Testing the Witten-Veneziano Formula on the Lattice

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Veneziano-Witten formula

The light meson spectrum exhibits a peculiar pattern:

- $\pi^\pm, \pi^0, K^\pm, K^0, \bar{K}^0, \eta$ ("octet mesons") have masses from $\sim 135 \ldots 548$ MeV
- In addition there is a "flavor-singlet", the $\eta'$
- If $m_u = m_d = m_s$ all 9 mesons should have the same mass

Surprisingly, $M_{\eta'} \approx 958$ MeV $\gg M_{\text{octet}}$

Solution to this puzzle:

- Large $M_{\eta'}$ is caused by the QCD vacuum structure and the $U(1)_A$ anomaly.
  
  Weinberg (1975), Belavin et al. (1975), t’Hooft (1976), Witten (1979), Veneziano (1979)

The $U(1)$ axial current is anomalously broken, i.e. even for $m_f = 0$:

$$\partial_\mu A^0_\mu = \frac{N_f g^2}{32 \pi^2} G^{\alpha}_{\mu \nu} \tilde{G}^{\alpha, \mu \nu} \neq 0$$

Adler (1969), Jackiw and Bell (1969)

- However, contribution to $M_{\eta'}$ vanishes to all orders in perturbation theory
- Instantons with non-trivial topology provide non-perturbative explanation

Belavin et al. (1975), t’Hooft (1976)
Assume we can compute masses (decay constants, mixing parameters) for $\pi$, $K$, $\eta$, $\eta'$ ...how can one establish a relation to the $U(1)_A$ anomaly?

- For large $N_c$, $g^2 N_c = \text{const}$, $N_f = \text{const}$ and $m_q = 0$ one can derive the Veneziano-Witten formula:

$$\frac{4N_f}{f_0^2} \chi_\infty = M_{\eta'}^2 + \mathcal{O}(1/N_c^2)$$

Veneziano (1979)

- $\chi_\infty$ is the susceptibility in pure Yang-Mills theory
- $f_0 \neq f_\pi$ is the singlet decay constant
- The flavor-singlet $\eta'$ is not a Goldstone boson in the chiral limit
- Including $m_q \neq 0$ effects one has

$$\frac{4N_f}{f_0^2} \chi_\infty = M_{\eta'}^2 + M_\eta^2 - 2M_K^2$$

Veneziano (1979)

- From a modern perspective the formula is obtained in $\chi$PT at LO for a combined power counting scheme

$$\mathcal{O}(p^2) \sim \mathcal{O}(m_q) \sim \mathcal{O}(1/N_c)$$
The Veneziano-Witten formula

\[ \chi_\infty = \frac{f_0^2}{4N_f} (M_{\eta'}^2 + M_\eta^2 - 2M_K^2) \]

connects:

- Topological susceptibility \( \chi_\infty \) in pure Yang-Mills gauge theory
- Meson masses \( M_K, M_\eta, N_{\eta'} \)
- Singlet decay constant \( f_0 \) ← main focus of this talk; need to look into \( \eta, \eta' \)–mixing!

Calculate all these quantities (for \( N_c = 3 \)) using:

- Dedicated simulations in the quenched setup for \( \chi_\infty \)
- \( N_f = 2 + 1 + 1 \) WtmLQCD configurations provided by ETMC


Can we (successfully) test the VW formula on the lattice?
Quenched computation of $\chi_\infty$

Can compute $\chi_\infty$ from (stochastically estimated) density chains

$$\chi_t = m_1 \cdot \ldots \cdot m_5 \cdot a^{16} \sum_{x_1, \ldots, x_4} \langle P_{31}(x_1)S_{12}(x_2)S_{23}(x_3) \times P_{54}(x_4)S_{45}(0) \rangle_c$$

where $S_{ij}, P_{ij}$ denote scalar and pseudoscalar densities, respectively.

