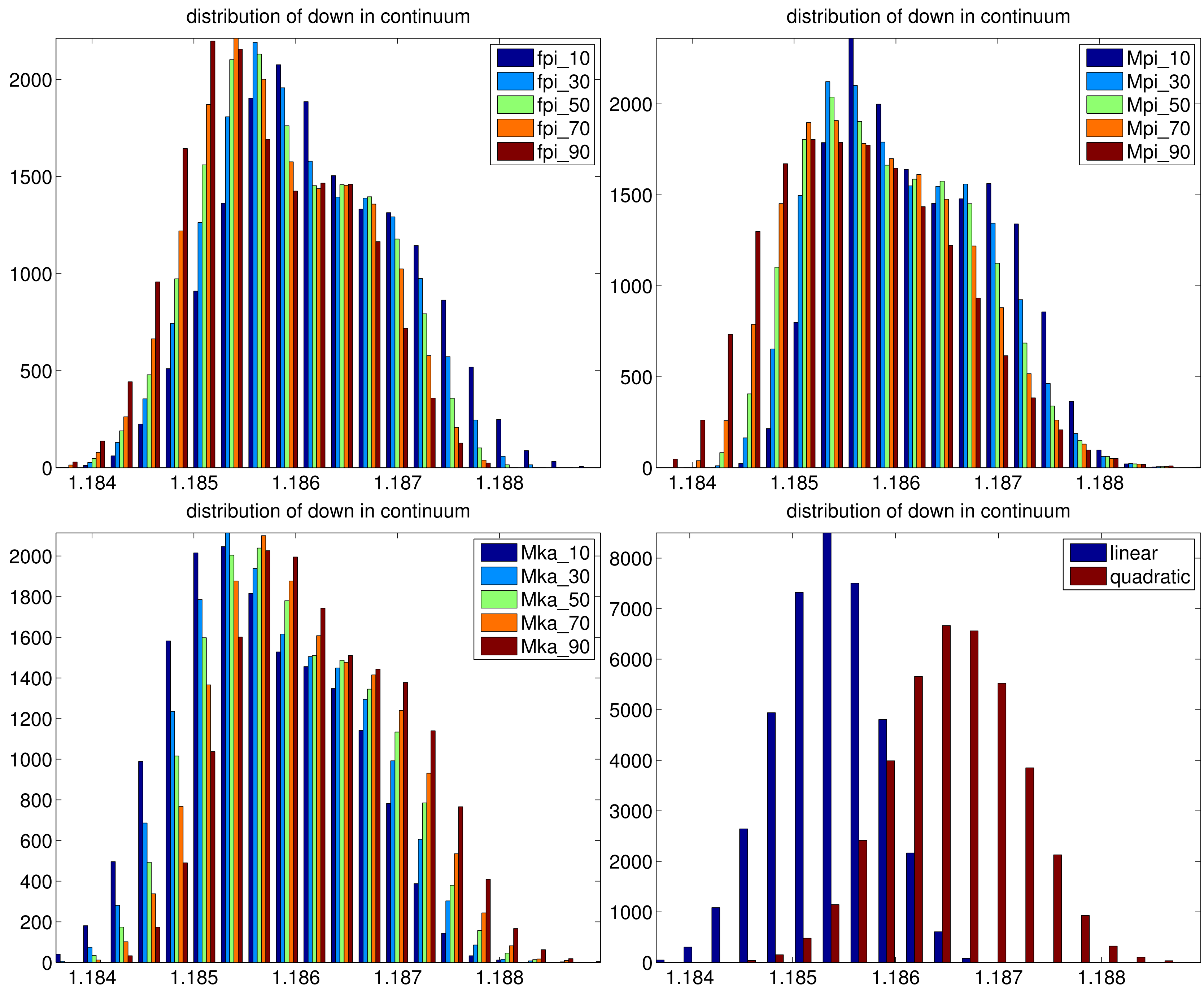
Final analysis:
Symanzik range (2 choices: 5 with $O(a^2)$ versus 6 with $O(a^2+a^4)$ cut-off effects)
Possibly isospin breaking correction (2 choices: via ChPT, via compensation point)

Overall $16\cdot 250\cdot\{1,2\}=\{4000,8000\}$ independent (complete and valid) analyses.
Systematic uncertainty of central value and statistical error by considering 15.865th percentile, 50th percentile, 84.135th percentile (with symmetrization).
Error budget by leaving out any single loop of the systematic variation (complement).

