17 July 2015

Study of entropy production in Yang-Mills theory with use of Husimi function

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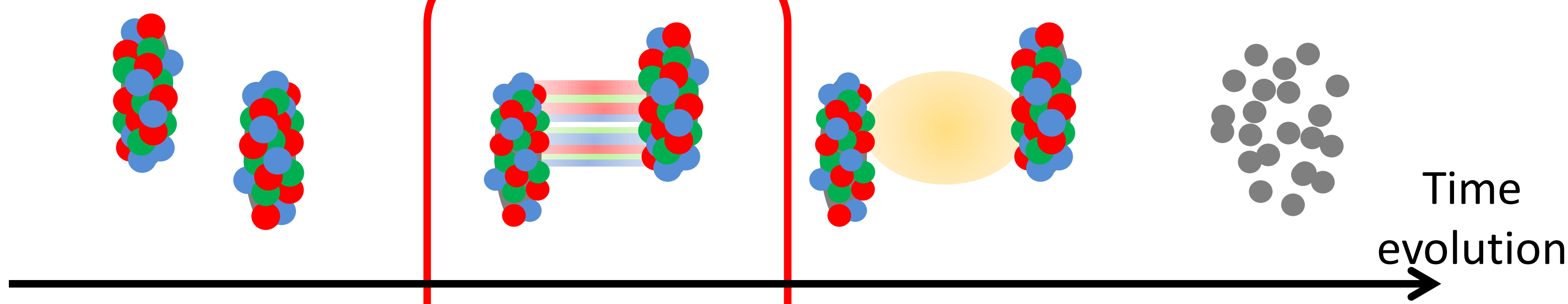
Lattice2015@Kobe

Motivation

Relativistic heavy ion collisions



Color Glass Condensate(CGC) — Classical approx. is valid. [McLerran, Venugopalan(1993)]

Glasma — Classical approx. [Lappi, McLerran(2006)]

Quark Gluon Plasma (QGP)

Hadron

Ideal liquid

Early thermalization [Heinz(2002)]

Large amount of entropy is produced in early stage of the collision.

Previous works

[T.Kunihiro, B.Muller, A.Ohnishi, A.Schafer, T.T.Takahashi, A.Yamamoto, PRD **82**, 114015(2010)]
[H.Iida, T.Kunihiro, B.Muller, A.Ohnishi, A.Schafer, T.T.Takahashi, PRD **88**, 094006(2013)]

Kolmogorov-Sinai entropy (entropy production rate) is positive in classical Yang-Mills field.
Initial fluctuations trigger the chaotic behavior.

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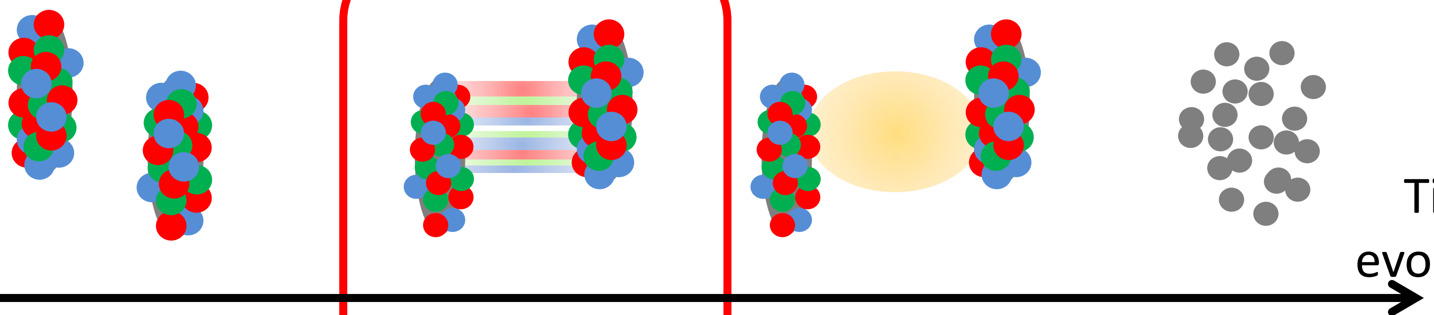

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Kolmogorov-Sinai entropy (entropy production rate) is positive in classical Yang-Mills field.
Initial fluctuations trigger the chaotic behavior.

We calculate entropy directly with initial fluctuations.

Contents

Entropy production in pure state

Von-Neumann entropy $S_{\mathrm{vN}} = -\mathrm{Tr}\rho \log \rho$ does not change by the quantum time evolution. Coarse-graining can be responsible for entropy production.

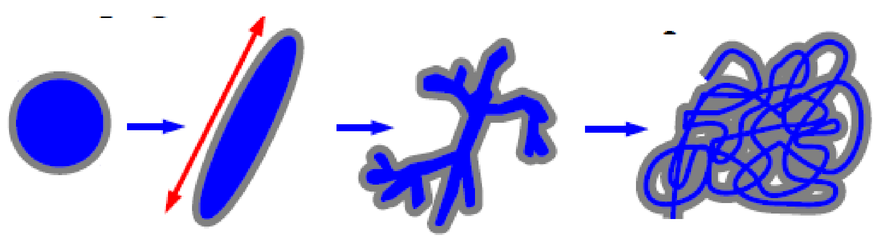
Ex) Partial trace is one way of coarse-graining.

$$\rho_A = \mathrm{Tr}_{\bar{A}}\rho$$

→Entanglement entropy $S_A = -\mathrm{Tr}\rho_A \log \rho_A$

It is considered as a probe of confinement [I.R.Klebanov et. al.(2008)] and calculated in lattice simulation.[Velytsky(2008)][Buividovich,Polikarpov(2008)] [Y.Nakagawa et. al.(2008,2010)][S.Aoki et. al.(2015)] etc.

In this talk, we propose another way of coarse-graining in the phase space to evaluate instabilities or chaotic behavior of a system.



Quantum distribution function

Wigner function [Wigner(1932)]

$$f_W(\vec{p},\vec{q};t) = \int d\vec{\eta} \exp(-i\vec{p}\cdot\vec{\eta}/\hbar)\langle \vec{q}+\vec{\eta}/2|\rho|\vec{q}-\vec{\eta}/2\rangle$$

Wigner function is the density matrix in Wigner representation.

(Quasi-)distribution function

$$\langle \hat{A} \rangle = \int \frac{d\vec{p}d\vec{q}}{(2\pi\hbar)^n} f_W(\vec{p},\vec{q};t) A_W(\vec{p},\vec{q};t)$$

Wigner function has a problem in serving as a quantum distribution function.
It is **not positive definite**.

Husimi function

We consider Gaussian smeared Wigner function, which leads to Husimi function.

Husimi function [Husimi(1940)]

$$f_H(\vec{p},\vec{q};t) = \int \frac{d\vec{p'}d\vec{q'}}{(\pi\hbar)^n} \exp(-\frac{1}{\Delta\hbar}(\vec{p}-\vec{p'})^2 - \frac{\Delta}{\hbar}(\vec{q}-\vec{q'})^2) f_W(\vec{p'},\vec{q'};t)$$

More generally, it is written in terms of a coherent state $|\vec{\alpha}\rangle$

$$f_H(\vec{p},\vec{q};t) = \langle\vec{\alpha}|\hat{\rho}|\vec{\alpha}\rangle$$

$$= |\langle\vec{\alpha}|\phi\rangle|^2 \geq 0$$

For the pure state $\rho = |\phi\rangle\langle\phi|$

Husimi function is semi-positive definite and is considered as a quantum distribution function.

Husimi-Wehrl entropy

Now we can define entropy in terms of Husuimi function —Husimi-Wehrl entropy—.

