What is QFT? Resurgent trans-series, Lefschetz thimbles, and new exact saddles

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Physical motivation

- Non-perturbative definition of non-trivial QFT in continuum. (Is there a useful formalism with *practical utility,* not formal, to be compared with lattice field theory?)

- Can we make semi-classical method exact? Resurgent expansion, Lefschetz thimble decomposition, (When is it possible?)

Asking above questions, you find yourself addressing

- IR-renormalon puzzle in QFT: Perturbation theory ill-defined

- Multi-instanton calculus: How to do it properly? Is it well-defined?

- Meaning of non-instanton saddles (there are many)

Mathematical motivation

- Resurgence: "New" idea in mathematics(Ecalle, 1980s, roots in Stokes 1860s)

- Resurgence= Unification of perturbation theory and non-perturbative sectors (ODE, non-linear ODE, Integrals, PDEs,...)

- Perturbation theory: divergent (asymptotic) series

- Series expansion $\Rightarrow$ trans-series expansion (Does it add something new?)

- Transseries expansion well-defined under analytic continuation (naturally incorporates Stokes phenomena, and Lefschetz cycle decomposition)

- **Main promise:** Perturbative and non-perturbative sectors entwined very deeply. (In contrast to what we learn in school.)

- Philosophical shift: **View semi-classical expansion as potentially exact.**

Outline

- Motivation from QFT
- Resurgence in quantum mechanics
- QFT and IR-renormalon puzzle
- Resurgent transseries $\Leftarrow$ Lefschetz thimble decomposition
- Uniform WKB and resurgent transseries
- Complexification of path integrals, and a possible paradigm shift

Motivation from QFT: e.g. CP(N-1) puzzles

verbatim in **4d QCD +more**

An asymptotically free theory with a complex projective target space.
Large-N, successful. Many problems unresolved at finite-N.

1) Pert. theory is an asymptotic (divergent) expansion even after regularization and renormalization. Is there a meaning to pert. theory?

2) Invalidity of the semi-classical dilute instanton gas approximation on R2. DIG assumes inter-instanton separation much larger than the instanton size, but the latter is a moduli, hence no meaning to the assumption. "Infrared embarrassment" Coleman's lectures.

3) A resolution of 2) was suggested by considering the theory in a small thermal box. But in the weak coupling regime, the theory always lands on the deconfined regime. (Affleck, 80) No semi-classical approx. for the confined regime.

4) Incompatibility of large-N with DIG. It better be so, we trust former and not the latter. (Witten, Jevicki, 79)

5) The renormalon ambiguity (technical, but deeper, to be explained), ('t Hooft,79)

Simpler question: Can we make sense of the semi-classical expansion of QFT?

Argyres, MÜ,
Dunne, MÜ, 2012

$$f(\lambda\hbar) \sim \sum_{k=0}^{\infty} c_{(0,k)}\, (\lambda\hbar)^k + \sum_{n=1}^{\infty} (\lambda\hbar)^{-\beta_n}\, e^{-n\, A/(\lambda\hbar)} \sum_{k=0}^{\infty} c_{(n,k)}\, (\lambda\hbar)^k$$

pert. th. n-instanton factor pert. th. around n-instanton

All series appearing above are asymptotic, i.e., divergent as $c_{(0,k)} \sim k!$. The combined object is called trans-series following resurgence terminology.

Borel resummation idea: If $P(\lambda) \equiv P(g^2) = \sum_{q=0}^{\infty} a_q g^{2q}$ has convergent Borel transform

$$BP(t) := \sum_{q=0}^{\infty} \frac{a_q}{q!} t^q$$

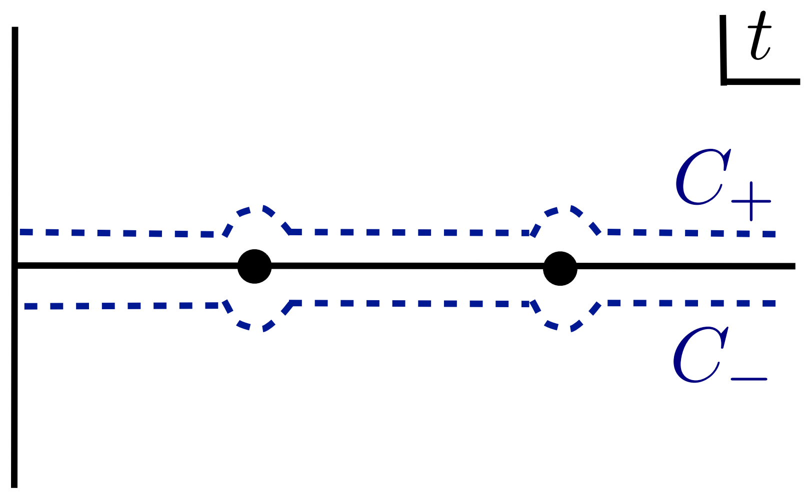
in neighborhood of $t = 0$, then

$$\mathbb{B}(g^2) = \frac{1}{g^2} \int_0^{\infty} BP(t) e^{-t/g^2} dt \ .$$

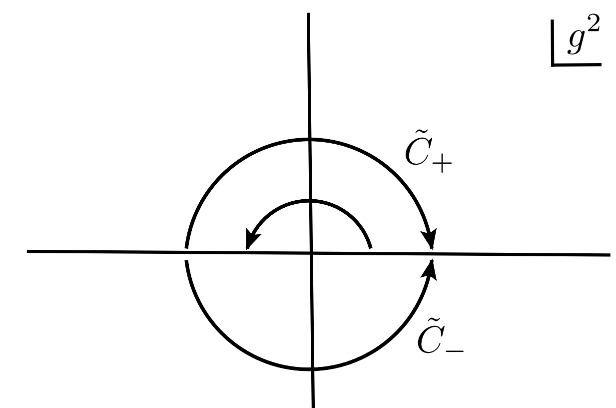
formally gives back $P(g^2)$, but is ambiguous if $BP(t)$ has singularities at $t \in \mathbb{R}^+$:

