# Chiral Symmetry Braeking in Bosonic Partition Functions

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C. Lehner, M. Ohtani, J.J.M. Verbaarschot and T. Wettig, Topology and Chiral Random Matrix Theory at Nonzero Imaginary Chemical Potential, Phys. Rev. D90 (2009) 074016 [arXiv:0902.2640 [hep-th]].

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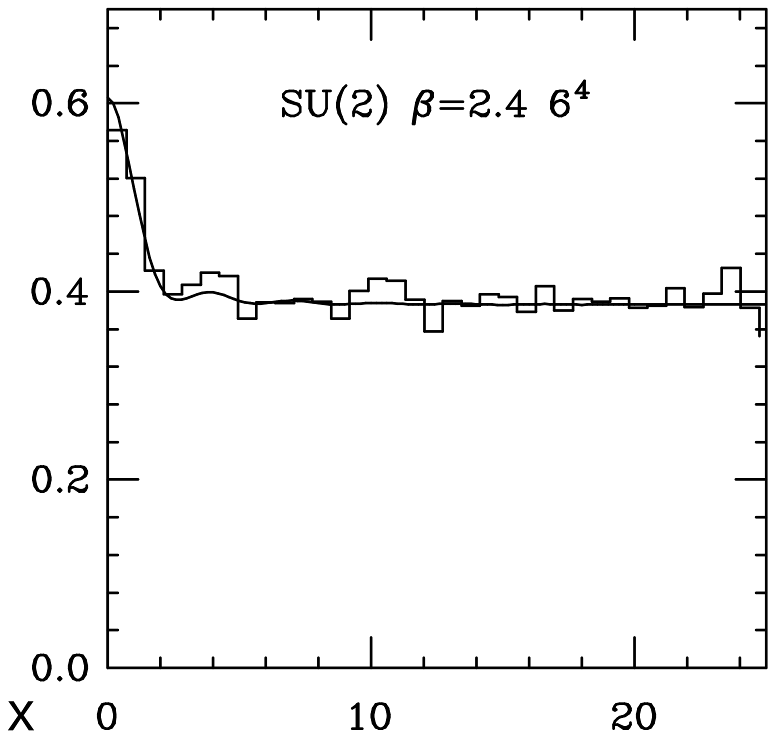
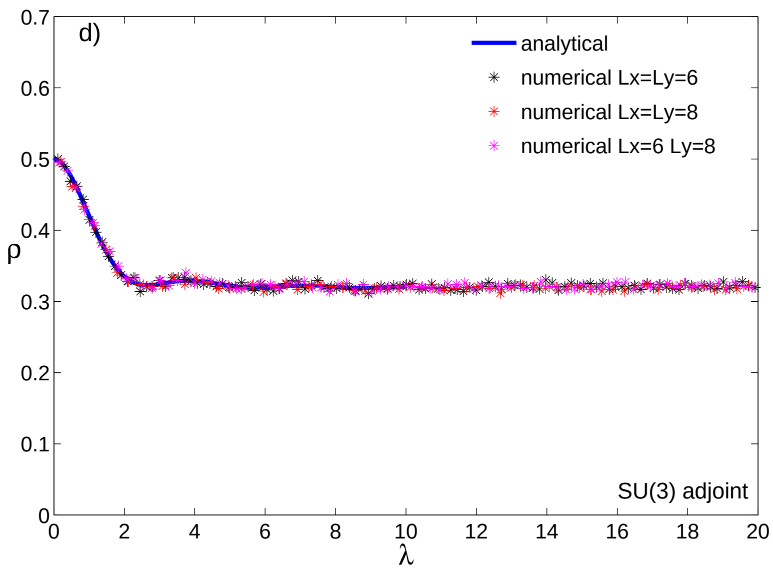



Microscopic spectral density of quenched staggered Dirac operator in 4 dimensions for QCD with two colors and quarks in the adjoint representation.