Husimi-Wehrl entropy [Wehrl(1978)]

$$S_{HW}(t) = -\int \frac{d\vec{p}d\vec{q}}{(2\pi\hbar)^n} f_H(\vec{p},\vec{q};t) \log f_H(\vec{p},\vec{q};t)$$

The entropy is not constant with time evolution because of Gaussian smearing even in the case of a pure state.

Problem : Integral over large dimensions

We propose two numerical methods for that, which is first applied to and checked in quantum mechanical systems with a few degree of freedoms. Next we apply them to Yang-Mills field.

Numerical method in semi-classical approximation

Based on

H.T., H.Iida, T.Kunihiro, A.Ohnishi, and T.T.Takahashi,

arXiv:1505.04698

to be published in Prog. Theor. Exp. Physics.

Semi-classical time evolution of Wigner function

H.T., H.Iida, T.Kunihiro, A.Ohnishi, and T.T.Takahashi, PTEP in press

In the case of $H = \frac{\vec{p}^2}{2m} + V(\vec{q})$, the time evolution of Wigner function is given by;

$$\frac{\partial}{\partial t} f_W = \sum_i^n \frac{\partial V}{\partial q_i} \frac{\partial f_W}{\partial p_i} - \sum_i^n \frac{p_i}{m} \frac{\partial f_W}{\partial q_i} + O(\hbar^2)$$

The semi-classical solution leads to

$$\frac{d}{dt} f_W(\vec{p}, \vec{q}; t) = 0$$ with **classical EOM** $$\dot{q}_i = \frac{p_i}{m}, \dot{p}_i = -\frac{\partial V}{\partial q_i}$$

This means that Wigner function is constant along the classical trajectory;

$$f_W(\vec{p}(t), \vec{q}(t); t) = f_W(\vec{p}(0), \vec{q}(0); t = 0) = \text{const.}$$

We can obtain the semi-classical time evolution of Wigner function by solving classical EOM.

Numerical methods

H.T., H.Iida, T.Kunihiro, A.Ohnishi, and T.T.Takahashi, PTEP in press

Two step Monte-Carlo method : direct Monte-Carlo evaluation

$$S_{HW}(t) = -\int \frac{d\vec{p}d\vec{q}}{(2\pi\hbar)^n} \exp(-\frac{1}{\Delta\hbar}\vec{p}^2 - \frac{\Delta}{\hbar}\vec{q}^2) \int \frac{d\vec{p'}d\vec{q'}}{(\pi\hbar)^n} f_W(\vec{p'}, \vec{q'}; t)$$
$$\times \log \int \frac{d\vec{p''}d\vec{q''}}{(\pi\hbar)^n} \exp(-\frac{1}{\Delta\hbar}(\vec{p} + \vec{p'} - \vec{p''})^2 - \frac{\Delta}{\hbar}(\vec{q} + \vec{q'} - \vec{q''})^2) f_W(\vec{p''}, \vec{q''}; t)$$

Test particle method : Wigner function is a sum of delta functions

Numerical methods

H.T., H.Iida, T.Kunihiro, A.Ohnishi, and T.T.Takahashi, PTEP in press

<u>Two step Monte-Carlo method</u> : direct Monte-Carlo evaluation

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$$\times \log \int \frac{d\vec{p''}d\vec{q''}}{(\pi\hbar)^n} \exp(-\frac{1}{\Delta\hbar}(\vec{p}+\vec{p'}-\vec{p''})^2 - \frac{\Delta}{\hbar}(\vec{q}+\vec{q'}-\vec{q''})^2) f_W(\vec{p''}, \vec{q''}; t)$$

Liouville's theorem

<u>Test particle method</u> : Wigner function is a sum of delta functions

Numerical methods

H.T., H.Iida, T.Kunihiro, A.Ohnishi, and T.T.Takahashi, PTEP in press

Two step Monte-Carlo method : **direct Monte-Carlo evaluation**

$$S_{HW}(t) = -\int \frac{d\vec{p}d\vec{q}}{(2\pi\hbar)^n}\exp(-\frac{1}{\Delta\hbar}\vec{p}^2 - \frac{\Delta}{\hbar}\vec{q}^2)\int\frac{d\vec{p'}d\vec{q'}}{(\pi\hbar)^n}f_W(\vec{p'},\vec{q'};t)$$

$$\times\log\int\frac{d\vec{p''}d\vec{q''}}{(\pi\hbar)^n}\exp(-\frac{1}{\Delta\hbar}(\vec{p}+\vec{p'}-\vec{p''})^2 - \frac{\Delta}{\hbar}(\vec{q}+\vec{q'}-\vec{q''})^2)f_W(\vec{p''},\vec{q''};t)$$

$$= -\frac{1}{N_{12}}\sum_i^{N_{12}}\log\frac{1}{N_3}\sum_j^{N_3}f_W(\vec{p''_j},\vec{q''_j};t)$$

Test particle method : **Wigner function is a sum of delta functions**

Numerical methods

H.T., H.Iida, T.Kunihiro, A.Ohnishi, and T.T.Takahashi, PTEP in press

Two step Monte-Carlo method : direct Monte-Carlo evaluation

$$S_{HW}(t) = -\int \frac{d\vec{p}d\vec{q}}{(2\pi\hbar)^n} \exp(-\frac{1}{\Delta\hbar}\vec{p}^2 - \frac{\Delta}{\hbar}\vec{q}^2) \int \frac{d\vec{p'}d\vec{q'}}{(\pi\hbar)^n} f_W(\vec{p'}, \vec{q'}; t)$$

$$\times \log \int \frac{d\vec{p''}d\vec{q''}}{(\pi\hbar)^n} \exp(-\frac{1}{\Delta\hbar}(\vec{p} + \vec{p'} - \vec{p''})^2 - \frac{\Delta}{\hbar}(\vec{q} + \vec{q'} - \vec{q''})^2) f_W(\vec{p''}, \vec{q''}; t)$$

$$= -\frac{1}{N_{12}} \sum_i^{N_{12}} \log \frac{1}{N_3} \sum_j^{N_3} f_W(\vec{p''_j}, \vec{q''_j}; t)$$

Test particle method :

$$f_W(\vec{p}, \vec{q}; t) = \frac{(2\pi\hbar)^n}{N} \sum_i^N \delta^{(n)}(\vec{p} - \vec{\bar{p}}^i(t)) \delta^{(n)}(\vec{q} - \vec{\bar{q}}^i(t))$$

$$S_{HW}(t) = -\int \frac{d\vec{p}d\vec{q}}{(\pi\hbar)^n} \exp[-\frac{1}{\Delta\hbar}\vec{p}^2 - \frac{\Delta}{\hbar}\vec{q}^2] \frac{1}{N} \sum_i^N$$

$$\times \log \frac{2^n}{N} \sum_j^N \exp[-\frac{1}{\Delta\hbar}(\vec{p} + \vec{\bar{p}}^i(t) - \vec{\bar{p}}^j(t))^2 - \frac{\Delta}{\hbar}(\vec{q} + \vec{\bar{q}}^i(t) - \vec{\bar{q}}^j(t))^2]$$