Perturbation theory: Borel plane, lateral Borel sums, ambiguity

Directional (sectorial) Borel sum. $\mathcal{S}_\theta P(g^2) \equiv \mathbb{B}_\theta(g^2) = \frac{1}{g^2}\int_0^{\infty\cdot e^{i\theta}} BP(t)\, e^{-t/g^2}\, dt$



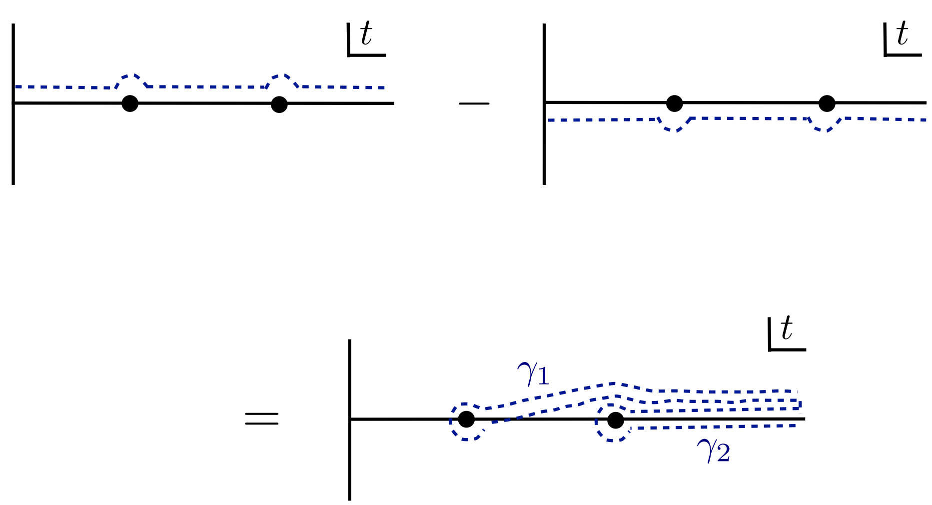
left and right Borel sums



Same structure by analytic continuation

$$\mathbb{B}_{0\pm}(|g^2|) = \mathrm{Re}\,\mathbb{B}_0(|g^2|) \pm i\,\mathrm{Im}\,\mathbb{B}_0(|g^2|), \qquad \mathrm{Im}\,\mathbb{B}_0(|g^2|) \sim e^{-2S_I} \sim e^{-2A/g^2}$$

The *non-equality* of the left and right Borel sum means the series is *non-Borel summable or ambiguous*. The ambiguity has the same form of a 2-instanton factor (not 1). The measure of ambiguity (Stokes automorphism/jump in g-space interpretation):



$$\mathcal{S}_{\theta^+} = \mathcal{S}_{\theta^-} \circ \underline{\mathfrak{S}}_\theta \equiv \mathcal{S}_{\theta^-} \circ (\mathbf{1} - \mathrm{Disc}_{\theta^-})\,,$$

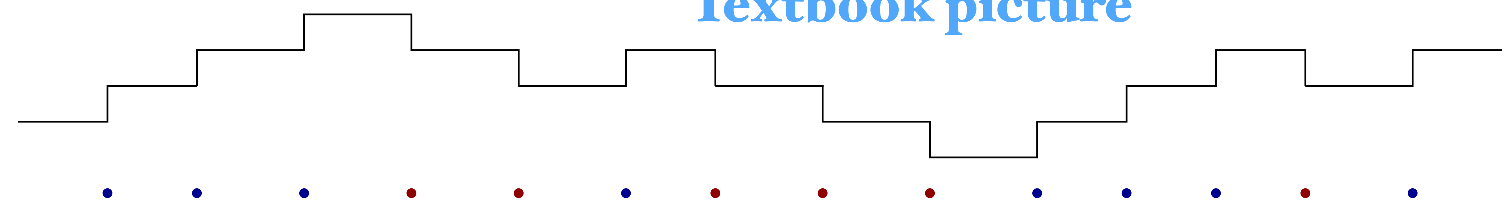
$$\mathrm{Disc}_{\theta^-}\,\mathbb{B} \sim e^{-t_1/g^2} + e^{-t_2/g^2} + \ldots \qquad t_i \in e^{i\theta}\mathbb{R}^+$$

Ecalle, 80s

Instantons and Bogomolny--Zinn-Justin (BZJ) prescription

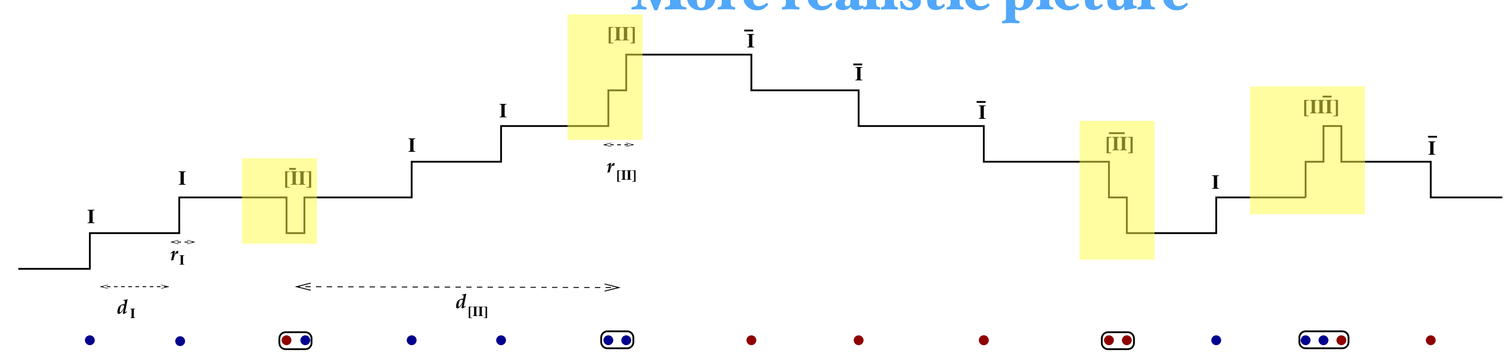
BZJ, QM (80s): for double well potential,
Here, we work with a periodic potential.
Dilute instanton, molecular instanton gas.