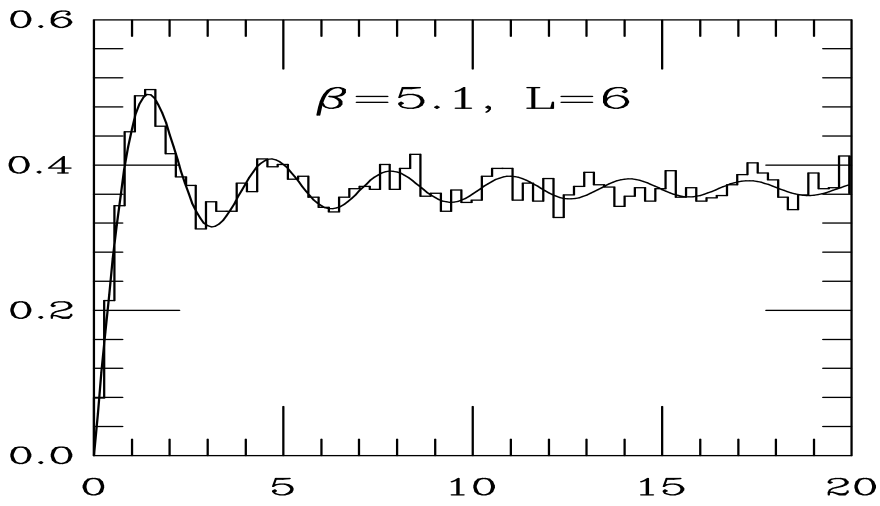
Edwards-Heller-Narayanan-1999

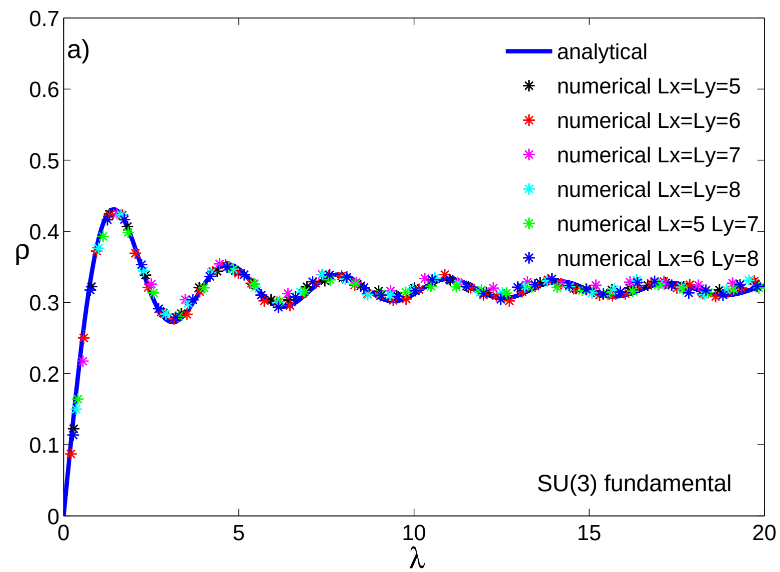



Microscopic spectral density of the quenched QCD Dirac operator in 2 dimensions for QCD with three colors and adjoint quarks ( $\beta = \infty$ ).

Kieburg-JV-Zafeiropoulos-2013

# Two and Four Dimensional Dirac Spectra for $\beta_D = 2$

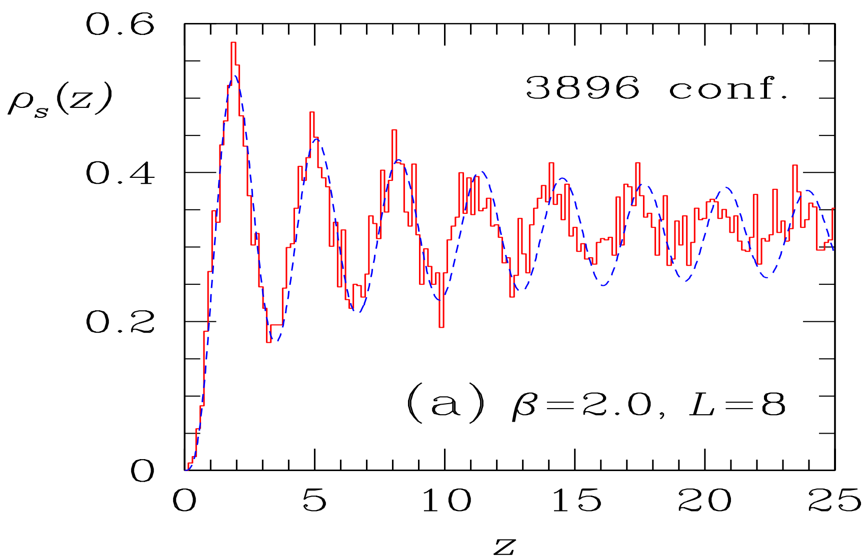



Microscopic spectral density of quenched staggered Dirac operator in 4 dimensions for QCD with three colors. Wettig-etal-1999

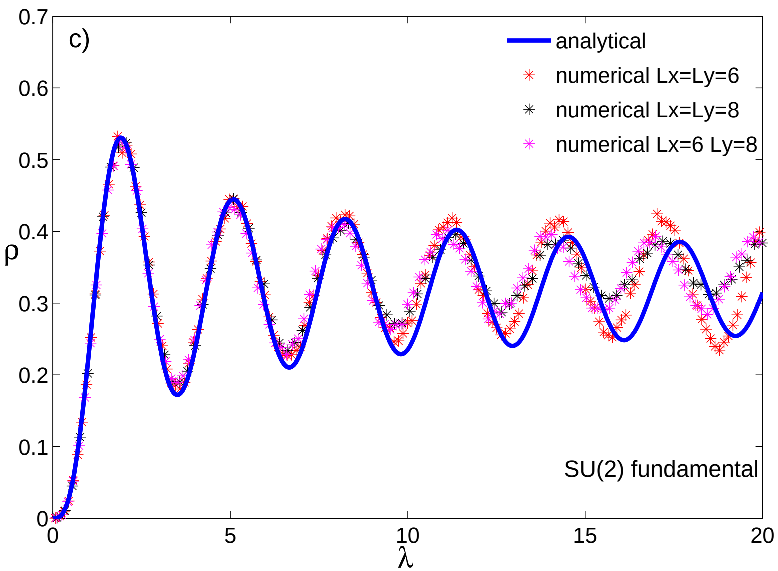
Microscopic spectral density of the quenched QCD Dirac operator in 2 dimensions for QCD with three colors and fundamental quarks ( $\beta = \infty$ ). Kieburg-JV-Zafeiropoulos-2013

# Two and Four Dimensional Dirac Spectra for $\beta_D = 4$




Microscopic spectral density of quenched staggered Dirac operator in 4 dimensions for QCD with two colors and fundamental quarks.

Wettig-JV-etal-1999




Microscopic spectral density of the quenched QCD Dirac operator in 2 dimensions for QCD with two colors and fundamental quarks ( $\beta = \infty$ ).

Kieburg-JV-Zafeiropoulos-2013

# Puzzle

- There is no difference in the quality of agreement with chRMT between two and four dimensional theories.
- Such agreement implies the validity of (partial quenched) chiral perturbation theory and thus the spontaneous breaking of chiral symmetry.
- Yet because of the Coleman-Mermin-Wagner continuous symmetries cannot be broken spontaneously in two dimensions.