Systematic uncertainty (2): histograms for $f_K^{\text{symm}}/f_\pi^{\text{symm}}$



Systematic uncertainty (3): histograms for $f_K^{\rm comp}/f_\pi^{\rm symm}$



Physics results (1): Numerical value of f_K

- Results for isospin-symmetric / charged decay constant ratios have 0.66 permille and 1.55 permille precision, respectively, viz.

$$f_K/f_\pi = 1.1904(05)_{\rm stat}(06)_{\rm syst}\;, \qquad f_{K^\pm}/f_{\pi^\pm} = 1.1869(09)_{\rm stat}(16)_{\rm syst}\;. \tag{12}$$

- Multiply result for $f_{K^\pm}/f_{\pi^\pm}$ with the PDG value $f_{\pi^\pm} = 130.41(20)\,\mathrm{MeV}$ to obtain

$$f_{K^\pm} = 154.78(34)\,\mathrm{MeV}\;. \tag{13}$$

- Plug experimental averages $\Gamma(\pi^+ \to \mu\bar{\nu}_\mu) = 3.8408(8)\,10^7 s^{-1}$ and $\Gamma(K^+ \to \mu\bar{\nu}_\mu) = 5.133(12)\,10^7 s^{-1}$ from PDG into eqn. (2) and take into account electromagnetic corrections with $\delta_{\rm em} = -0.0070(18)$ to obtain the fully evaluated relation

$$\frac{|V_{us}|}{|V_{ud}|}\,\frac{f_{K^\pm}}{f_{\pi^\pm}} = 0.27598(43)_{\rm exp} \tag{14}$$

where the right-hand side has 1.56 permille accuracy.

Physics results (2): V_{us} and first-row CKM unitarity test

- Combining this with our charged decay constant ratio yields

$$\frac{|V_{us}|}{|V_{ud}|} = 0.23252(36)_{\mathrm{exp}}(36)_{\mathrm{lat}} = 0.23252(51)\ . \tag{15}$$

- Current best estimate of $|V_{ud}|$ is the Hardy-Towner value $|V_{ud}| = 0.97425(22)_{\mathrm{nuc}}$ which has 0.23 permille accuracy. Upon combining it with (15) we find