Textbook picture



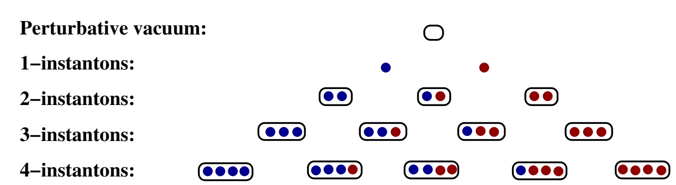
a) Dilute gas of 1−instantons

More realistic picture



b) Dilute gas of 1−instantons, 2−instantons, and other molecular−events

$$
\begin{array}{ccccccc}
r_{\rm I} & \ll & r_{\rm [II]} \sim \ell_{\rm qzm} & \ll & d_{\rm I} & \ll & d_{\rm [II]}\,, \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
L & \ll & L \,\log\left(\frac{1}{g^2}\right) & \ll & L\, e^{S_0} & \ll & L\, e^{2S_0}\,.
\end{array}
$$



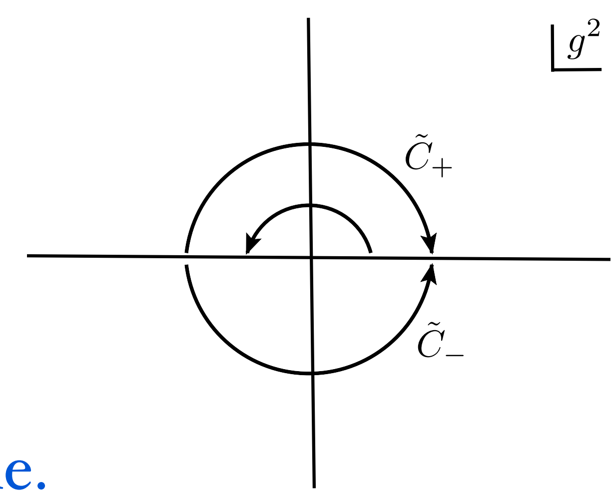
c) Representatives of n−instanton events, sketched according to the resurgence triangle.

How to make sense out of correlated events?

$[\mathcal{I}\mathcal{I}]$ Evaluate quasi-zero mode integral.Easy.

$[\mathcal{I}\bar{\mathcal{I}}]$ Naive calculation meaningless* at $g^2 > 0$.
The quasi-zero mode integral is dominated at small-separations where a molecular instanton is meaningless. BZJ: Continue to $g^2 < 0$, evaluate there, and continue back to $g^2 > 0$: two fold-ambiguous!

$$[\mathcal{I}\bar{\mathcal{I}}]_{\theta=0^\pm} = \mathrm{Re}\,[\mathcal{I}\bar{\mathcal{I}}] + i\,\mathrm{Im}\,[\mathcal{I}\bar{\mathcal{I}}]_{\theta=0^\pm}$$



*: Retrospectively, it better be so, because we are on a Stokes line.

Borel-Ecalle summability

Remarkable fact: Leading ambiguities cancel.

Non-Borel summable. But a generalized notion of summability exits.
An elementary incidence of Borel-Ecalle summability.

$$\mathrm{Im}\,\mathbb{B}_{0,\theta=0^\pm} + \mathrm{Im}\,[\mathcal{I}\overline{\mathcal{I}}]_{\theta=0^\pm} = 0\,, \qquad \text{up to } O(e^{-4S_I})$$

Data from P.T. Data from N.P. sector

Borel-Ecalle summability

- periodic potential: $V(x) = \frac{1}{g^2}\,\sin^2(g\,x)$
- vacuum saddle point

$$c_n \sim n!\left(1 - \frac{5}{2}\cdot\frac{1}{n} - \frac{13}{8}\cdot\frac{1}{n(n-1)} - \dots\right)$$

- instanton/anti-instanton saddle point:

$$\mathrm{Im}\,E \sim \pi\,e^{-2\frac{1}{2g^2}}\left(1 - \frac{5}{2}\cdot g^2 - \frac{13}{8}\cdot g^4 - \dots\right)$$

- double-well potential: $V(x) = x^2(1-gx)^2$
- vacuum saddle point

$$c_n \sim 3^n n!\left(1 - \frac{53}{6}\cdot\frac{1}{3}\cdot\frac{1}{n} - \frac{1277}{72}\cdot\frac{1}{3^2}\cdot\frac{1}{n(n-1)} - \dots\right)$$

- instanton/anti-instanton saddle point:

$$\mathrm{Im}\,E \sim \pi\,e^{-2\frac{1}{6g^2}}\left(1 - \frac{53}{6}\cdot g^2 - \frac{1277}{72}\cdot g^4 - \dots\right)$$

resurgence: fluctuations about the instanton/anti-instanton saddle are determined by those about the vacuum saddle.