# Possible Solutions

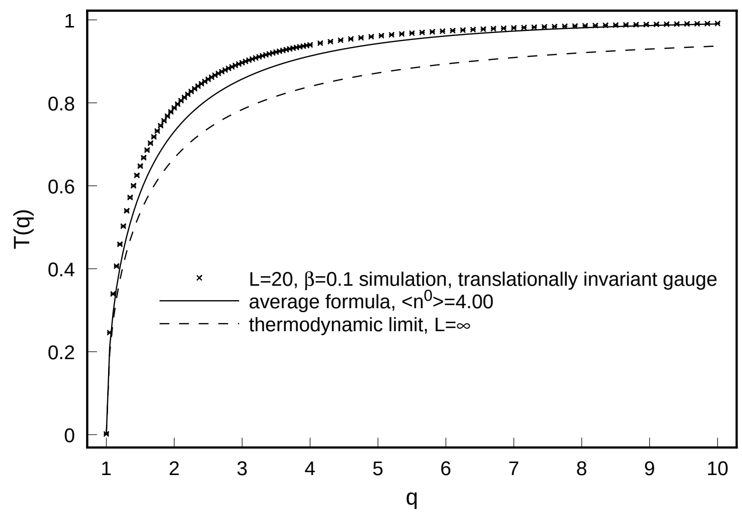
The generating function for the resolvent is given by

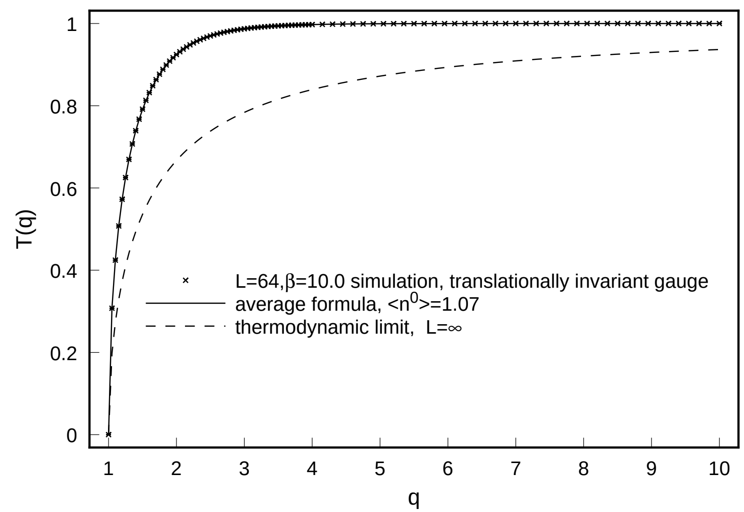
$$G(z) = \frac{d}{dz}\Big|_{z'=z} \left\langle \det{}^{N_f}(D+m) \frac{\det(D+z)}{\det(D+z')} \right\rangle .$$

- This partition function has both fermionic and bosonic “ghost”-quarks.
- The flavor group is a supergroup and the boson-boson part has to be to be noncompact. Otherwise the integrals in the partition function diverge.
- A trivial or invariant ground state cannot exist for a noncompact Goldstone manifold because the integration over the noncompact group will be divergent.

Zirnbauer, Spencer-Zirnbauer-2004, Niedermaier-Seiler-2003

# Hyperbolic $\sigma$ model




Expectation value of $T(q) = \langle \tanh e(q) \cdot n \rangle$ with $e(q) = \sqrt{q^2 - 1}, e_2, e_3)$ depends on the value of $q$ for a SO(2,1) two-dimensional hyperbolic $\sigma$ -model (the spin components are normalized as $n_0^2 - n_1^2 - n_2^2 = 1$ .

Duncan-Niedermaier-Seiler-2004

The claim is that the Coleman-Mermin-Wagner theorem is not valid for noncompact symmetries or more generally for non-amenable symmetries (these are symmetry groups for which a invariant mean does not exist).

# Mermin-Wagner-Coleman Theorem

- If the claim by Niedermaier, Seiler, Spencer and Zirnbauer is correct, it would imply that in two dimensions the compact part of the symmetry group is not broken spontaneously while the non-compact part is broken.
- In four dimensions both parts are broken spontaneously and one would expect different universal eigenvalue correlations, or at least a different scaling domain of the universal correlations, if this is not the case. At this moment it is not yet clear what is going on.

# Condensed Matter Philosophy

In the condensed matter it is assumed that, whether or a continuous symmetry is broken spontaneously, we can bosonize the partition function in terms of the “Goldstone” degrees of freedom.

To find out if there is spontaneous symmetry breaking one has to study the behavior of the order parameter under renormalization.

We will ask a simpler question and study the difference between the bosonic and fermionic partition functions with the same parameters.

# Bosonic versus Fermionic Partition Functions

We will consider three examples:

- Phase quenched partition function at nonzero chemical potential in the $\epsilon$ -domain.
- The $\epsilon$ –domain of one flavor QCD.
- One flavor chiral random matrix model at imaginary chemical potential.
- The chiral condensate remains constant for quark masses beyond this critical value.