Numerical methods

H.T., H.Iida, T.Kunihiro, A.Ohnishi, and T.T.Takahashi, PTEP in press

Two step Monte-Carlo method : **direct Monte-Carlo evaluation**

$$S_{HW}(t) = -\int \frac{d\vec{p}d\vec{q}}{(2\pi\hbar)^n} \exp(-\frac{1}{\Delta\hbar}\vec{p}^2 - \frac{\Delta}{\hbar}\vec{q}^2) \int \frac{d\vec{p'}d\vec{q'}}{(\pi\hbar)^n} f_W(\vec{p'},\vec{q'};t)$$
$$\times \log \int \frac{d\vec{p''}d\vec{q''}}{(\pi\hbar)^n} \exp(-\frac{1}{\Delta\hbar}(\vec{p}+\vec{p'}-\vec{p''})^2 - \frac{\Delta}{\hbar}(\vec{q}+\vec{q'}-\vec{q''})^2) f_W(\vec{p''},\vec{q''};t)$$
$$= -\frac{1}{N_{12}} \sum_i^{N_{12}} \log \frac{1}{N_3} \sum_j^{N_3} f_W(\vec{p''_j},\vec{q''_j};t)$$

Test particle method : $f_W(\vec{p},\vec{q};t) = \frac{(2\pi\hbar)^n}{N} \sum_i^N \delta^{(n)}(\vec{p}-\vec{\bar{p}}^i(t))\delta^{(n)}(\vec{q}-\vec{\bar{q}}^i(t)))$

$$S_{HW}(t) = -\int \frac{d\vec{p}d\vec{q}}{(\pi\hbar)^n} \exp[-\frac{1}{\Delta\hbar}\vec{p}^2 - \frac{\Delta}{\hbar}\vec{q}^2] \frac{1}{N} \sum_i^N$$
$$\times \log \frac{2^n}{N} \sum_j^N \exp[-\frac{1}{\Delta\hbar}(\vec{p}+\vec{\bar{p}}^i(t)-\vec{\bar{p}}^j(t))^2 - \frac{\Delta}{\hbar}(\vec{q}+\vec{\bar{q}}^i(t)-\vec{\bar{q}}^j(t))^2]$$
$$= -\frac{1}{N_{MC}} \sum_k^{N_{MC}} \frac{1}{N} \sum_i^N \log \frac{2^n}{N} \sum_j^N \exp[-\frac{1}{\Delta\hbar}(\vec{p}_k+\vec{\bar{p}}^i(t)-\vec{\bar{p}}^j(t))^2 - \frac{\Delta}{\hbar}(\vec{q}_k+\vec{\bar{q}}^i(t)-\vec{\bar{q}}^j(t))^2]$$

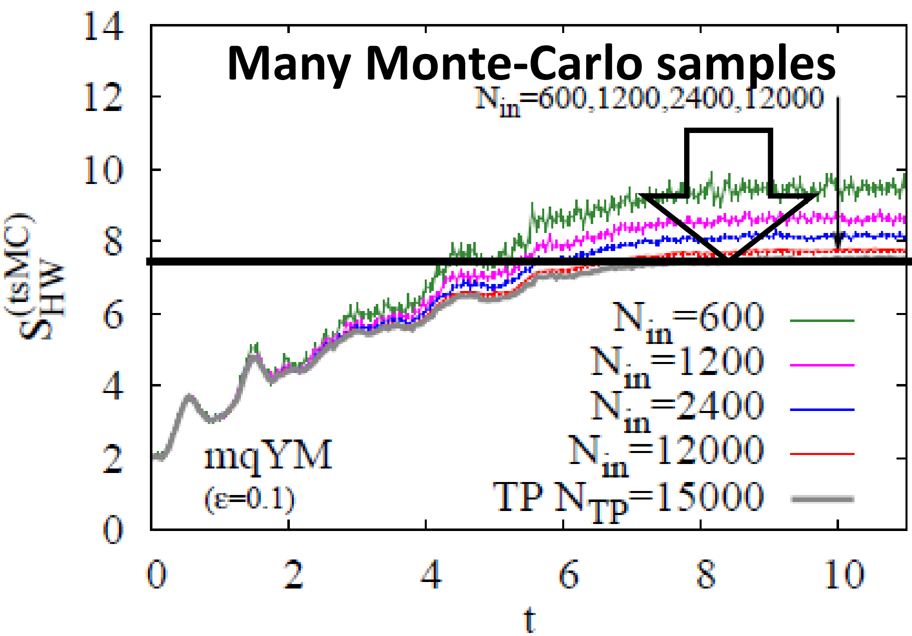
Examples in quantum mechanics

H.T., H.Iida, T.Kunihiro, A.Ohnishi, and T.T.Takahashi, PTEP in press

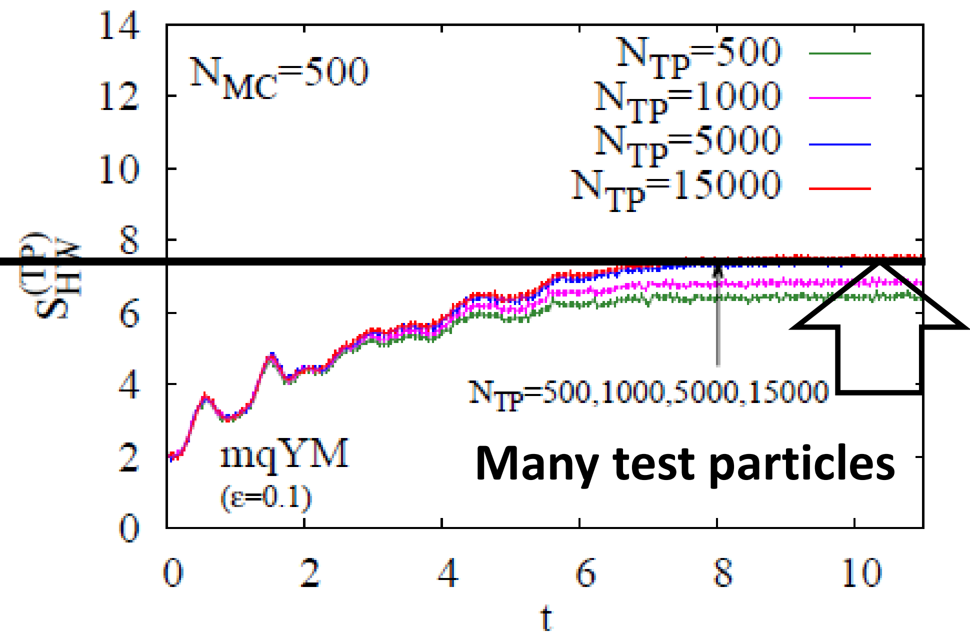
$$H = \frac{p_1^2}{2m} + \frac{p_2^2}{2m} + \frac{1}{2}g^2 q_1^2 q_2^2 + \frac{\epsilon}{4}q_1^4 + \frac{\epsilon}{4}q_2^4$$

We set $m = 1, g = 1, \epsilon = 0.1$

Two step Monte-Carlo method



Test particle method



Two numerical method describe the entropy production.

Both results are consistent within error bars.

Extension to Yang-Mills field

Classical Yang-Mills field

H.T., H.Iida, T.Kunihiro, A.Ohnishi, and T.T.Takahashi , work in progress

We will work in temporal gauge $A_0^a = 0$

Then Hamiltonian in a non-compact formalism is given by

$$H = \frac{1}{2}\sum_{x,a,i} E_i^a(x)^2 + \frac{1}{4}\sum_{x,a,i,j} F_{ij}^a(x)^2$$

$$F_{ij}^a = \partial_i A_j^a(x) - \partial_j A_i^a(x) + \sum_{b,c} f^{abc} A_i^b(x) A_j^c(x)$$

Canonical variables are $(A_i^a(x), E_i^a(x))$

EOM is

$$\dot{A}_i^a(x) = E_i^a(x)$$

$$\dot{E}_i^a(x) = \sum_j \partial_j F_{ij}^a(x) + \sum_{b,c,j} f^{abc} A_j^b(x) F_{ji}^c(x)$$

For the extension, we consider

$$(q,p) \rightarrow (A_i^a(x), E_i^a(x))$$

c.f. S. Mrowczynski, B. Muller(1994) (in a scalar field case)

Product ansatz

H.T., H.Iida, T.Kunihiro, A.Ohnishi, and T.T.Takahashi , work in progress

In higher dimension, we need a larger number of samples and test particles. We consider product ansatz to converge numerical results.