→ Theoretical sound definition; needs only multiplicative renormalization factors

- Use Wilson twisted mass valence quarks ⇒ Automatic $O(a)$–improvement
- Box length fixed to 2.8 fm
- Four values of lattice spacing $a = 0.07$ fm to $a = 0.14$ fm
- Linear scaling in $a^2$ as expected
- Continuum limit $r_0 \chi_\infty = 0.049(6)_{\text{stat+sys}}$

$JHEP$ 0903:013 (2009)
Masses

- $M_\eta, M_{\eta'}$ have been computed in previous study
  
  \[ JHEP 1211 (2012) 048 \]
  \[ Phys. Rev. Lett. 111 (2013) 18, 181602 \]

- Extrapolation compatible with experimental results

- $M_{\eta'}$ has still sizable errors due to large disconnected contributions

- $\Delta M_{\eta'}$ yields significant contribution to overall uncertainty

- $M_K$ has been computed in same study

  - Requires only connected diagrams $\Rightarrow$ tiny errors; irrelevant to overall error budget

**Remark:** Fermionic quantities are correlated on each ensemble (e.g. $M_\eta, M_{\eta'}$)
Decay constants $f_P^i$ are defined from axial-vector matrix elements (amplitudes)

$$\langle 0 | A^i_\mu | P (p) \rangle = i f^i_P p_\mu, \quad P = \eta, \eta', \quad$$

On the lattice: **quark flavor basis** (i=l,s) is "natural" choice

$$A^l_\mu = \frac{1}{\sqrt{2}}(\bar{u}\gamma_\mu \gamma_5 u + \bar{d}\gamma_\mu \gamma_5 d), \quad A^s_\mu = \bar{s}\gamma_\mu \gamma_5 s.$$

$\eta$ and $\eta'$ are not flavor eigenstates; most general parametrization:

$$\begin{pmatrix} f^l_\eta & f^s_\eta \\ f^l_\eta' & f^s_\eta' \end{pmatrix} = \begin{pmatrix} f_l \cos \phi_l & -f_s \sin \phi_s \\ f_l \sin \phi_l & f_s \cos \phi_s \end{pmatrix}$$

From $\chi$PT one expects $|\phi_l - \phi_s|$ to be small, i.e. $\frac{|\phi_l - \phi_s|}{|\phi_l + \phi_s|} \ll 1$, $\phi \approx \phi_l \approx \phi_s$

- Confirmed in previous lattice study

*Phys.Rev.Lett. 111 (2013) 18, 181602*

Unfortunately, the axial vector turned out to be too noisy to determine $f_{l,s}$ or $\phi / \phi_{l,s}$ directly
Consider pseudoscalar matrix element instead of axial vector

\[ h^i_P = 2m_i < 0|P^i|P >, \quad P = \eta, \eta', \]

which can be related to axial vector via the \textbf{anomaly equation} using \( \chi \text{PT} \):

\[
\begin{pmatrix}
  h^l_\eta & h^s_\eta \\
  h^l_{\eta'} & h^s_{\eta'}
\end{pmatrix}
= \begin{pmatrix}
  \cos \phi & -\sin \phi \\
  \sin \phi & \cos \phi
\end{pmatrix}
\text{diag} \left( f_i M^2_{PS}, f_s \left( 2M^2_K - M^2_{PS} \right) \right).
\]

\[ \rightarrow \text{Residual } \chi \text{PT-dependence compared to axial-vector approach} \]

However, Veneziano-Witten formula is only LO \( \chi \text{PT} \) as well ...
Definition of $f_0$

$f_0$ is defined in different basis (octet-singlet basis):

\[
A^0_\mu = \frac{1}{\sqrt{6}}(\bar{u}\gamma_\mu\gamma_5u + \bar{d}\gamma_\mu\gamma_5d + \bar{s}\gamma_\mu\gamma_5s),
\]

\[
A^8_\mu = \frac{1}{\sqrt{3}}(\bar{u}\gamma_\mu\gamma_5u + \bar{d}\gamma_\mu\gamma_5d - 2\bar{s}\gamma_\mu\gamma_5s).
\]

- Parametrization similar to quark flavor basis:

\[
\begin{pmatrix}
    f_{\eta}^8 & f_{\eta}^0 \\
    f_{\eta'}^8 & f_{\eta'}^0
\end{pmatrix}
= 
\begin{pmatrix}
    f_8 \cos \phi_8 & -f_0 \sin \phi_0 \\
    f_8 \sin \phi_8 & f_0 \cos \phi_0
\end{pmatrix}
\]