# II. Phase Quenched QCD

Pion Condensation

# Chiral Condensate of Phase Quenched QCD

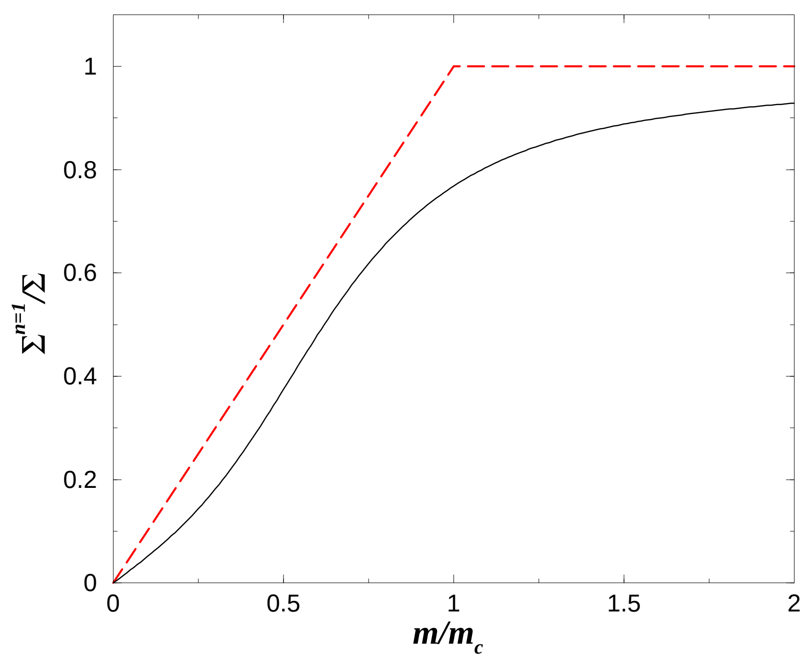
Phase quenched QCD is defined by the partition function

$$\langle \det(D + m + \mu\gamma_0) \det(D + m - \mu\gamma_0) \rangle$$

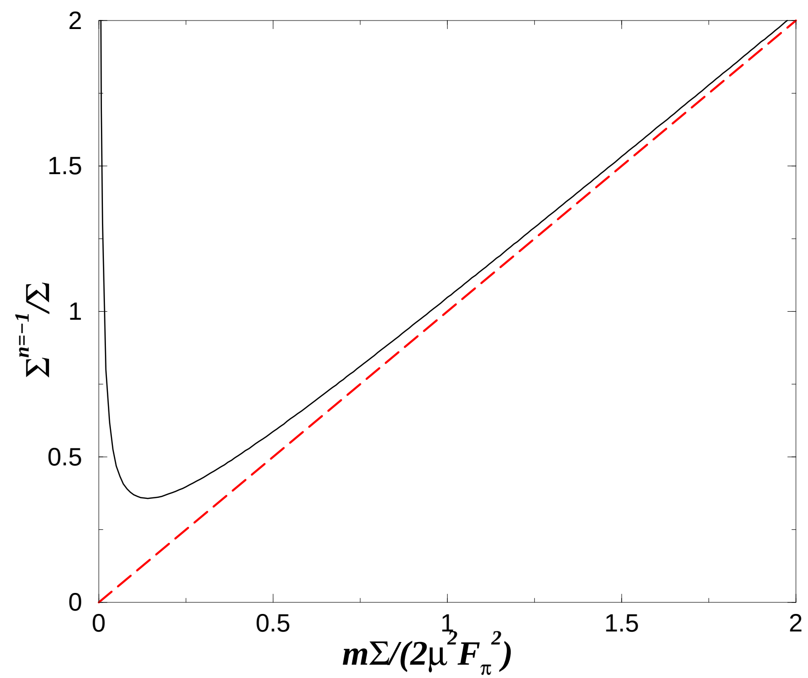
and therefore $\mu$ can be interpreted as an isospin chemical potential.

At low temperatures, a phase transition to a phase of Bose-condensed pions takes place at $\mu = m_\pi/2$ .

# Fermionic Versus Bosonic Phase Quenched QCD




Fermionic partition function




Bosonic partition function

$$\langle \det(D + m + \mu\gamma_0) \det(D + m - \mu\gamma_0) \rangle$$

$$\left\langle \frac{1}{\det(D + m + \mu\gamma_0) \det(D + m - \mu\gamma_0)} \right\rangle$$

Splittorff-JV-2005

The mean field result is given by the red dashed curve.

# What is Going on with the Bosonic Partition Function

- This behavior occurs because the contribution of a single eigenvalue close to the mass diverges logarithmically in the regularization parameter.
- The regularized bosonic partition function is defined by

$$\left\langle \det{}^{-1} \begin{pmatrix} \epsilon & D + m - \mu\gamma_0 \\ D + m + \mu\gamma_0 & \epsilon \end{pmatrix} \right\rangle .$$

- This partition function has a charged Goldstone boson with mass $\sim \sqrt{\epsilon}$ which condenses already for an infinitesimal isospin chemical potential.
- This is a generic result. For example one can easily check that the same phenomenon occurs for compact the $U(1)$ bosonic phase quenched lattice QCD partition function in one dimension.

# III. Microscopic Limit of One Flavor QCD

What the mass dependence of the partition function of QCD with one bosonic flavor?

# One Flavor QCD

$$Z^{N_f=1}(m) = \langle \det(D+m) \rangle.$$

- Axial symmetry is broken explicitly by the anomaly
- For $m\Lambda_{QCD} \ll 1/\sqrt{V}$ the partition function is given by

$$Z^{N_f=1}(m,\theta) \quad = \quad e^{mV\Sigma\cos\theta} = \sum_\nu e^{i\nu\theta} I_\nu(mV\Sigma).$$

Leutwyler-Smilga-1996

- The chiral condensate is constant as a function of the mass.
- The sum over topology $\nu$ always converges because for large $\nu$ we have $I_\nu(x) \sim x^\nu/\nu!$ .

Damgaard-2001,Lehner-Ohtani-JV-Wettig-2006

# Bosonic One Flavor Partition Function

$$Z = \left\langle \frac{1}{\det(D+m)} \right\rangle = \left\langle \int d\phi e^{-\phi^*(D+m)\phi}. \right\rangle$$

$$\phi^*(D+m)\phi = \begin{pmatrix} \phi_1^* \\ \phi_2^* \end{pmatrix} \begin{pmatrix} m & id \\ -id^\dagger & m \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}.$$

To maintain complex conjugation required for convergence the axial symmetry is given by

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \to \begin{pmatrix} e^s\phi_1 \\ e^{-s}\phi_2 \end{pmatrix}$$

So instead of the axial U(1) symmetry for the fermionic partition function we have the noncompact Gl(1)/U(1) symmetry.