We assume that Husimi function is decomposed into the product of that of 1-dim degree of freedom.

$$f_H(q,p;t) = \prod_i^D h_i(q_i,p_i;t)$$

But we solve a equation of motion of full degrees of freedom unlike Hartree approximation.

Then Husimi-Wehrl entropy is written by

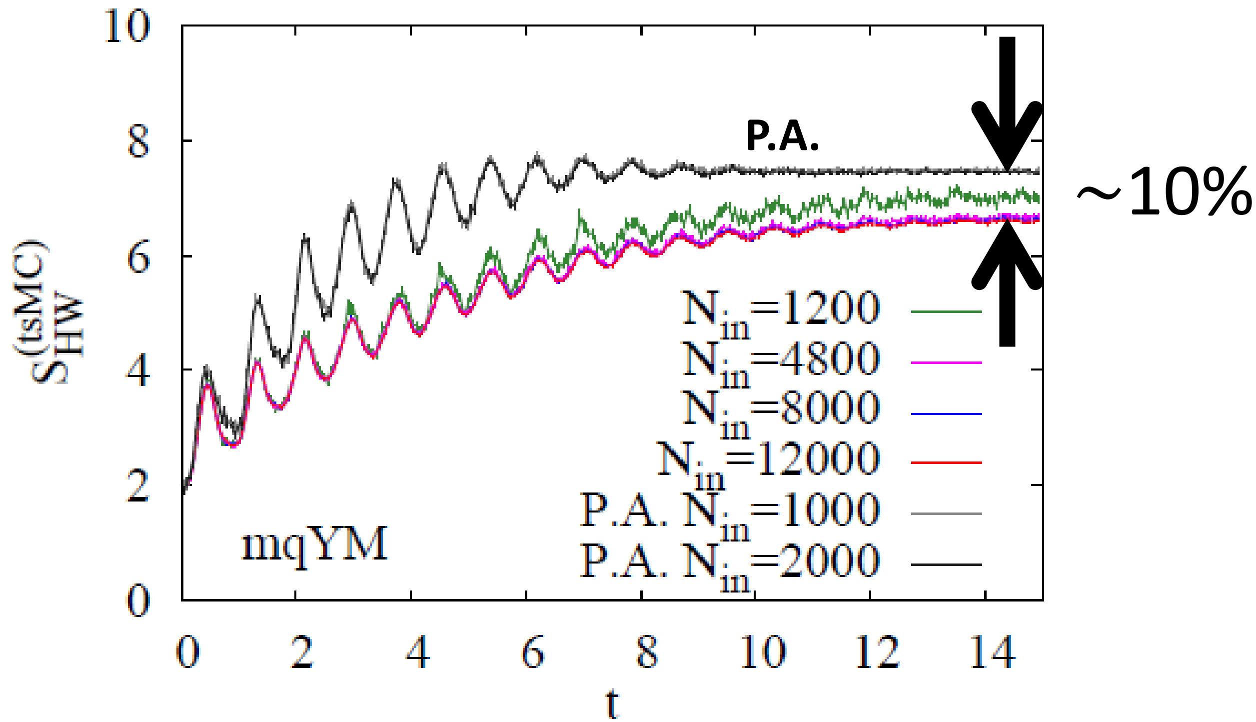
$$S_{HW} \simeq -\sum_i^D \int \frac{dq_i dp_i}{2\pi\hbar} h(q_i,p_i;t) \log h(q_i,p_i;t)$$

Check in the case of quantum mechanical systems

H.T., H.Iida, T.Kunihiro, A.Ohnishi, and T.T.Takahashi , work in progress

2-dim case ($\epsilon = 1$)

in two step Monte-Carlo method

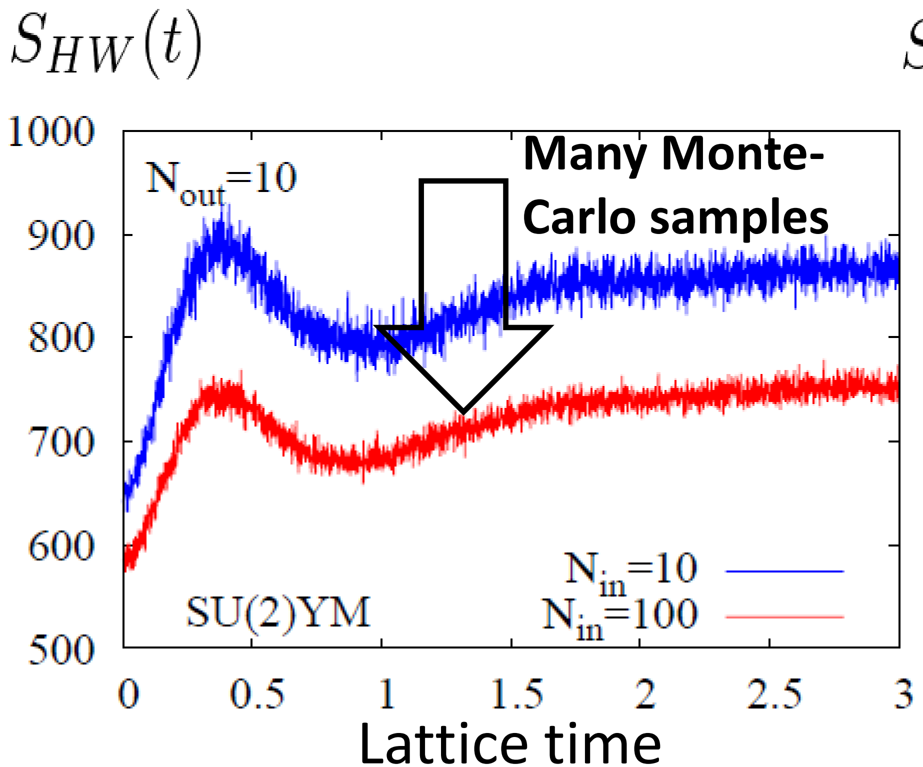


Product ansatz gives consistent results within 10% error bar.