- Again, $f_{0,8}$ and $f_{l,s}$ can be related by continuum $\chi$PT to the given order, e.g.

\[
f_0^2 = -\frac{7}{6}f_\pi^2 + 2\frac{3}{3}f_K^2 + 3\frac{2}{2}f_l^2,
\]

\[
f_0^2 = +\frac{1}{3}f_\pi^2 - 4\frac{3}{3}f_K^2 + f_l^2 + f_s^2,
\]

\[
f_0^2 = +8\frac{3}{3}f_\pi^2 - 16\frac{3}{3}f_K^2 + 3f_s^2,
\]

\[
\rightarrow \text{Not unambiguous}; \text{ they will have different systematics}
\]
Technical aside: $\eta, \eta'$ in WtmLQCD

We work in the Wilson twisted mass $N_f = 2 + 1 + 1$ (unitary) setup:

- In the physical basis 2 $\gamma$-combinations ($i\gamma_5$, $i\gamma_0\gamma_5$) available; consider only $i\gamma_5$:

  \begin{align*}
  \text{phys basis:} & & \eta^\text{phys}_l &= \frac{1}{\sqrt{2}} \bar{\psi}_l i\gamma_5 \psi_l, & \eta^\text{phys}_{c,s} &= \bar{\psi}_h \left( \frac{1 \pm \tau^3}{2} i\gamma_5 \right) \psi_h = \left\{ \begin{array}{c}
  \bar{c}i\gamma_5 c \\
  \bar{s}i\gamma_5 s
  \end{array} \right. \\
  \text{tm basis:} & & \eta^\text{tm}_l &= \frac{1}{\sqrt{2}} \bar{\chi}_l (-\tau^3) \chi_l & \eta^\text{tm}_{c,s} &= \frac{1}{2} \bar{\chi}_h \left( -\tau^1 \pm i\gamma_5 \tau^3 \right) \chi_h.
  \end{align*}

  $\Rightarrow$ Heavy operators are a sum of scalars and pseudoscalars

- Considering renormalization we have (up to an irrelevant global factor)

  \begin{align*}
  \eta^\text{tm}_{c,s,\text{renormalized}} &= Z \left( \bar{\chi}_c i\gamma_5 \chi_c - \bar{\chi}_s i\gamma_5 \chi_s \right) / 2 + (\bar{\chi}_s \chi_c + \bar{\chi}_c \chi_s) / 2 \\
  \eta^\text{tm}_{s,\text{renormalized}} &= Z \left( \bar{\chi}_s i\gamma_5 \chi_s - \bar{\chi}_c i\gamma_5 \chi_c \right) / 2 - (\bar{\chi}_s \chi_c + \bar{\chi}_c \chi_s) / 2.
  \end{align*}

- Need ratio $Z = \frac{Z_P}{Z_S}$ of (non-singlet!) scalar and pseudoscalar renormalization constants

- Use $Z$ from two methods ($M1$ and $M2$) $\Rightarrow$ different $O(a^2)$–effects for observables

*Remark:* $M_\eta$, $M_{\eta'}$, and $\phi$ do NOT depend on $Z$, but ONLY $f_l$, $f_s$ (and hence $f_0$)
\[ f_l \text{ only mildly affected by choice of } Z = Z_P/Z_S \text{ (M1 left panel, M2 right panel)} \]

- Non-linear \((r_0 M_{PS})^2\) dependence
- Strange quark mass dependence unclear
- Mildly affected by lattice artifacts
Decay constants – $f_s$

- Strongly affected by choice of $Z$ (M1 left panel, M2 right panel)
- Sizable lattice artifacts and $m_s$ dependence for M1
- Similar effect observed for $f_K$, which is also required for $f_0$!