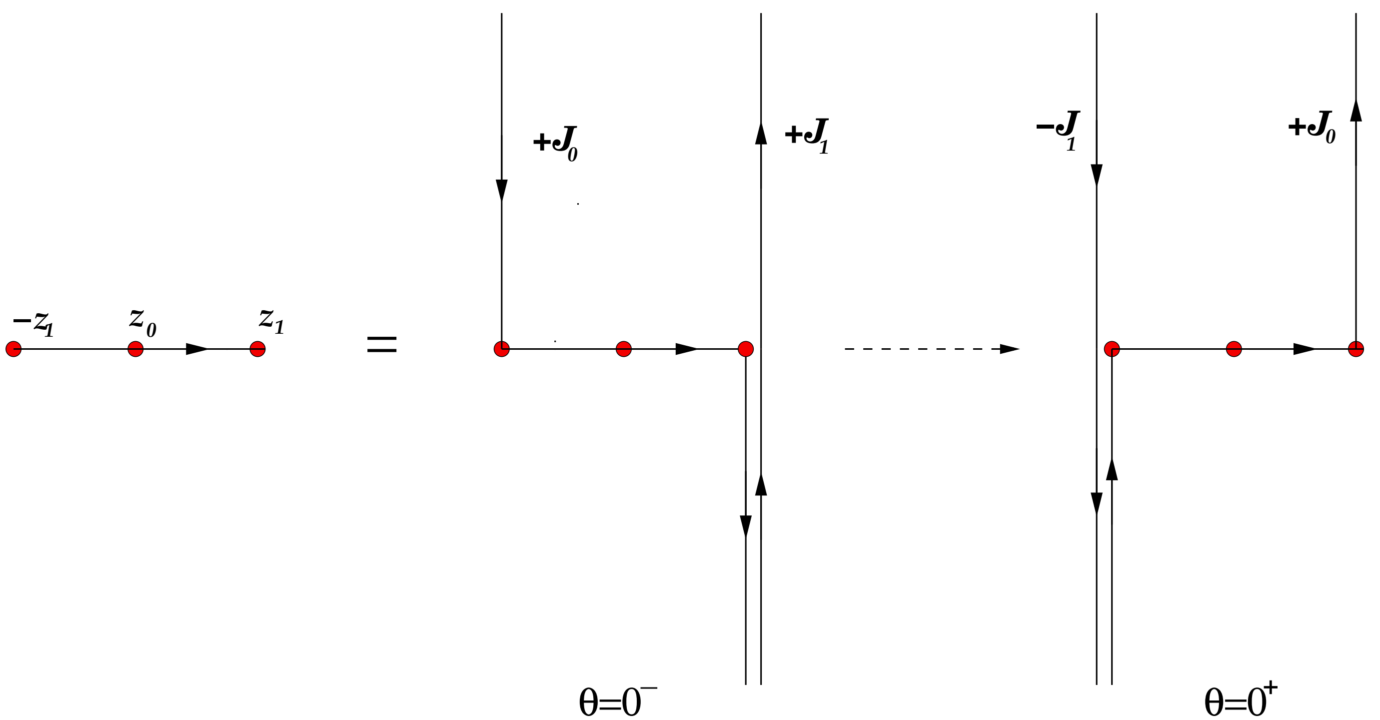
Why is this happening? d=0 prototype, Stokes phenomena

Zero dimensional toy example in steepest descent (semi-classical) method:

$$Z^{0\mathrm{d}}(\lambda) = \frac{1}{\sqrt{\lambda}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dx\, e^{-\frac{1}{2\lambda}\sin^2(x)}$$

To each saddle, there is, in general, a unique steepest descent path (generalization to multi-dimension is Lefschetz thimble). These thimbles form natural basis for integration and analytic continuation. P-saddle and NP-saddle. Original cycle = linear combination of these thimbles.

But on the Stokes line, thimble decomposition is multi-fold ambiguous!



$$I = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \longrightarrow \Sigma = \begin{cases} \mathcal{J}_0(0^-) + \mathcal{J}_1(0^-) \\ \mathcal{J}_0(0^+) - \mathcal{J}_1(0^+) \end{cases}$$

0: P-saddle, 1: NP-saddle

Determination of thimble:

Complex gradient-flow (or Picard-Lefschetz) equations.

Complexify everything, because thimbles lives in C!