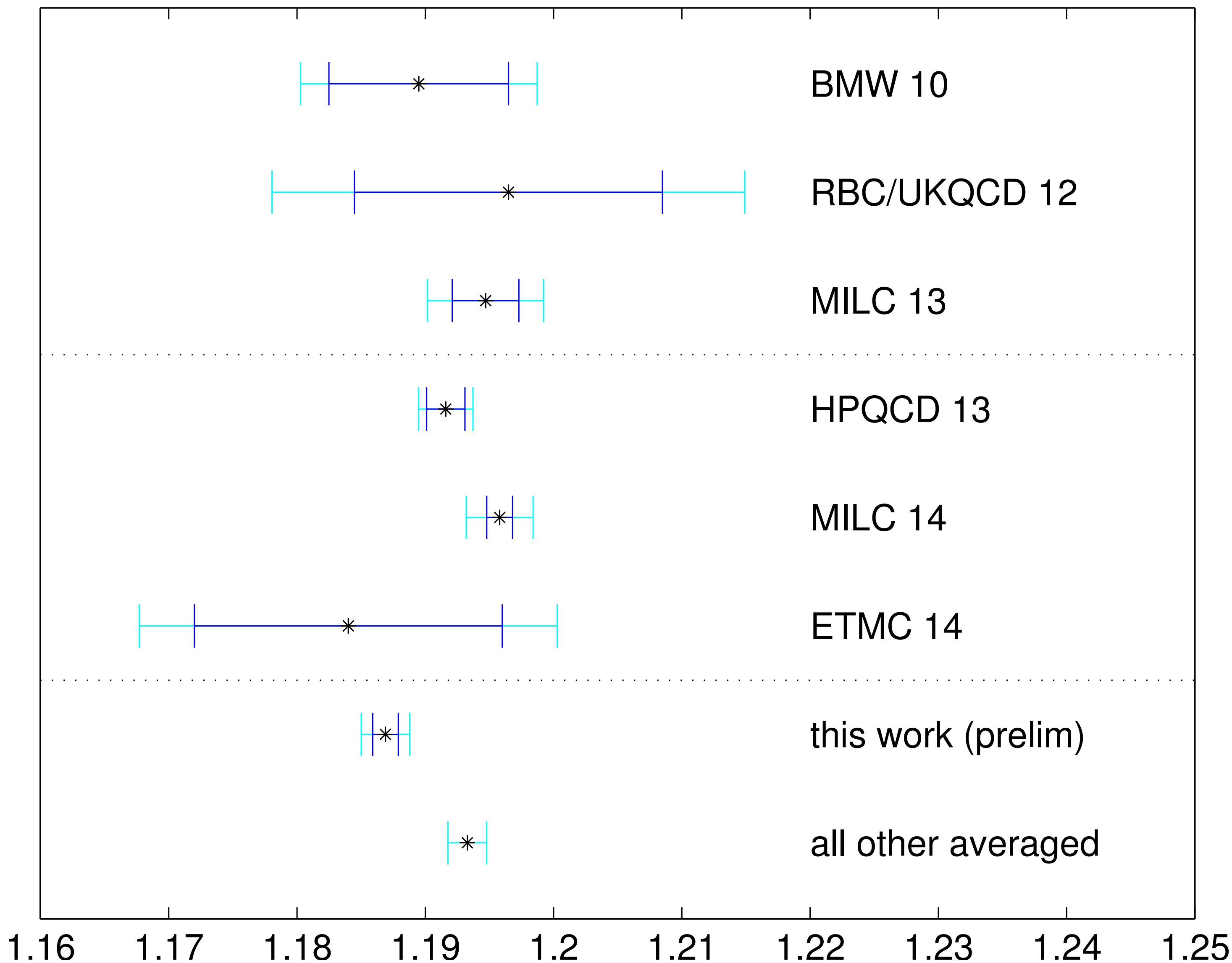
$$|V_{us}| = 0.22653(35)_{\mathrm{exp}}(35)_{\mathrm{lat}}(05)_{\mathrm{nuc}} = 0.22653(50) \tag{16}$$

as well as $|V_{ud}|^2 \cdot (1 + |V_{us}|^2/|V_{ud}|^2) = 1.00048(50)$. Upon adding the square of $|V_{ub}| = 0.00415(49)$ [PDG] to the latter, we end up with a first-row unitarity test

$$|V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 = 1.00050(50) \tag{17}$$

which is perfectly consistent with the unitarity constraint of the Standard Model.

Physics results (3): Comparison with other collaborations



Discrepancy with HPQCD is $1.1916(21) - 1.1869(18) = 0.0047(28)$, i.e. at 1.7σ.

Summary

- $4{+}4{+}5{+}5{+}5{+}4{+}4 = 31$ ensembles at 7 lattice spacings near physical mass point.
- 6 additional ensembles at one $(\beta, m_{ud}, m_s, m_c)$ for dedicated finite-volume study.
- Interpolation to physical mass point performed separately at each β.
- Systematic error by "histogram method" includes uncertainty of input f_π, M_π, M_K, while dominant source of uncertainty is isospin correction.
- Preliminary $f_{K^\pm}/f_{\pi^\pm}$ with 1.6 permille overall uncertainty, slight inconsistency with HPQCD and MILC, subsequent V_{us} still satisfies CKM first-row unitarity relation.