# Bosonic Partition Function in the Microscopic Domain

The fermionic one-flavor partition function for $m \ll 1/\sqrt{V}$ is given by

$$
\begin{aligned}
Z_\nu^{N_f=1}(m) &= \int_{U\in U(1)} dU \det{}^\nu U e^{\frac{1}{2}mV\sigma \mathrm{Tr}(U+U^{-1})} \\
&= \int d\theta e^{i\nu\theta} e^{mV\Sigma\cos\theta} \\
&= I_\nu(mV\Sigma).
\end{aligned}
$$

The bosonic partition function is obtained by replacing the U(1) integral by a Gl(1)/U(1) integral and is thus given by

$$
\begin{aligned}
Z_\nu^{N_f=-1} &= \int_{U\in Gl(1)/U(1)} dU \det{}^\nu U e^{mV\Sigma \mathrm{Tr}(U+U^{-1})} \\
&= \int ds e^{\nu s} e^{mV\Sigma\cosh s} \\
&= K_\nu(mV\Sigma).
\end{aligned}
$$

# Bosonic One-Flavor Partition Function at $\theta = 0$

$$Z^{N_f=-1}(m) = \sum_\nu P(\nu) K_\nu(mV\Sigma)$$

- For large $\nu$ at fixed $x$ we have that $K_\nu(x) \sim \nu!/x^\nu$ .
- The sum is still convergent because of the distribution of the topological charge set by the gauge field, $P(\nu) \sim \exp(-\nu^2/2\chi V)$ .
- For large $mV\Sigma$ we can use $K_\nu(x) \sim e^{-x}/\sqrt{|x|}$ and

$$Z^{N_f=-1}(m) \sim \sqrt{\chi V}\frac{e^{-mV\Sigma}}{\sqrt{|m|V\Sigma}}$$

Note that the $\sqrt{m}$ factor is not canceled as is the case for the fermionic partition function where $\chi = m\Sigma$ .

# IV. Random Matrix Model at Imaginary Chemical Potential

Random Matrix Model

Chiral Condensate

Distribution over topological sectors

# Chiral Random Matrix Model at Imaginary Chemical Potential

Dirac Operator

$$D = \begin{pmatrix} m & id + t \\ id^\dagger + t & m \end{pmatrix}$$

and $d$ is a $n \times (n + \nu)$ matrix from a Gaussian distribution.

Generically, $D$ has $\nu$ zero modes. In the thermodynamic limit we keep $N \equiv 2n + \nu$ fixed.

Jackson-JV-2001, Lehner-Ohtani-JV-Wettig-2006

The chiral condensate is normalized to one and the critical imaginary chemical potential for chiral symmetry restoration is also at $t = 1$ .

# Solution of the Random Matrix Model

Bosonic partition function

$$Z_\nu^{N_f=-1}(z,t) \quad = \quad z^{-\nu}\int_0^\infty \frac{ds}{s^{\nu+1}} e^{-sNz^2/2} \frac{1}{(1/s+1/\Sigma^2)^{(N+\nu)/2}} e^{-\frac{1}{2}Nt^2/(1/s+1/\Sigma^2}$$

Kellerstein-JV-2015

Fermionic partition function

$$Z_\nu^{N_f=1}(z,t) = \int_0^\infty ds s^{\nu+1} I_\nu(zNs\Sigma)(s^2+t^2)^{(N-\nu)/2} e^{-Ns^2-nz^2/2}.$$

Halasz-Jackson-JV-2004, Lehner-Ohtani-JV-Wettig-2006

# Microscopic Limit

In the microscopic limit $mN$ fixed for $N \to \infty$ the integrals can be evaluated analytically

$$Z_\nu^{N_f=-1} \sim (1 - t^2\Sigma^2)^{\nu/2} K_\nu(Nm\Sigma\sqrt{1 - t^2\Sigma^2}).$$

$$Z_\nu^{N_f=1} \sim \frac{1}{(1 - t^2\Sigma^2)^{\nu/2}} I_\nu(Nm\Sigma\sqrt{1 - t^2\Sigma^2}).$$

Lehner-Ohtani-JV-Wettig-2006

For QCD one would expect that in this limit the dependence partition function on the imaginary chemical potential is only through a modified chiral condensate. This is only the case if we introduce an additional normalization factor

$$\mathcal{N}^{N_f=1} = \frac{1}{(1 - \Sigma^2 t^2)^{\nu/2}}, \qquad \mathcal{N}^{N_f=-1} = (1 - \Sigma^2 t^2)^{\nu/2}.$$

# Normalization Factor

The partition function at $\theta = 0$ thus requires a nontrivial normalization factor

