Testing the Witten-Veneziano Formula on the Lattice

K. Cichy[a,b,c], E. Garcia-Ramos[d], K. Jansen[b,d], K. Ottnad[e], C. Urbach[e]

for ETM Collaboration

[a] Goethe-Universität, Institut für Theoretische Physik, Frankfurt a.M., Germany
[b] NIC, DESY Zeuthen, Germany
[c] Adam Mickiewicz University, Faculty of Physics, Poland
[d] Humboldt Universität zu Berlin, Germany
[e] Institut für Strahlen- und Kernphysik and Bethe Center for Theoretical Physics, Universität Bonn, Germany

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Veneziano-Witten formula

The light meson spectrum exhibits a peculiar pattern:

- $\pi^\pm$, π^0, $K^\pm$, K^0, $\bar{K}^0$, η ("octet mesons") have masses from ~ 135 ... 548 MeV
- In addition there is a "flavor-singlet", the η'
- If $m_u = m_d = m_s$ all 9 mesons should have the same mass

Surprisingly, $M_{\eta'} \approx 958$ MeV $\gg M_{octet}$

Solution to this puzzle:

- Large $M_{\eta'}$ is caused by the QCD vacuum structure and the $U(1)_A$ anomaly.

Weinberg (1975), Belavin et al. (1975), t'Hooft (1976), Witten (1979), Veneziano (1979)

The $U(1)$ axial current is anomalously broken, i.e. even for $m_f = 0$:

$$\partial_\mu A^0_\mu = \frac{N_f g^2}{32\pi^2} G^a_{\mu\nu} \tilde{G}^{a,\mu\nu} \neq 0$$

Adler (1969), Jackiw and Bell (1969)

- However, contribution to $M_{\eta'}$ vanishes to all orders in perturbation theory
- Instantons with non-trivial topology provide non-perturbative explanation

Belavin et al. (1975) , t'Hooft (1976)

Assume we can compute masses (decay constants, mixing parameters) for π, K, η, η'

...how can one establish a relation to the $U(1)_A$ anomaly?

- For large N_c, $g^2 N_c = const$, $N_f = const$ and $m_q = 0$ one can derive the Veneziano-Witten formula:

$$\frac{4N_f}{f_0^2}\chi_\infty = M_{\eta'}^2 + \mathcal{O}(1/N_c^2)$$

Witten (1979)

- χ_∞ is the susceptibility in pure Yang-Mills theory
- $f_0 \neq f_\pi$ is the singlet decay constant
- The flavor-singlet η' is **not** a Goldstone boson in the chiral limit
- Including $m_q \neq 0$ effects one has

$$\frac{4N_f}{f_0^2}\chi_\infty = M_{\eta'}^2 + M_\eta^2 - 2M_K^2$$

Veneziano (1979)

- From a modern perspective the formula is obtained in χPT at LO for a combined power counting scheme

$$\mathcal{O}(p^2) \sim \mathcal{O}(m_q) \sim \mathcal{O}(1/N_c)$$

The Veneziano-Witten formula

$$\chi_\infty = \frac{f_0^2}{4N_f}(M_{\eta'}^2 + M_\eta^2 - 2M_K^2)$$

connects:

- Topological susceptibility χ_∞ in pure Yang-Mills gauge theory
- Meson masses M_K, M_η, $N_{\eta'}$
- Singlet decay constant f_0 ← main focus of this talk; need to look into η,η'–mixing!

Calculate all these quantities (for $N_c = 3$) using:

- Dedicated simulations in the quenched setup for χ_∞
- $N_f = 2 + 1 + 1$ WtmLQCD configurations provided by ETMC

JHEP 0108:058 (2001), Nucl. Phys. Proc. Suppl.1258 (2004), JHEP 1006:111 (2010)

Can we (successfully) test the VW formula on the lattice?