$Z$ factor enters through operator mixing ($\mathcal{P}$ and $\mathcal{S}$) for heavy quarks ($f_l$, $f_s$, $f_k$) and $\mu_s$ ($f_K$)

$$f_K = (\mu_l + \mu_s) \frac{\langle 0 | \bar{\rho}^+_{\text{neutral}}^{+, tm} | K \rangle}{M_K^2}, \quad \mu_{c,s} = \mu_\sigma \pm Z \mu_\delta$$
Decay constants – $f_0$

$f_0$ from definition D1 and two choices of $Z$ (M1 left panel, M2 right panel)

- Clearly affected by choice of $Z$ (M1 left panel, M2 right panel)
- Visible strange quark mass dependence and non-linear $(r_0 M_{PS})^2$–dependence
- Unclear how to fit

⇒ **Don’t fit $f_0$ itself, but compute r.h.s of VW-formula first, i.e.** $\frac{f_0^2}{4N_f}(M_{\eta'}^2 + M_\eta^2 - 2M_K^2)$
Results

Plots show results for $r_0^4 \chi_\infty$ for the three definitions of $f_0$:

- Lattice artifacts depend on choice of $f_0$ definition
- Lattice artifacts mostly (much) smaller than for $f_0$ itself
- $m_s$–dependence unclear; possibly difficult to disentangle from $a^2$-effects
- $M_{PS}$–dependence small
- Relative stat. errors differ slightly for different definitions of $f_0$
- Const. fit does not describe the data well in some cases ($\chi^2/dof = 1.7...4.5$)

However, can try to estimate a systematic error from all six const. fits ...
Weight fits with p-values, $w = 1 - 2|p - 0.5|$ to obtain final result

Use mean absolute deviation from central value as sys. error (dotted lines)

$\Rightarrow r_0^4 \chi^{\text{dyn}} = 0.047(3)_{\text{stat}}(11)_{\text{sys}}$

in agreement with quenched

$r_0^4 \chi^{\text{YM}} = 0.049(6)_{\text{stat+sys}}$

Comparison in physical units problematic because in general $r_0^{\text{dyn}} \neq r_0^{\text{YM}}$

$r_0^{\text{YM}} = 0.5 \text{ fm}$ and $r_0^{\text{dyn}} = 0.474(14) \text{ fm}$ yields good agreement

Including linear terms in $(r_0 M_{PS})^2$, $(r_0 M_K)^2$ and $(a/r_0)^2$ improves $\chi^2/\text{dof}$ ...

BUT:

Many terms compatible with zero; larger errors but similar result:

$r_0^4 \chi^{\text{dyn,lin}} = 0.051(23)_{\text{stat}}$

Decay constants – $f_K$

- Left panel shows $r_0 f_K$ computed using $Z = Z_P/Z_S$ from method M1, right panel for M2
- $Z$ factor enters through operator mixing ($\mathcal{P}$ and $\mathcal{S}$) in the heavy quark sector and strange quark mass

$$f_K = (\mu_l + \mu_s) \left| \frac{\langle 0 | \tilde{\mathcal{P}}^{+ \, \text{tm}}_{\text{neutral}} | K \rangle}{M_K^2} \right|,$$

$$\mu_{c,s} = \mu_\sigma \pm Z \mu_\delta$$
Decay constant ratio – $f_s/f_K$

- Left panel shows $f_s/f_K$ computed using $Z$ from method M1, right panel for M2

$Z$ factor enters through:

- Operator mixing in the heavy quark sector ($f_s$ and $f_K$)
- Strange quark mass ($f_K$)

$f_s$ and $f_K$ show similar lattice artifacts and $\mu_s$–dependence $\rightarrow$ ratio cancels most effects
Decay constants – $f_0$

From left to right: definitions D1, D2, D3 for $f_0$

Upper row: $Z$ from method M1; Lower row: $Z$ from method M2
Overview $\chi_\infty$ from dynamical simulations

From left to right: definitions D1, D2, D3 for $f_0$

Upper row: $Z$ from method M1; Lower row: $Z$ from method M2
Results for fits including terms $\sim (r_0 M_{PS})^2$, $\sim (r_0 M_{PS})^2$ and $\sim (a/r_0)^2$

- $\chi^2$/dof good in most cases $\lesssim 1.5$
- Most additional terms compatible with zero within (large) errors
- Overall stat. error becomes very large $\rightarrow$, but central value does not change
- Adding only $\sim (a/r_0)^2$–term is not sufficient
- Fits become unstable; adding further terms not possible