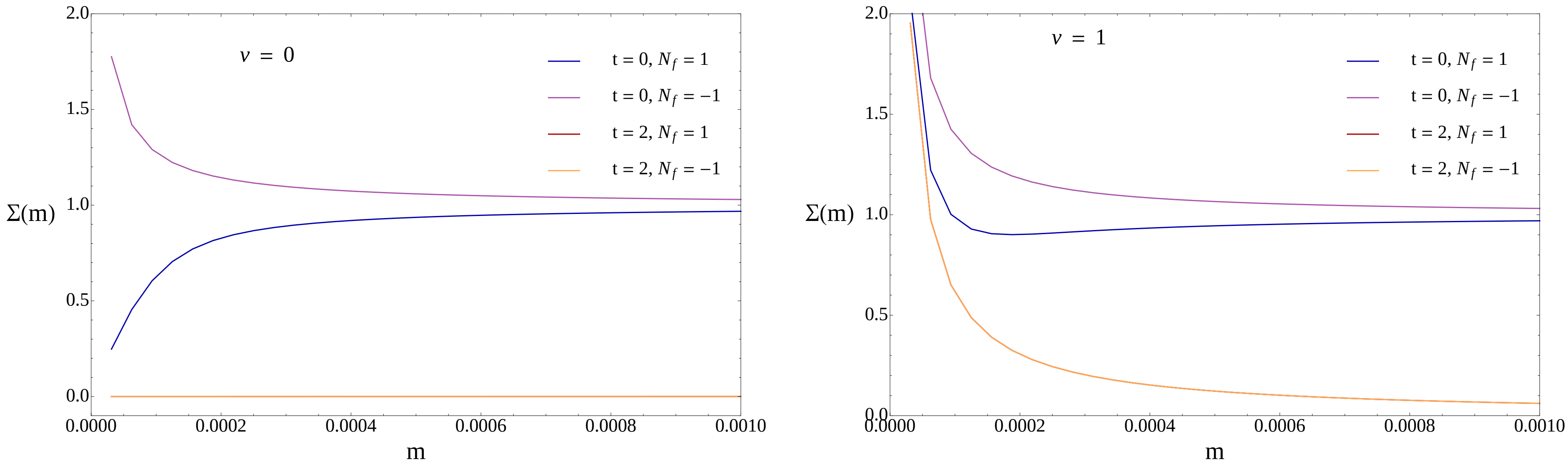
$$Z(m, \theta = 0) = \sum_{\nu} \mathcal{N}_{\nu} Z_{\nu}.$$

In QCD $\mathcal{N}_{\nu}$ is a normal is taken to be equal unity.

In lattice QCD one would also expect a $\nu$ -dependent normalization factor.

Lehner-Ohtani-JV-Wettig-2006

# Comparison of Fermionic and Bosonic Partition Functions

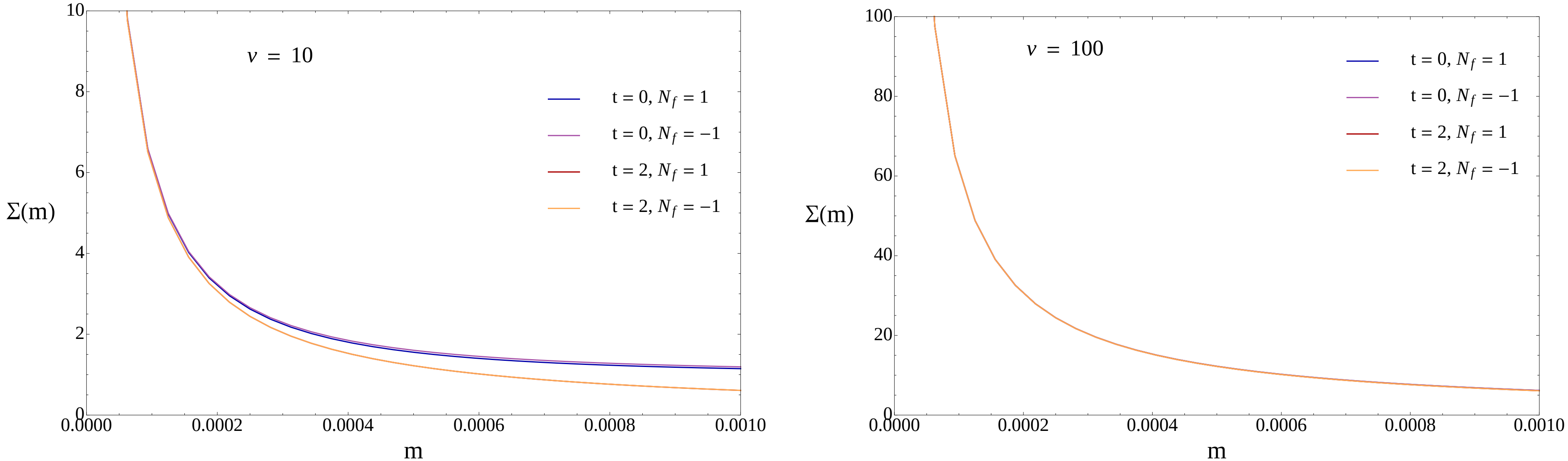



For $\nu = 0$ the probability to find a single eigenvalue at $\lambda = 0$ diverges in the chiral limit for the bosonic partition function while it vanishes for the fermionic partition function.

For $\nu \neq 0$ both the fermionic and the bosonic partition function are dominated by the zero modes in the chiral limit.