Quenched computation of χ_∞

Can compute χ_∞ from (stochastically estimated) density chains

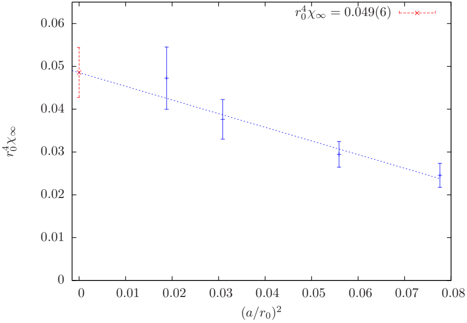
$$\chi_t = m_1 \cdot \ldots m_5 \cdot a^{16} \sum_{x_1,\ldots,x_4} \langle P_{31}(x_1) S_{12}(x_2) S_{23}(x_3) \times P_{54}(x_4) S_{45}(0) \rangle_c$$

JHEP 0903:013 (2009)

where S_{ij}, P_{ij} denote scalar and pseudoscalar densities, respectively.

→ **Theoretical sound definition**; needs only multiplicative renormalization factors

- Use Wilson twisted mass valence quarks
  ⇒ Automatic $\mathcal{O}(a)$–improvement
- Box length fixed to 2.8 fm
- Four values of lattice spacing $a = 0.07\,\mathrm{fm}$ to $a = 0.14\,\mathrm{fm}$
- Linear scaling in a^2 as expected
- Continuum limit $r_0\chi_\infty = 0.049(6)_{\mathrm{stat+sys}}$



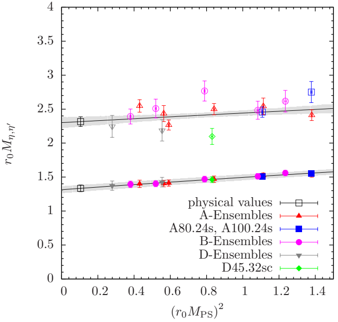
Masses

- M_η, $M_{\eta'}$ have been computed in previous study

  JHEP 1211 (2012) 048

  Phys.Rev.Lett. 111 (2013) 18, 181602

- Extrapolation compatible with experimental results
- $M_{\eta'}$ has still sizable errors due to large disconnected contributions
- $\Delta M_{\eta'}$ yields significant contribution to overall uncertainty



- M_K has been computed in same study
- Requires only connected diagrams $\Rightarrow$ tiny errors; irrelevant to overall error budget

Remark: Fermionic quantities are correlated on each ensemble (e.g. M_η, $M_{\eta'}$)

η,η'–Mixing

Decay constants f^i_{P} are defined from axial-vector matrix elements (amplitudes)

$$\langle 0| A^i_\mu |\mathrm{P}(p)\rangle = i f^i_{\mathrm{P}} p_\mu \,, \qquad \mathrm{P} = \eta, \eta' \,,$$

On the lattice: **quark flavor basis** (i=l,s) is "natural" choice

$$A^l_\mu = \frac{1}{\sqrt{2}}(\bar{u}\gamma_\mu\gamma_5 u + \bar{d}\gamma_\mu\gamma_5 d) \,, \qquad A^s_\mu = \bar{s}\gamma_\mu\gamma_5 s \,.$$

η and η' are not flavor eigenstates; most general parametrization:

$$\begin{pmatrix} f^l_\eta & f^s_\eta \\ f^l_{\eta'} & f^s_{\eta'} \end{pmatrix} = \begin{pmatrix} f_l \cos\phi_l & -f_s \sin\phi_s \\ f_l \sin\phi_l & f_s \cos\phi_s \end{pmatrix}$$

From χPT one expects $|\phi_l - \phi_s|$ to be small, i.e. $\frac{|\phi_l - \phi_s|}{|\phi_l + \phi_s|} \ll 1$, $\phi \approx \phi_l \approx \phi_s$