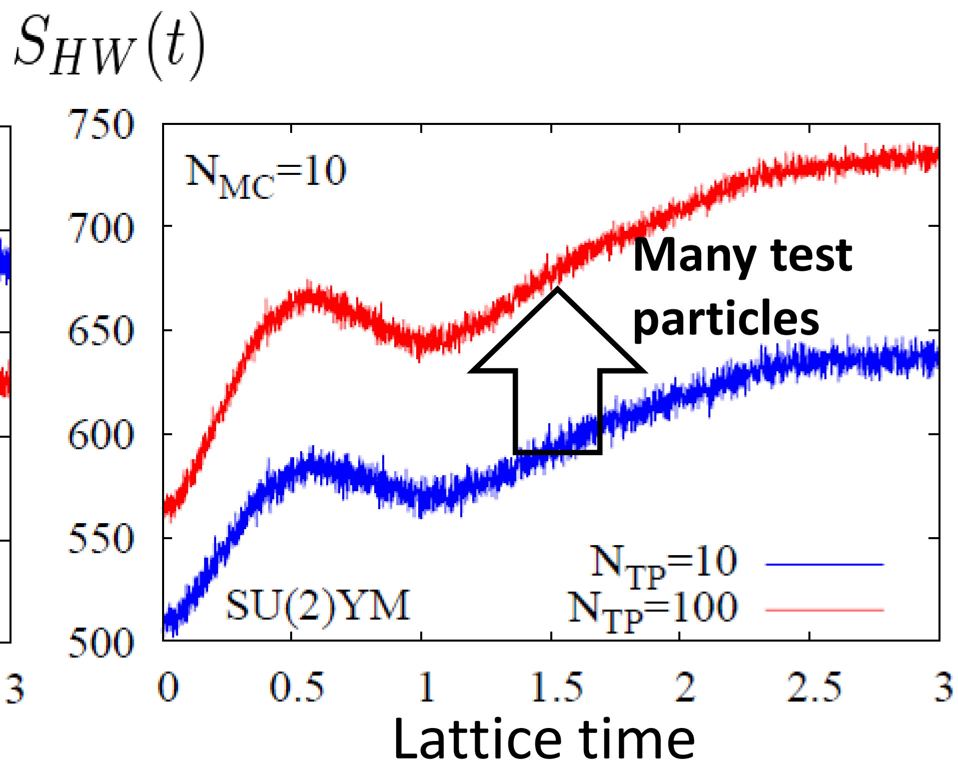
4^3 lattice SU(2) Yang-Mills field

H.T., H.Iida, T.Kunihiro, A.Ohnishi, and T.T.Takahashi , work in progress

Two step Monte-Carlo method



Test particle method

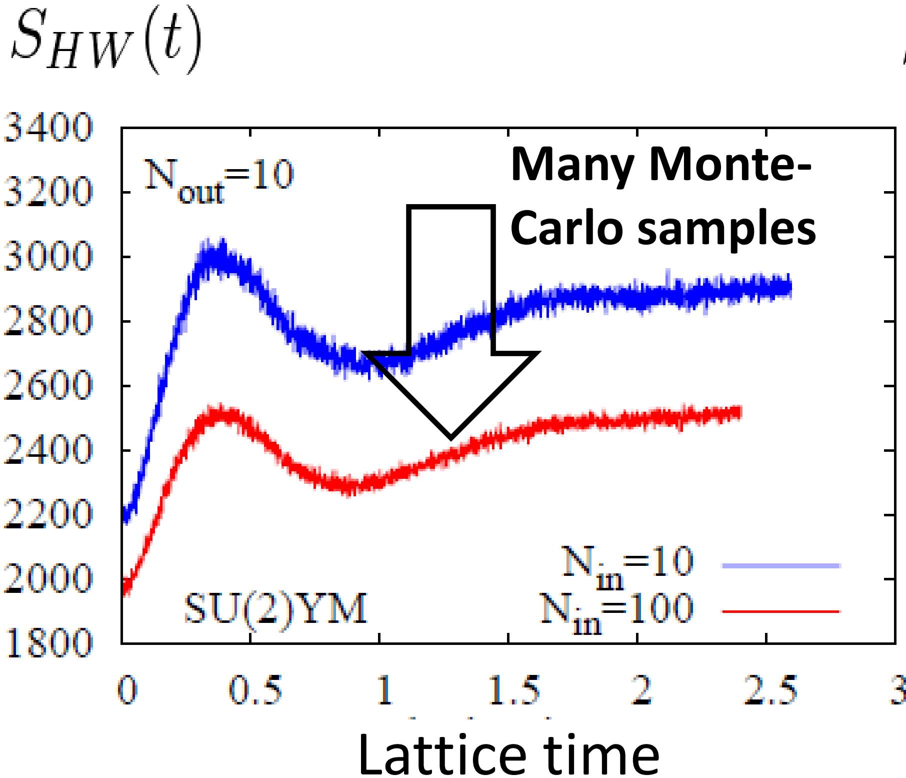


We see that the entropy as given by H-W entropy is created in Yang-Mills theory though in the product ansatz.

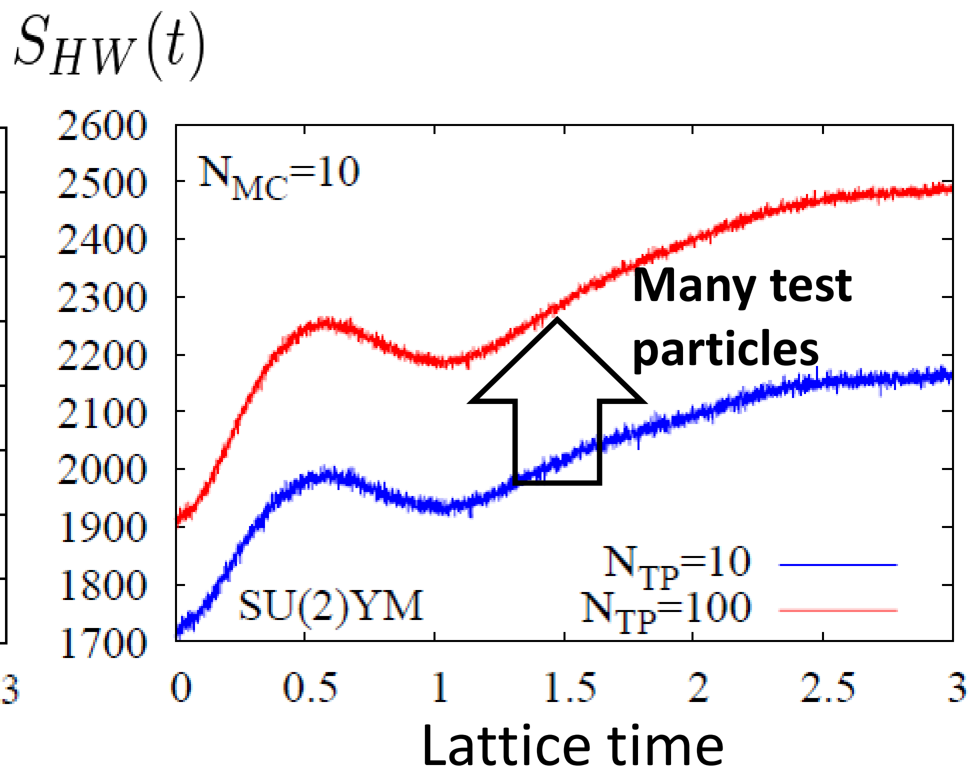
6^3 lattice SU(2) Yang-Mills field

H.T., H.Iida, T.Kunihiro, A.Ohnishi, and T.T.Takahashi , work in progress

Two step Monte-Carlo method



Test particle method



Husimi-Wehrl entropy is produced in a lager lattice size.
The behavior are the same as that in 4^3 lattice qualitatively.