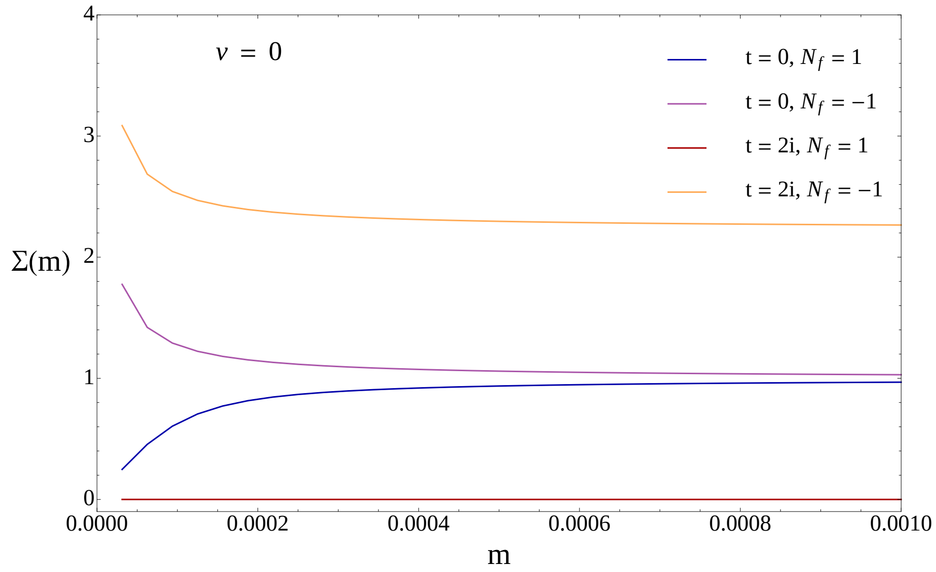
The parameters of the chiral random matrix model are equal to $N = 2^{14}$ and $\Sigma = 1$ .

# Chiral Condensate of Large Topological Charge



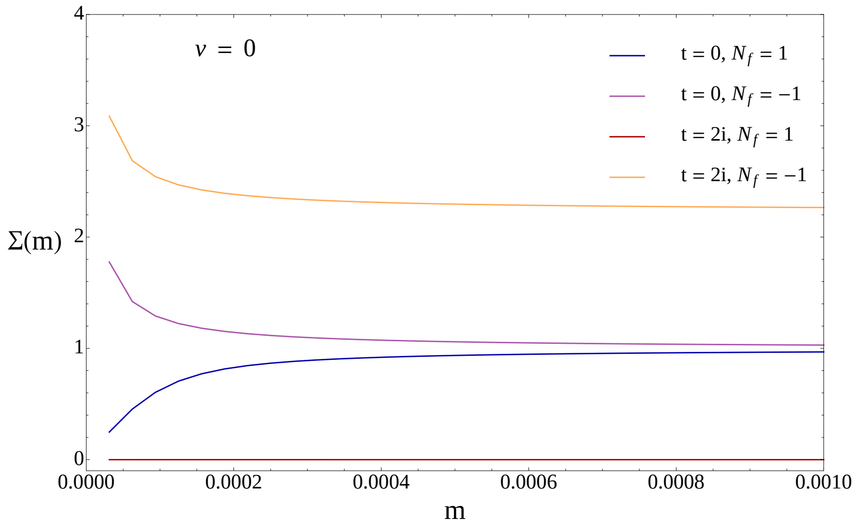

For large topological charge, the bosonic and fermionic partition function are almost identical for $m \ll 1/\sqrt{N}$ , and in this domain, the mass dependence of the chiral condensate becomes temperature independent.

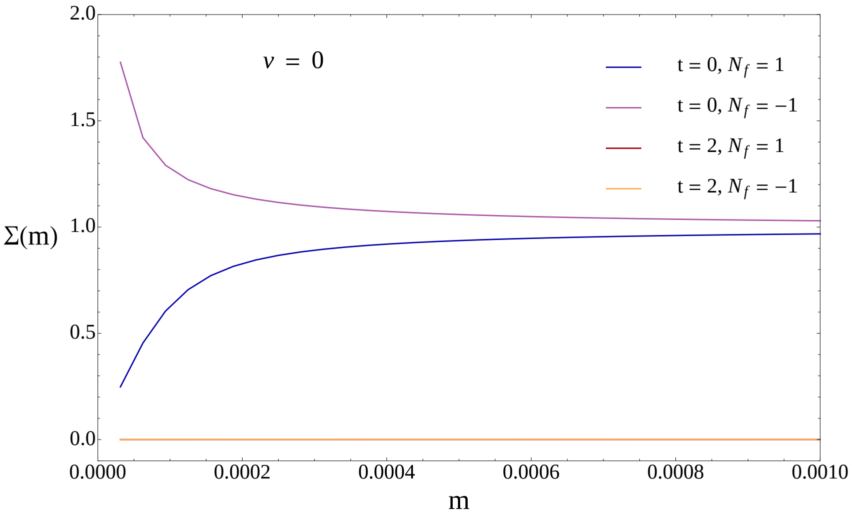
# Chiral Condensate for Real Chemical Potential




The expressions for both the fermionic and bosonic partition function can be continued to real chemical potential by the substitution $t^2 \to -t^2$ in our expresssions. For the fermion partition function this can be justified à priori, but the $derivation$ of the bosonic partition function is not valid for imaginary $t$. We are currently working out the meaning of this analytical continuation in terms of the original random matrix model.

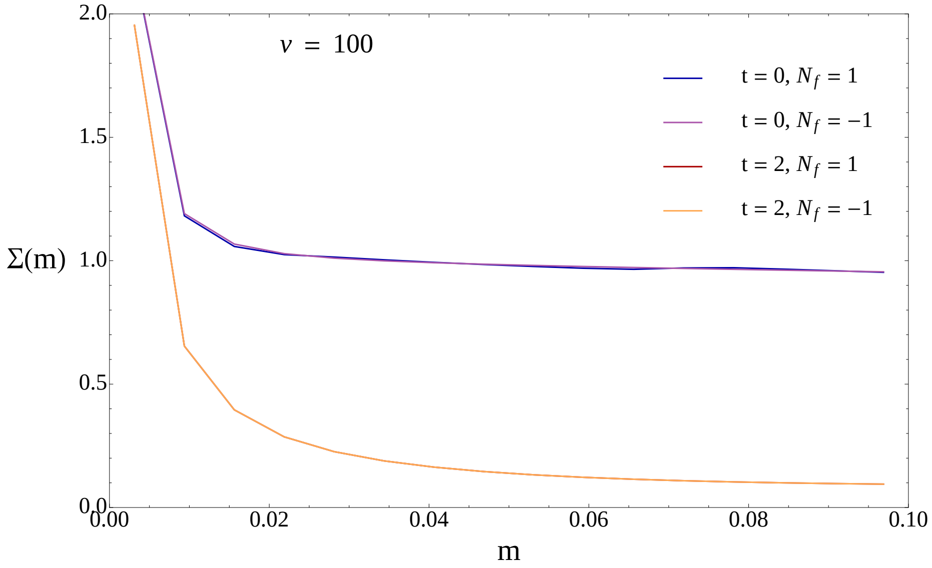
# Comparison of Real and Imaginary Chemical Potential