- Confirmed in previous lattice study

Phys.Rev.Lett. 111 (2013) 18, 181602

Unfortunately, the axial vector turned out to be too noisy to determine $f_{l,s}$ or ϕ / $\phi_{l,s}$ directly

Pseudoscalar amplitudes

Consider pseudoscalar matrix element instead of axial vector

$$h^i_{\mathrm{P}} = 2m_i < 0|P^i|\mathrm{P} > , \quad \mathrm{P} = \eta, \eta' \, ,$$

which can be related to axial vector via the **anomaly equation** using χPT:

$$\begin{pmatrix} h^l_\eta & h^s_\eta \\ h^l_{\eta'} & h^s_{\eta'} \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \mathrm{diag}\left(f_l M^2_{PS}, f_s \left(2M^2_K - M^2_{PS}\right)\right) \, .$$

Phys.Rev. D58 (1998) 114006, Phys.Lett. B449 (1999) 339-346

→ Residual χPT-dependence compared to axial-vector approach

However, Veneziano-Witten formula is only LO χPT as well ...

Definition of f_0

f_0 is defined in different basis (octet-singlet basis):

$$A^0_\mu = \frac{1}{\sqrt{6}}(\bar{u}\gamma_\mu\gamma_5 u + \bar{d}\gamma_\mu\gamma_5 d + \bar{s}\gamma_\mu\gamma_5 s)\,,$$
$$A^8_\mu = \frac{1}{\sqrt{3}}(\bar{u}\gamma_\mu\gamma_5 u + \bar{d}\gamma_\mu\gamma_5 d - 2\bar{s}\gamma_\mu\gamma_5 s)\,.$$

- Parametrization similar to quark flavor basis:

$$\begin{pmatrix} f^8_\eta & f^0_\eta \\ f^8_{\eta'} & f^0_{\eta'} \end{pmatrix} = \begin{pmatrix} f_8 \cos\phi_8 & -f_0 \sin\phi_0 \\ f_8 \sin\phi_8 & f_0 \cos\phi_0 \end{pmatrix}$$

- Again, $f_{0,8}$ and $f_{l,s}$ can be related by continuum χPT to the given order, e.g.

$$f_0^2 = -7/6 f_\pi^2 + 2/3 f_K^2 + 3/2 f_l^2\,, \qquad (D1)$$
$$f_0^2 = +1/3 f_\pi^2 - 4/3 f_K^2 + f_l^2 + f_s^2\,, \qquad (D2)$$
$$f_0^2 = +8/3 f_\pi^2 - 16/3 f_K^2 + 3 f_s^2\,, \qquad (D3)$$

→ **Not unambiguous**; they will have different systematics

Technical aside: η,η' in WtmLQCD

We work in the Wilson twisted mass $N_f = 2+1+1$ (unitary) setup:

- In the physical basis 2 γ-combinations ($i\gamma_5$, $i\gamma_0\gamma_5$) available; consider only $i\gamma_5$:

phys basis: $$\eta_l^{phys} = \frac{1}{\sqrt{2}}\bar\psi_l i\gamma_5\psi_l\,, \qquad \eta_{c,s}^{phys} = \bar\psi_h\left(\frac{1\pm\tau^3}{2}i\gamma_5\right)\psi_h = \begin{cases}\bar c i\gamma_5 c\\ \bar s i\gamma_5 s\end{cases},$$

tm basis: $$\eta_l^{tm} = \frac{1}{\sqrt{2}}\bar\chi_l\left(-\tau^3\right)\chi_l \qquad \eta_{c,s}^{tm} = \frac{1}{2}\bar\chi_h\left(-\tau^1 \pm i\gamma_5\tau^3\right)\chi_h\,.$$