Summary

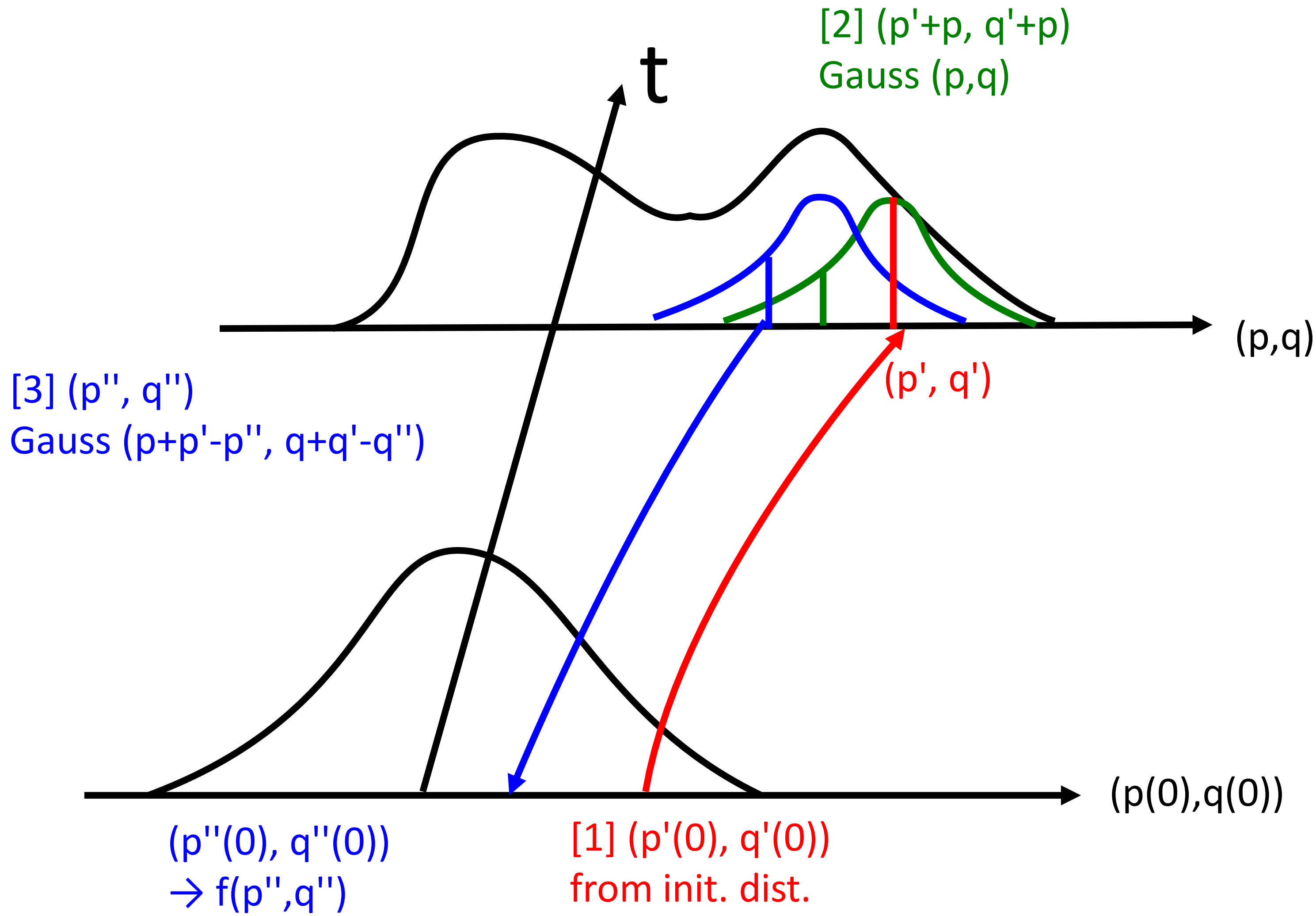
- We have proposed the entropy defined by a quantum distribution function and calculate Husimi-Wherl(H-W) entropy.
- We have checked that our numerical methods work in quantum mechanical systems and they have been applicable in Yang-Mills field theory with product ansatz.
- We have proposed product ansatz and found that it gives H-W entropy within 10% accuracy in quantum mechanical systems.
- We have calculated H-W entropy in Yang-Mills field theory on lattice and showed that H-W entropy has been produced. This result suggests that thermal entropy has been created in Yang-Mills theory.

Future work

- Consider the physical meaning of product ansatz.
- Calculate H-W entropy on a larger lattice.
- Check the entropy production in expanding geometry and discuss a relation to early thermalization.

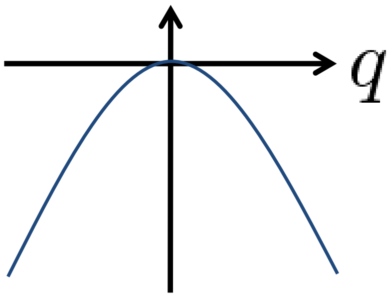
Back up

Sampling with Wigner function



Inverted harmonic oscillator

T. Kunihiro, B. Muller, A. Ohnishi, A. Shafer(2009) (Analytic solution)



Hamiltonian is

$$H = \frac{p^2}{2m} - \frac{1}{2}\lambda^2 q^2$$

Initial condition of Wigner function

$$f_W(p, q; t = 0) = 2\exp(-\frac{1}{\hbar\omega}p^2 - \frac{\omega}{\hbar}q^2)$$

Analytic solution of H-W entropy is given by

$$\boxed{S(t) = \frac{1}{2}\log\frac{A(t)}{4} + 1}$$

with $$A(t) = 2(\sigma\rho\cosh 2\lambda t + 1 + \delta\delta')$$

$$\sigma = \frac{\lambda^2 + \omega^2}{2\lambda\omega}, \delta = \frac{\lambda^2 - \omega^2}{2\lambda\omega}, \rho = \frac{\Delta^2 + \lambda^2}{2\Delta\lambda}, \delta' = \frac{\Delta^2 - \lambda^2}{2\Delta\lambda}$$

Inverted harmonic oscillator

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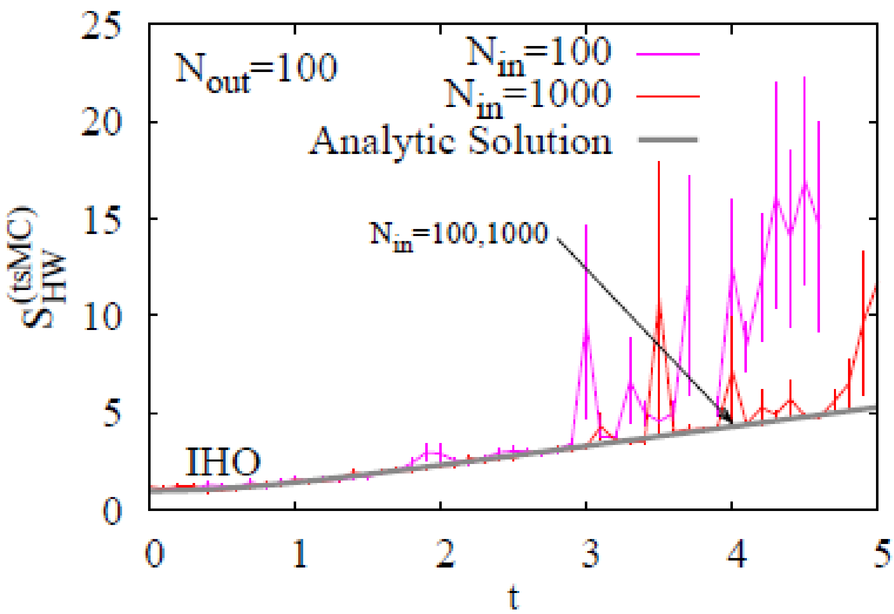
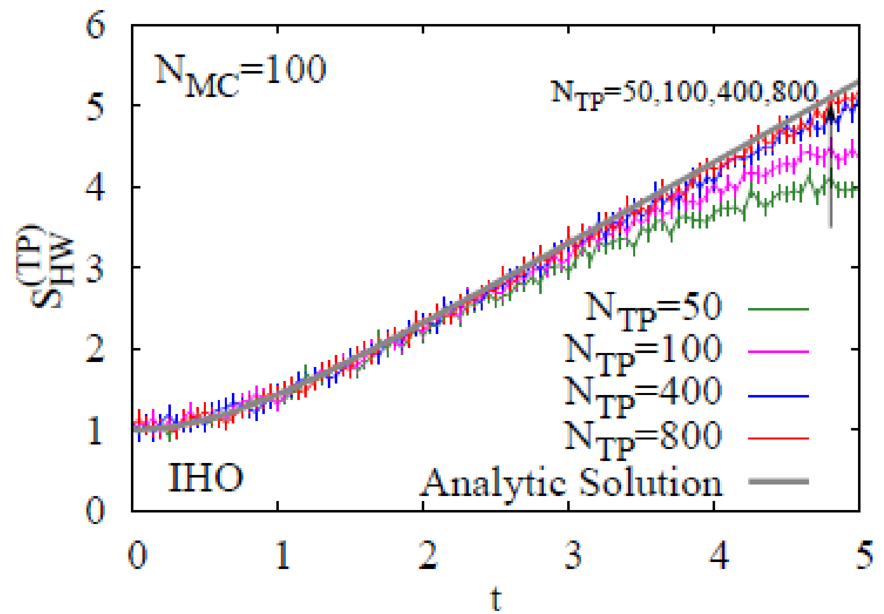
Two step Monte-Carlo method