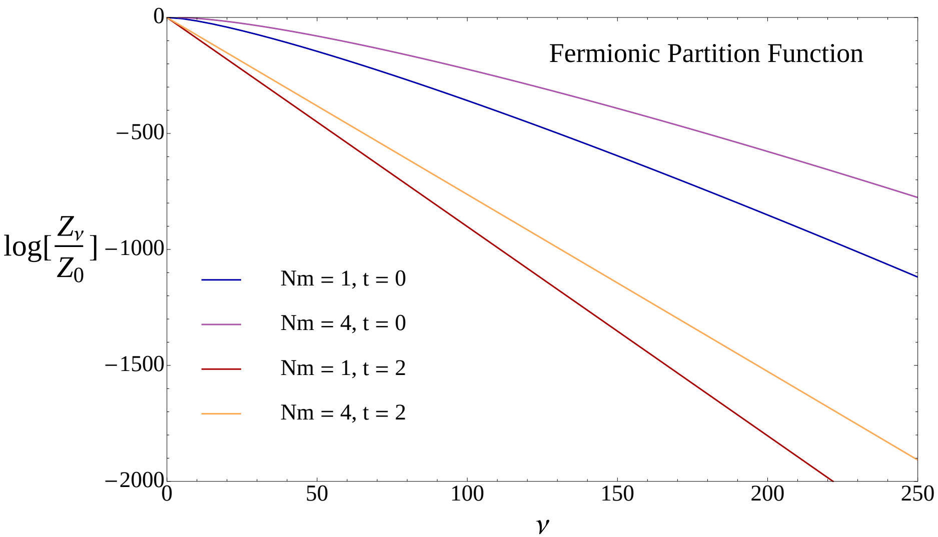

Comparison of the chiral condensate for real (left) and imaginary (right) chemical potential. The bosonic result does not have a phase transition to the chirally restored phase.

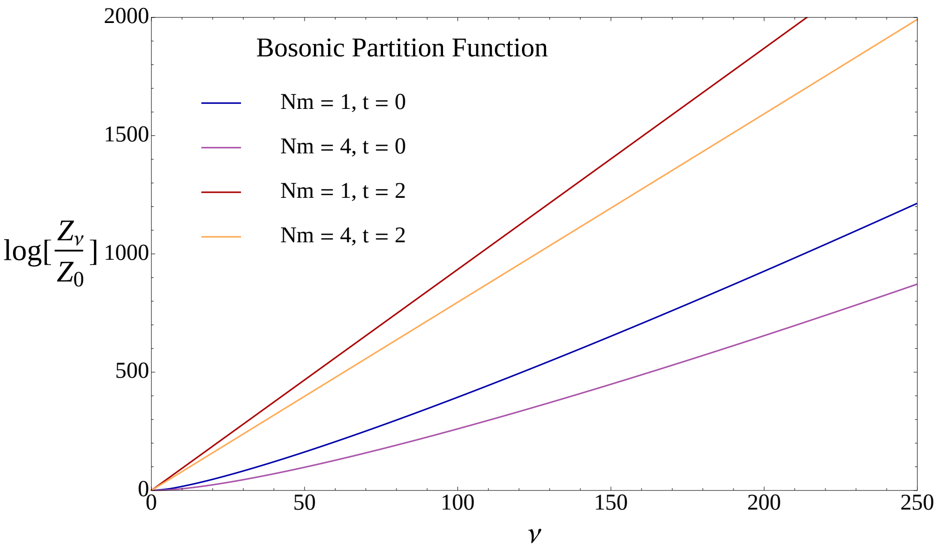
# Bosonic Partition Function is Dominated by topology




When $m$ is kept fixed in the thermodynamic limit, the axial symmetry is restored for large imaginary chemical potential, and for large topology, the fermionic and bosonic partition function show an identical mass dependence.

# Distribution of the Topological Charge




Because of the $m^\nu$ factor in the fermion determinant topology is suppressed for the fermionic partition function while it is enhanced for the bosonic partition function.

# Bosonic Partition Function is Dominated by topology

More quantitatively

$$Z_\nu^{N_f=-1} \sim e^{\alpha|\nu| - \nu^2/2\chi V}$$

The average topological charge is given by

$$\langle|\nu|\rangle = \frac{\sum_\nu |\nu| Z_\nu^{N_f=-1}}{\sum_\nu Z_\nu^{N_f=-1}} = \alpha\chi V.$$

The contribution from the zero modes to the chiral condensate cannot be neglected in the thermodynamic limit

$$\frac{1}{V}\frac{\langle|\nu|\rangle}{m} = \frac{\alpha\chi}{m}.$$

# IV. Conclusions

- Noncompact symmetries may be broken spontaneously for parameter values for which the corresponding compact symmetry is restored.

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- The bosonic one flavor partition function is dominated by configurations with topological charge that scales linearly in the volume.
- Nontrivial normalization factors may appear in the sum over topological sectors.
- The excellent agreement of two dimensional QCD Dirac spectra with random matrix theory remains a puzzle.