$\Rightarrow$ Heavy operators are a sum of scalars and pseudoscalars

- Considering renormalization we have (up to an irrelevant global factor)

$$\eta_{c,renormalized}^{tm} = Z\left(\bar\chi_c i\gamma_5\chi_c - \bar\chi_s i\gamma_5\chi_s\right)/2 + \left(\bar\chi_s\chi_c + \bar\chi_c\chi_s\right)/2$$
$$\eta_{s,renormalized}^{tm} = Z\left(\bar\chi_s i\gamma_5\chi_s - \bar\chi_c i\gamma_5\chi_c\right)/2 - \left(\bar\chi_s\chi_c + \bar\chi_c\chi_s\right)/2\,.$$

- Need ratio $Z = \frac{Z_P}{Z_S}$ of (non-singlet!) scalar and pseudoscalar renormalization constants
- Use Z from two methods (**M1** and **M2**) $\Rightarrow$ different $\mathcal{O}(a^2)$–effects for observables

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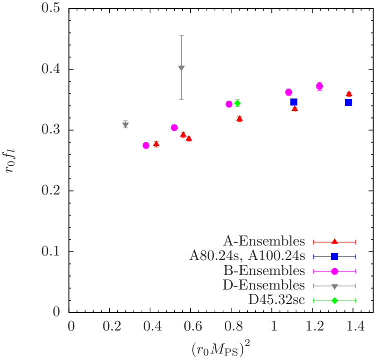
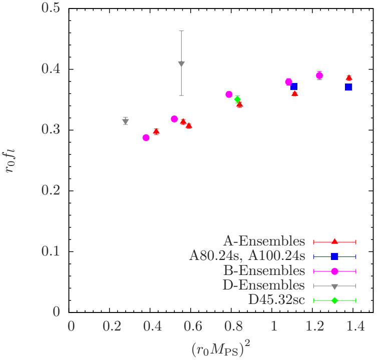
Remark: M_η, $M_{\eta'}$ and ϕ do NOT depend on Z, but ONLY f_l, f_s (and hence f_0)

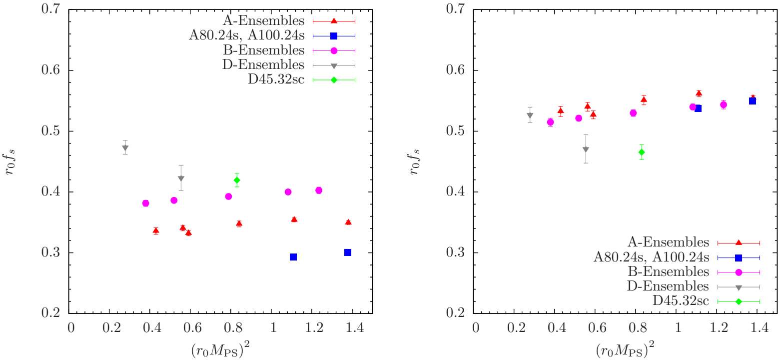
Decay constants – f_l





- f_l only mildly affected by choice of $Z = Z_P/Z_S$ (M1 left panel, M2 right panel)
- Non-linear $(r_0 M_{\rm PS})^2$ dependence
- Strange quark mass dependence unclear
- Mildly affected by lattice artifacts

Decay constants – f_s

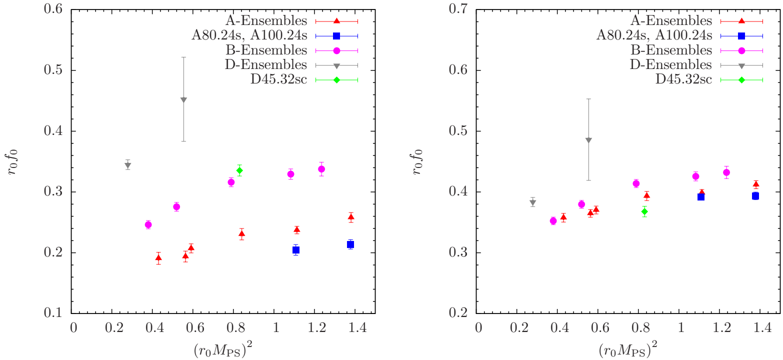


- Strongly affected by choice of Z (M1 left panel, M2 right panel)
- Sizable lattice artifacts and m_s dependence for M1
- Similar effect observed for f_K, which is also required for f_0!