$$\frac{dz}{dt} = -e^{-i\theta}\frac{\partial \bar{S}}{\partial \bar{z}}$$

$$\mathrm{Im}S(z)|_{\mathcal{J}_i} = \mathrm{Im}S(z_i),$$

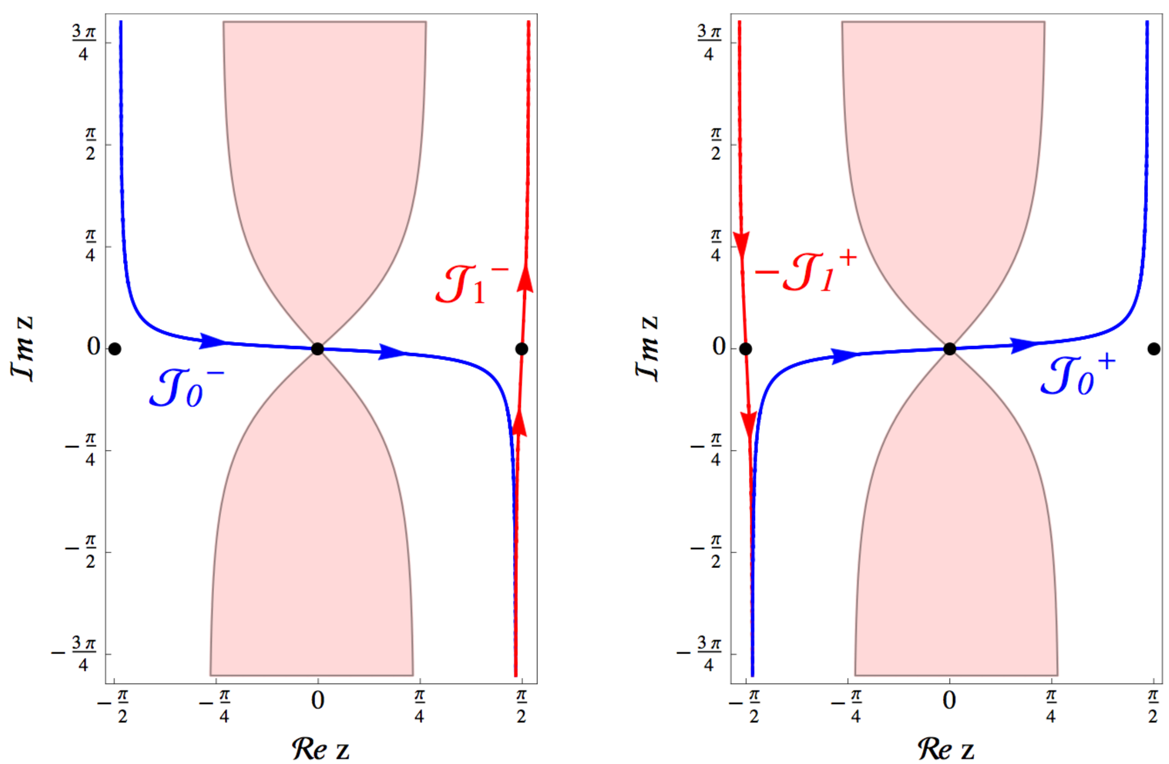
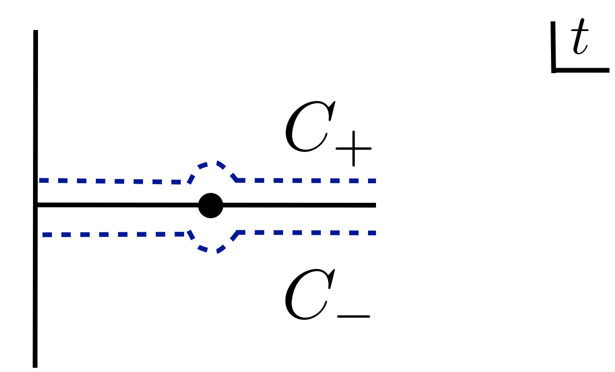


Figure 1. Left: Lefschetz thimbles at $\lambda = e^{i\theta}$ with $\theta = 0^-$: $\mathcal{J}_0 + \mathcal{J}_1$. **Right:** At $\theta = 0^+$. $\mathcal{J}_0 - \mathcal{J}_1$. We take $\theta = \mp 0.1$ to ease visualization.

Same ambiguity



$$\mathbb{B}_{0^\pm}(\lambda) = \int_{\mathcal{J}_0^\pm} e^{-\frac{1}{\lambda}S}$$

Giving an elegant geometric meaning to Borel analysis:

Left/right Borel sum = Integration over Lefschetz thimble!

Borel ambiguity=Ambiguity in the choice of the cycle on a Stokes line

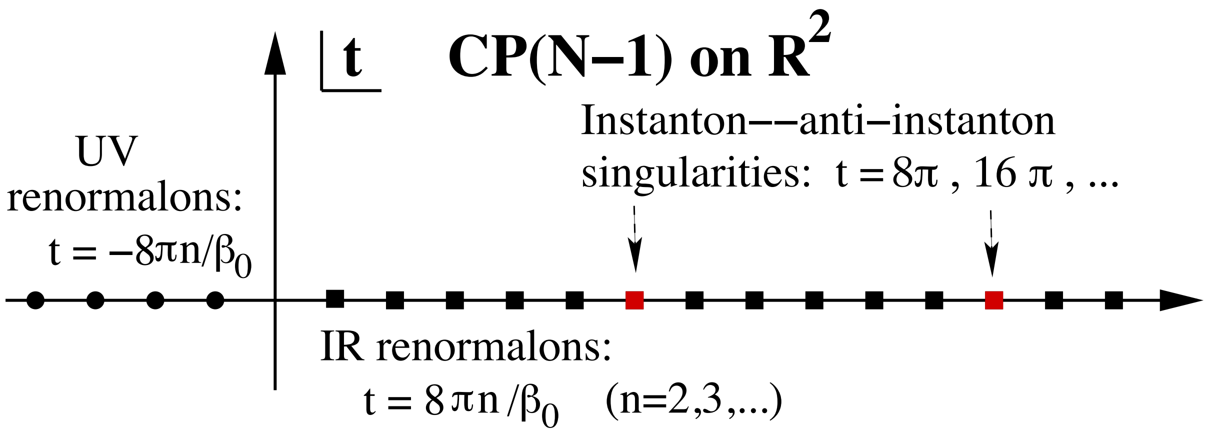
Can the mechanism in QM work in QFT? QCD on R4 or CP(N-1) on R2?

't Hooft(79) :***No,*** *on R4,* Argyres, MÜ: ***Yes,*** *on R3 x S1,*
F. David(84), Beneke(93) : ***No,*** *on R2.* Dunne, MÜ: ***Yes,*** *on R1 x S1*

Why doesn't it work, say for CP(N-1) on R2?

$[\mathcal{I}\bar{\mathcal{I}}]$ contribution, calculated in some way, gives an ±i exp[-2S_I].
Lipatov(77): Borel-transform BP(t) has singularities at t_n= 2n g^2 S_I.

BUT, BP(t) has other (more important) singularities closer to the origin of the Borel-plane. (not due to factorial growth of number of diagrams.)



't Hooft called these IR-renormalon singularities with the hope that they would be associated with a saddle point like instantons. **No such configuration is known!**

A real problem in QFT, means pert. theory, as is, ill-defined. How to cure starting from micro-dynamics?

Standard view emanating from late 70s,
e.g. : from Parisi(78)

If the theory is renormalizable, the Borel transform has new singularities which cannot be controlled by using semi-classical methods [5-8] .