Test particle method



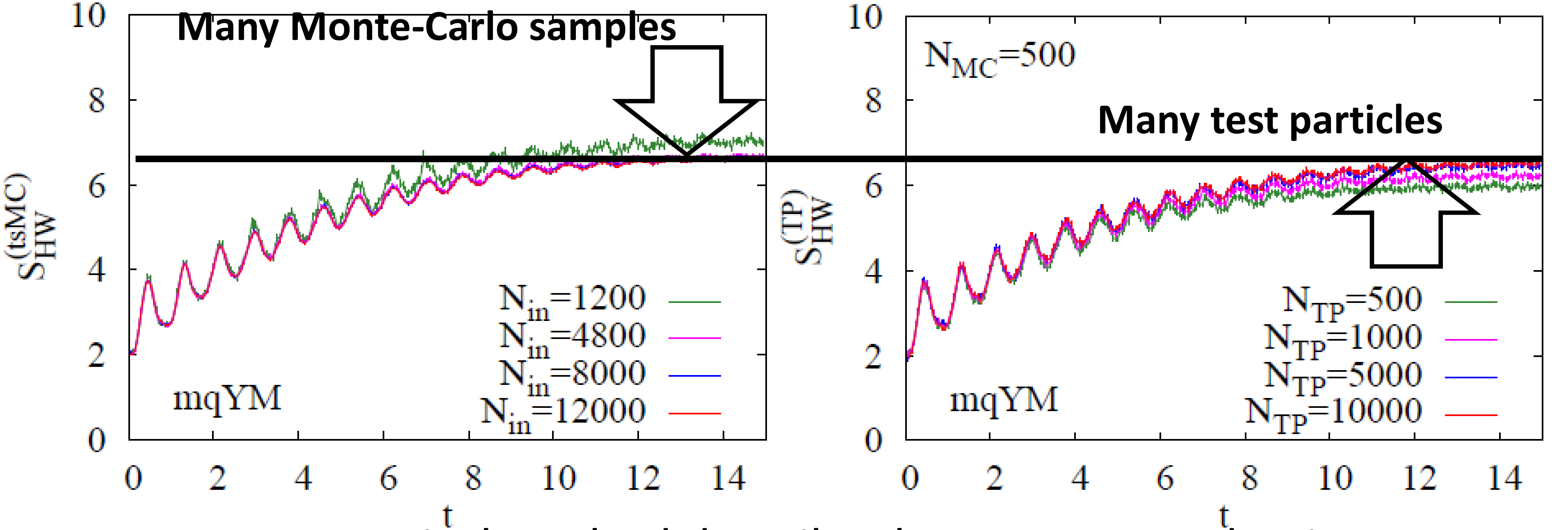
Numerical results are consistent with analytic solutions in large number of samples or test particles.

Simulations in quantum mechanics

$$H = \frac{p_1^2}{2m} + \frac{p_2^2}{2m} + \frac{1}{2}g^2 q_1^2 q_2^2 + \frac{\epsilon}{4}q_1^4 + \frac{\epsilon}{4}q_2^4$$

We set $m = 1, g = 1, \epsilon = 1$

Two step Monte-Carlo method **Test particle method**



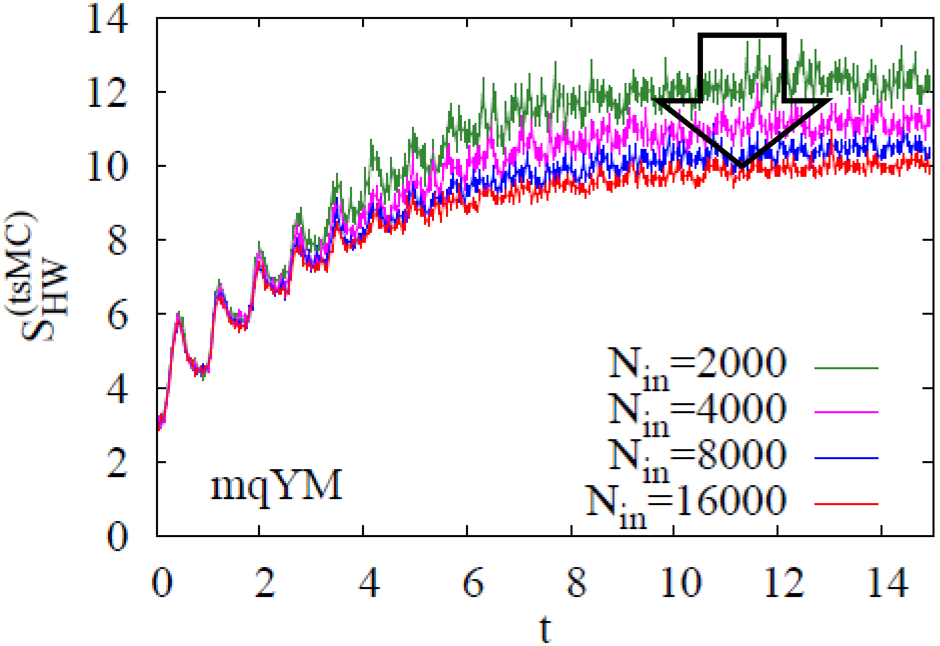
Two numerical method describe the entropy production.

Both results are consistent within error bars.

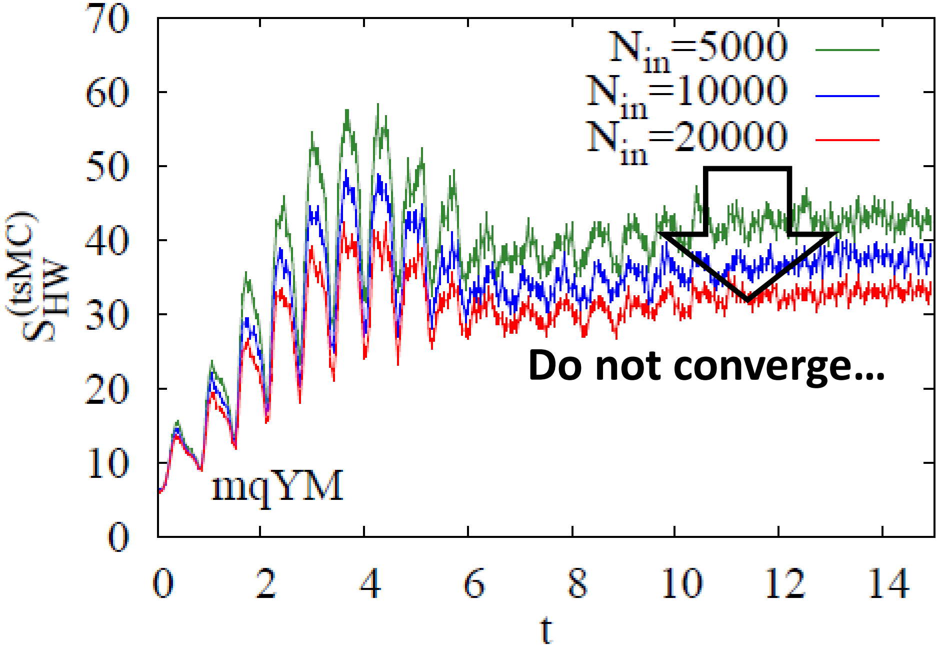
Quantum mechanics in higher dim.

$$H = \sum_{i}^{D} \frac{p_i^2}{2m} + \frac{g^2}{2} \sum_{i \neq j}^{D} q_i^2 q_j^2 + \frac{\epsilon}{4} \sum_{i}^{D} q_i^4$$

3-dim case



6-dim case



We need more samples and test-particles in higher dimension.
In a large dimensional case like quantum field theory, we apply some approximations.