Z factor enters through operator mixing ($\mathcal{P}$ and $\mathcal{S}$) for heavy quarks (f_l, f_s, f_k) and μ_s (f_K)

$$f_K = (\mu_l + \mu_s)\,\frac{\langle 0|\tilde{\mathcal{P}}^{+,tm}_{neutral}|K\rangle}{M_K^2}\,, \qquad \mu_{c,s} = \mu_\sigma \pm Z\mu_\delta$$

Decay constants – f_0

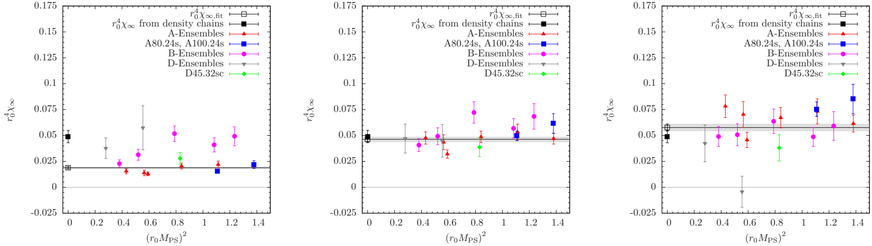


f_0 from definition D1 and two choices of Z (M1 left panel, M2 right panel)

- Clearly affected by choice of Z (M1 left panel, M2 right panel)
- Visible strange quark mass dependence and non-linear $(r_0 M_{\mathrm{PS}})^2$–dependence
- Unclear how to fit

$\Rightarrow$ **Don't fit f_0 itself, but compute r.h.s of VW-formula first, i.e.** $\frac{f_0^2}{4N_f}(M_{\eta'}^2 + M_\eta^2 - 2M_K^2)$

Results



Plots show results for $r_0^4\chi_\infty$ for the three definitions of f_0:

- Lattice artifacts depend on choice of f_0 definition
- Lattice artifacts mostly (much) smaller than for f_0 itself
- m_s–dependence unclear; possibly difficult to disentangle from a^2-effects
- M_{PS}–dependence small
- Relative stat. errors differ slightly for different definitions of f_0
- Const. fit does not describe the data well in some cases ($\chi^2/dof = 1.7 \ldots 4.5$)

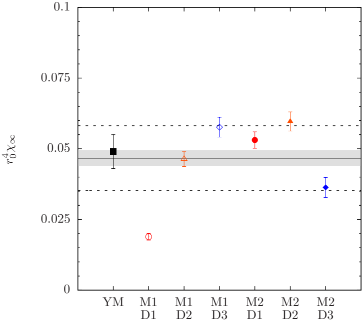
However, can try to estimate a systematic error from all six const. fits ...

- Weight fits with p-values, $w = 1 - 2|p - 0.5|$ to obtain final result
- Use mean absolute deviation from central value as sys. error (dotted lines)

$$\Rightarrow \quad r_0^4 \chi_\infty^{\rm dyn} = 0.047(3)_{\rm stat}(11)_{\rm sys}$$