5 G. 't Hooft, Lectures given at Erice (1977)

6 B. Lautrup, Phys. Lett. 69B (1977) 109

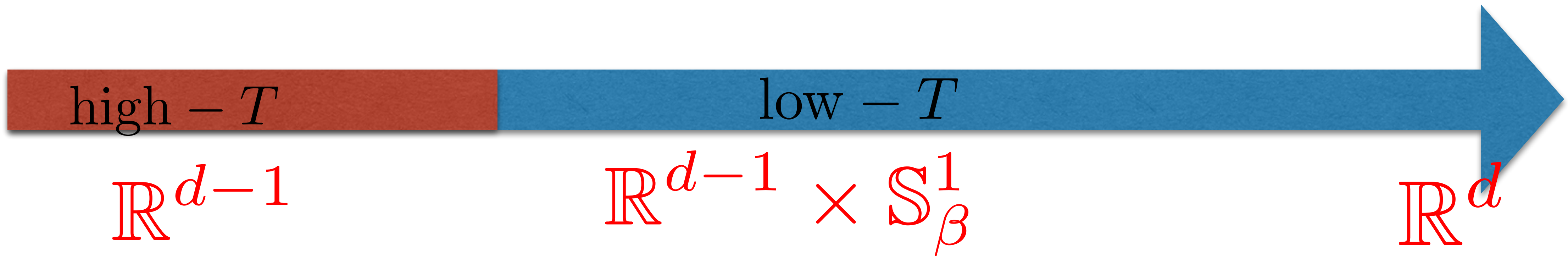
7 G. Parisi, Lectures given at the 1977 Cargèse Summer School

8 P. Olesen, Nordita preprint NBI HE 77.48 (1977)

Question: What happens if we can make the most interesting QFTs semi-classically calculable?

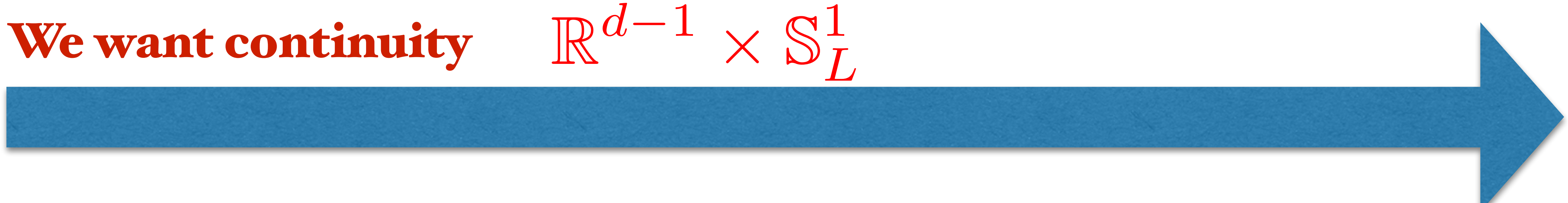
Is this even possible?

$\mathbb{CP}^{N-1}$ on $\mathbb{R}^1 \times \mathbb{S}^1_L$ and Continuity



Thermal: Rapid crossover at finite-N, phase transition at large-N $Z(\beta) = \mathrm{tr} e^{-\beta H}$

We want continuity $\mathbb{R}^{d-1} \times \mathbb{S}^1_L$



Prevent phase transition by using circle compactification or judicious matter choice or (twisted/non-thermal) boundary conditions.

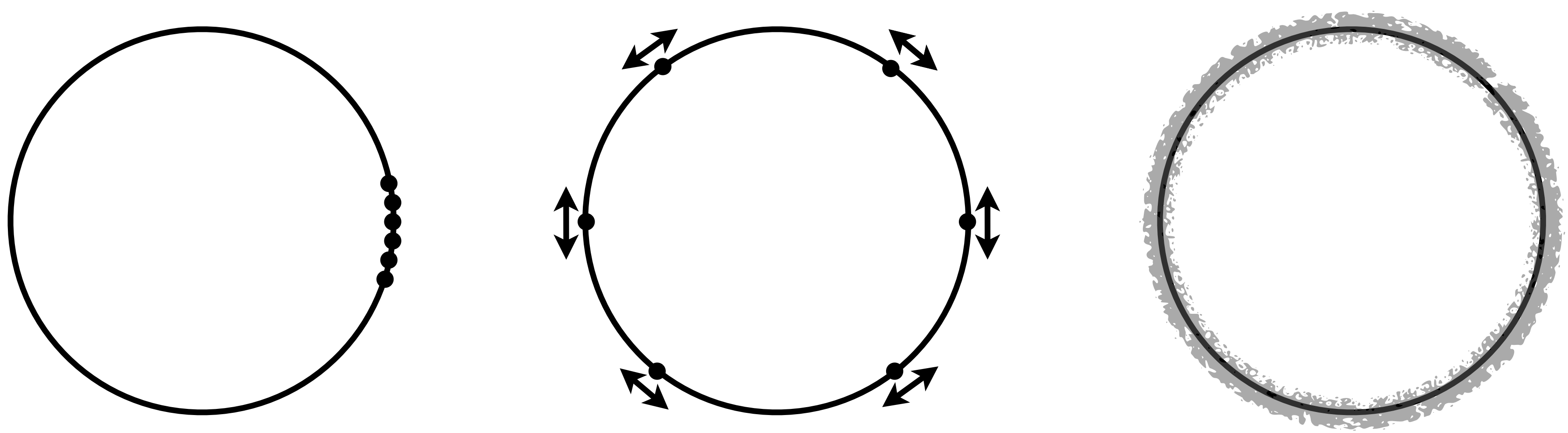
Supersymmetric theories: Continuity and analyticity (Witten,80).

$$Z(L) = \mathrm{tr}[e^{-LH}(-1)^F]$$

Non-supersymmetric theories, including QCD-like theories:
The possibility of is realized in 2007 (M.Ü., Yaffe07) and also see Ogilvie, Myers.).
Semi-classical version of the beautiful large-N reduction idea (Eguchi, Kawai 82)!