Product ansatz

We assume that Husimi function is decomposed into the production of that of 1-dim degree of freedom.

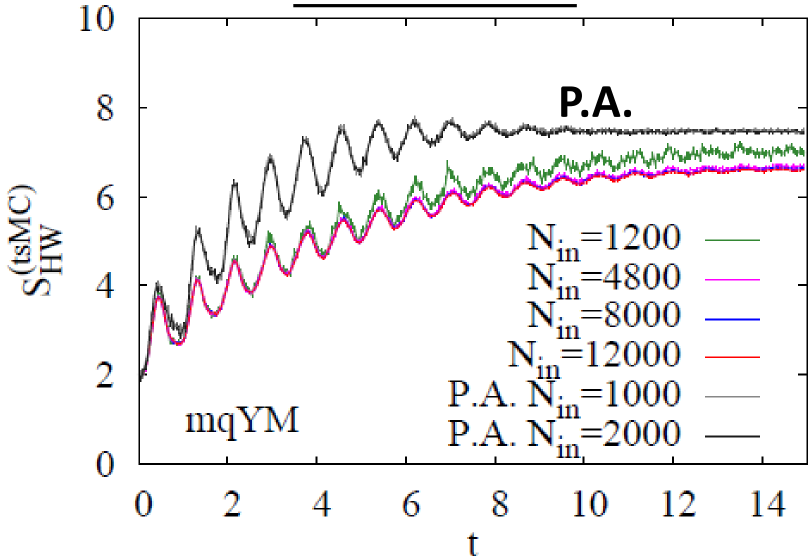
$$f_H(q,p;t) = \prod_i^D h_i(q_i,p_i;t)$$

Then Husimi-Wehrl entropy is written by

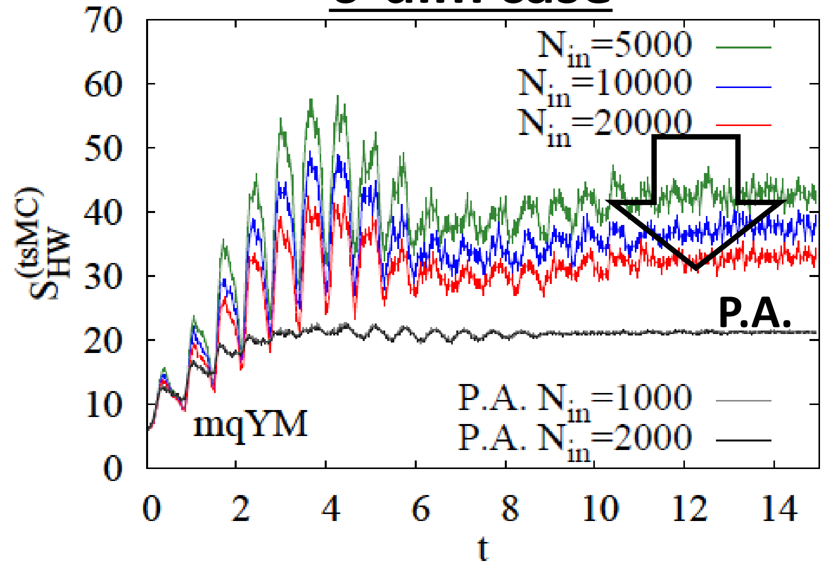
$$S_{HW} \simeq -\sum_i^D \int \frac{dq_i dp_i}{2\pi\hbar} h(q_i,p_i;t) \log h(q_i,p_i;t)$$

Check in the case of quantum mechanical systems.

2-dim case



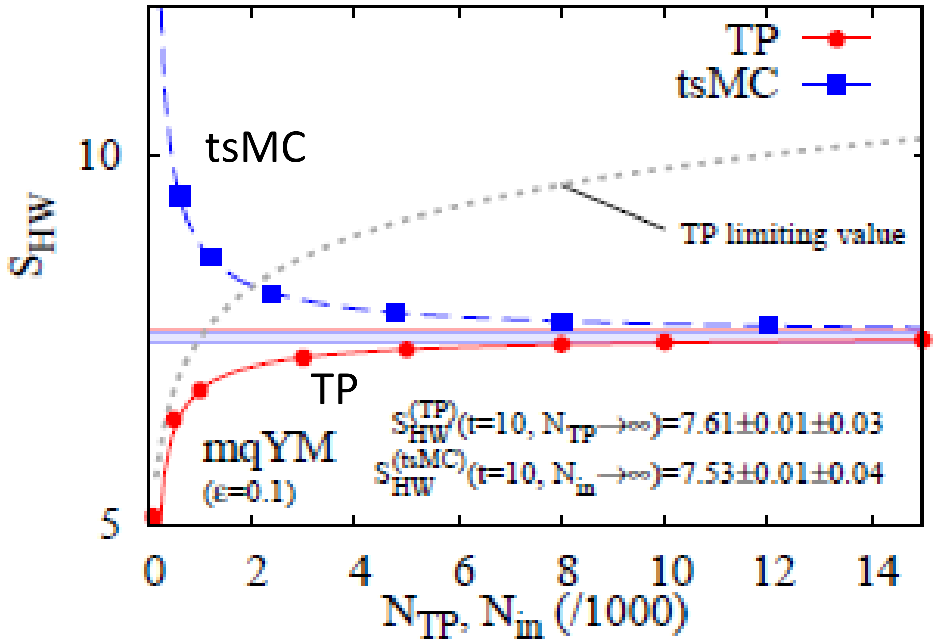
6-dim case



Product ansatz gives consistent results within 10% error bar.

The results of product ansatz converge even in higher dimension.

Large N value at time t=10



Both results are consistent within error bars.

Extension to Yang-Mills field

Tsukiji et. al. in progress

We set inner product of fields
The extension is straightforward.

$$AB = \sum_{i,a} \int d^3x A_i^a(x) B_i^a(x)$$

Husimi functional

$$f_H[A,E;t] = \int \frac{DA'DE'}{(\pi\hbar)^{N_D}} \exp[-\frac{1}{\hbar\Delta}(A-A')^2 - \frac{\Delta}{\hbar}(E-E')^2] f_W[A',E';t]$$

Husimi-Wehrl entropy

$$S_{HW}(t) = -\int \frac{DADE}{(2\pi\hbar)^{N_D}} f_H[A,E;t] \log f_H[A,E;t]$$

Initial condition of Wigner functional

$$f_W[A,E:t=0] = 2^{N_D} \exp[-\frac{1}{\hbar\omega}A^2 - \frac{\omega}{\hbar}E^2]$$