in agreement with quenched

$$r_0^4 \chi_\infty^{\rm YM} = 0.049(6)_{\rm stat+sys}$$



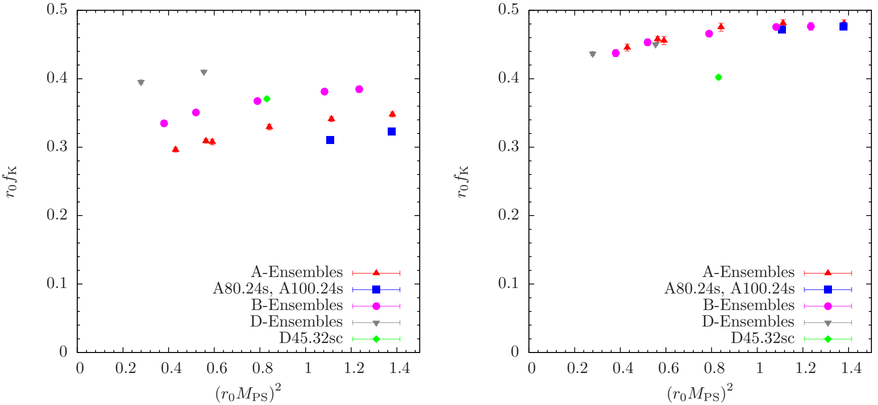
- Comparison in physical units problematic because in general $r_0^{dyn} \neq r_0^{\rm YM}$
- $r_0^{\rm YM} = 0.5\,{\rm fm}$ and $r_0^{\rm dyn} = 0.474(14)\,{\rm fm}$ yields good agreement *Nucl.Phys. B887 (2014) 19-68*
- Including linear terms in $(r_0 M_{\rm PS})^2$, $(r_0 M_K)^2$ and $(a/r_0)^2$ improves χ^2/dof ...

BUT:

- Many terms compatible with zero; larger errors but similar result:

$$r_0^4 \chi_\infty^{\rm dyn,lin} = 0.051(23)_{\rm stat}$$

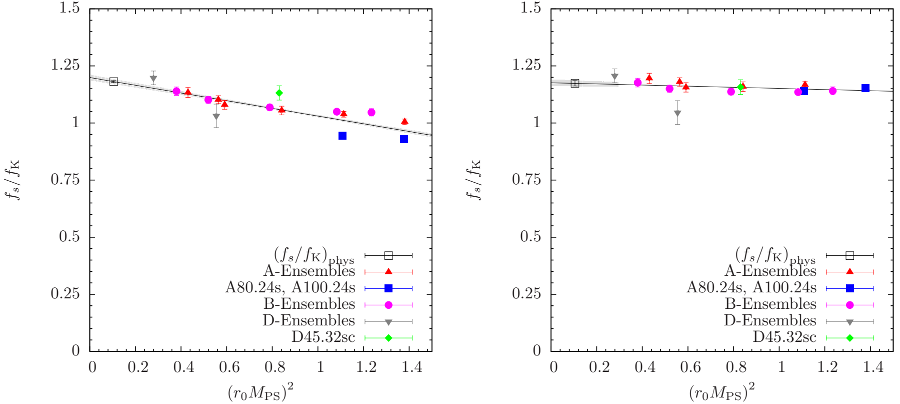
Decay constants – f_K



- Left panel shows $r_0 f_K$ computed using $Z = Z_P/Z_S$ from method M1, right panel for M2
- Z factor enters through operator mixing ($\mathcal{P}$ and $\mathcal{S}$) in the heavy quark sector and strange quark mass

$$f_K = (\mu_l + \mu_s)\,\frac{\langle 0|\tilde{\mathcal{P}}^{+,tm}_{neutral}|K\rangle}{M_K^2}\,, \qquad \mu_{c,s} = \mu_\sigma \pm Z\mu_\delta$$