Sigma connection holonomy

Define: sigma-connection holonomy $\Leftrightarrow$ Wilson line Polyakov loop (gauge th.)
=Twisted boundary conditions for dynamical fields
Dunne,MÜ, 2012 (not the U(1) holonomy, it is size-N matrix. Genuine counterpart of Wilson line!



(a)
Thermal:
Eigenvalue attraction

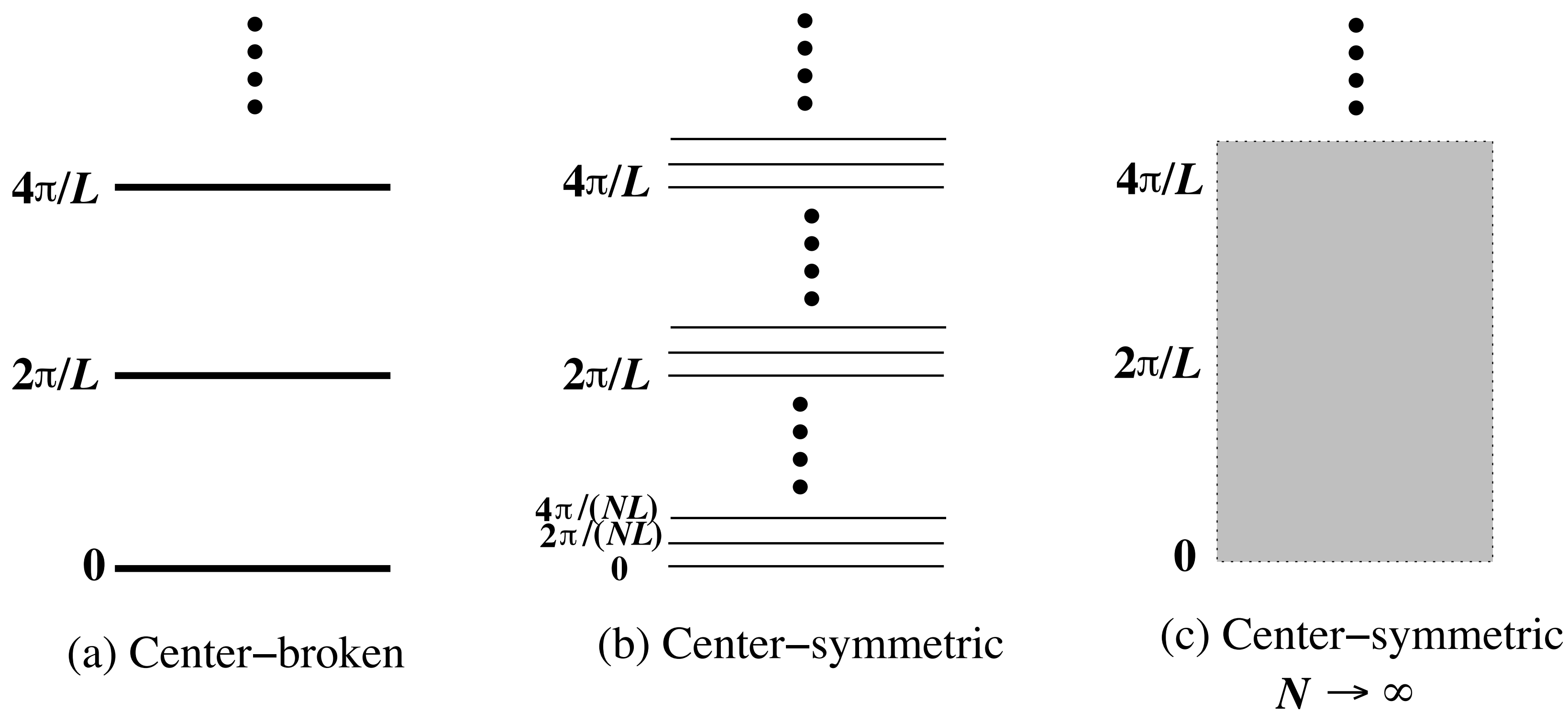
(b)
Spatial:
Eigenvalue repulsion

(c)
Strong coupling
Randomization

Difference of (a) and (b): van Baal, Kraan, Lee, Yi (97/98) in gauge th. on R_3 x S_1

(a) means a different "phase"(regime), not good for our purpose, good for thermal physics.
(b) means **semi-classical calculability!**
(c) means **incalculable:** Volume independence/large-N reduction demands (c)!

The dependence of perturbative spectrum to the sigma holonomy background

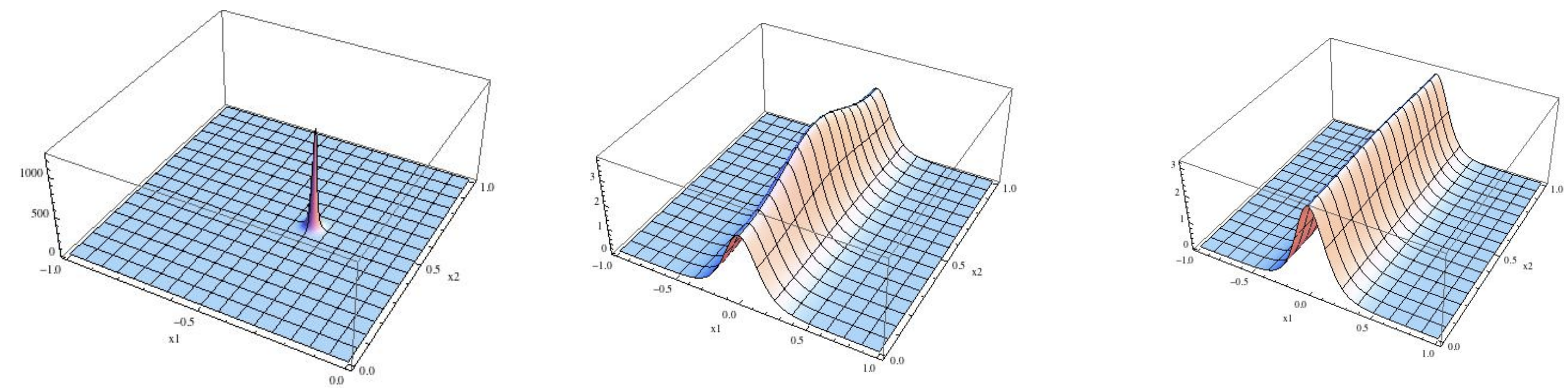


Same as gauge theory on R3 x S1: Spectrum become dense in the L=fixed, and N-large $\Longrightarrow$ Imprint of the large-N volume independence (large-N reduction).