Decay constant ratio – f_s/f_K



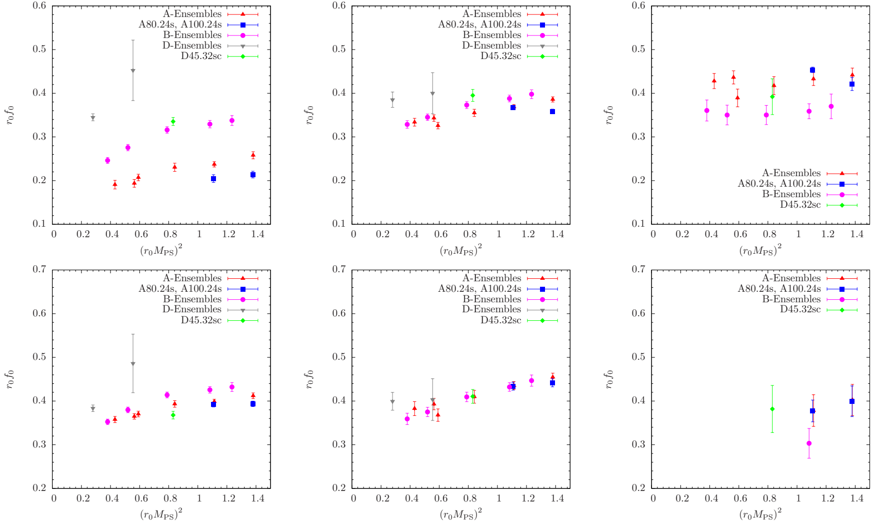
- Left panel shows f_s/f_K computed using Z from method M1, right panel for M2

Z factor enters through:

- Operator mixing in the heavy quark sector (f_s and f_K)
- Strange quark mass (f_K)

f_s and f_K show similar lattice artifacts and μ_s–dependence → ratio cancels most effects

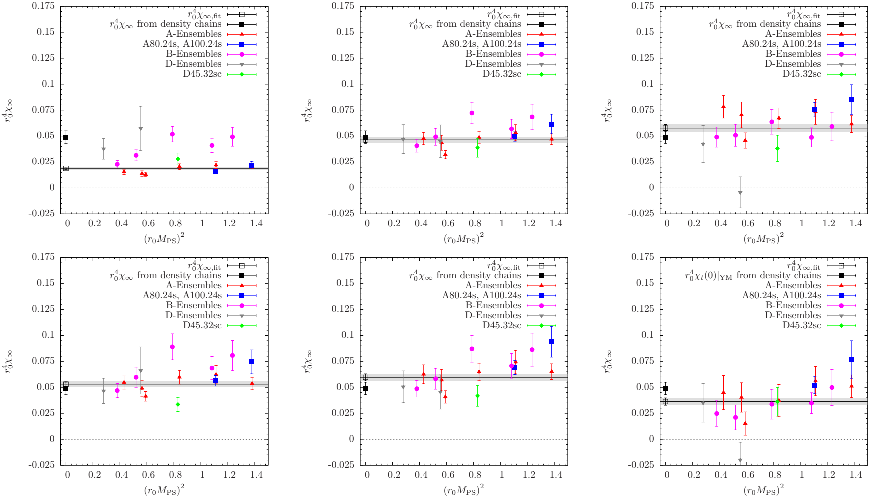
Decay constants – f_0



From left to right: definitions D1, D2, D3 for f_0

Upper row: Z from method M1; Lower row: Z from method M2

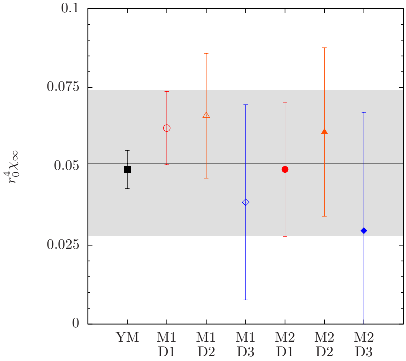
Overview χ_∞ from dynamical simulations



From left to right: definitions D1, D2, D3 for f_0

Upper row: Z from method M1; Lower row: Z from method M2

Results for fits including terms $\sim (r_0 M_{\rm PS})^2)$, $\sim (r_0 M_{\rm PS})^2$ and $\sim (a/r_0)^2$



- χ^2/dof good in most cases $\lesssim 1.5$
- Most additional terms compatible with zero within (large) errors
- Overall stat. error becomes very large $\rightarrow$, but central value does not change
- Adding only $\sim (a/r_0)^2$–term is not sufficient
- Fits become unstable; adding further terms not possible