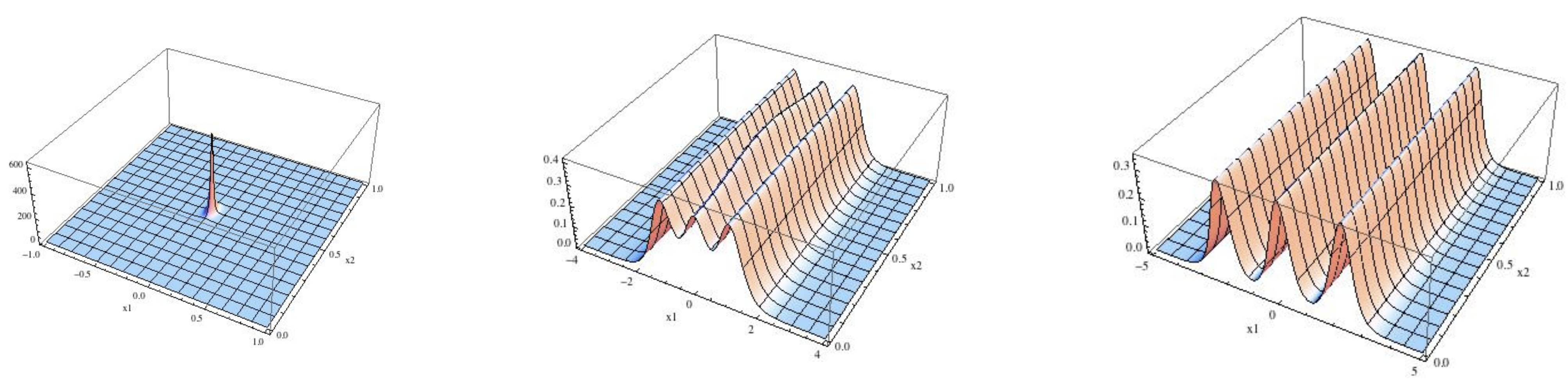
Here, we will study non-pert. effects in the long-distance effective theory within Born-Oppenheimer approx. in case (b) for finite-N.

Topological configurations, 1-defects

In thermal box, and high T, associated with trivial holonomy, the fractionalization does not occur (Affleck, 80s). Plot is for CP(2)



In spatial box, and small-L, associated with non-trivial holonomy, the fractionalization does occur. Large-2d BPST instanton in CP(2) fractionates into 3-types of kink-instantons.



Topological configurations, 1-defects, formally

Kink-instantons: (1d-instanton and twisted instantons) Associated with the N-nodes of the affine Dynkin diagram of SU(N) algebra. The twisted-instanton is present only because the theory is locally 2d! Also derived in Bruckmann et.al.(07, 09)

$$\widetilde{n} \longrightarrow \widetilde{n} + \alpha_i, \qquad \alpha_i \in \Gamma_r^\vee$$

$$\mathcal{K}_k : \qquad S_k = \frac{4\pi}{g^2} \times (\mu_{k+1} - \mu_k) = \frac{S_I}{N} \qquad , \qquad k = 1, \ldots, N$$

$$\mathcal{I}_{2d} \sim e^{-\frac{4\pi}{g^2}} = \left(\frac{\Lambda}{\mu}\right)^{\beta_0} \sim \prod_{j=0}^{\mathfrak{r}} [\mathcal{K}_j]^{k_j^\vee}, \qquad \beta_0 = h^\vee = \sum_{i=0}^{\mathfrak{r}} k_i^\vee, \qquad \mathfrak{r} = \mathrm{rank}[\mathfrak{su}(N)]$$

$k_i^\vee$ Dual Kac labels, all 1 for su(N) algebra, multiplicity of kink-instanton

Sum of Dual Kac labels= Dual Coxeter number= Beta function
Fractionalization formula of the instanton, or non-instanton saddle! (universal)

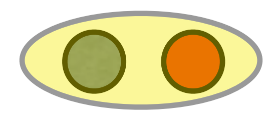
Gauge theory counter-part on R_3 x S_1:
Monopole-instantons or 3d-instanton and twisted instanton.
(caloron constituents) : van Baal, Kraan, (97/98), Lee-Yi (97)

Topological molecules: 2-defects

2-defects are also universal, dictated by Cartan matrix of Lie algebra: We call them charged and neutral bions

Charged bions: For each negative entry of the extended Cartan matrix $\widehat{A}_{ij} < 0$, there exists a bion $\mathcal{B}_{ij} = [\mathcal{K}_i \overline{\mathcal{K}}_j]$, associated with the correlated tunneling-anti-tunneling event

$$\widetilde{n} \longrightarrow \widetilde{n} + \alpha_i - \alpha_j \qquad \alpha_i \in \Gamma_r^\vee$$



Neutral bions: For each positive entry of the extended Cartan matrix $\widehat{A}_{ii} > 0$, there exists a neutral bion $\mathcal{B}_{ii} = [\mathcal{K}_i \overline{\mathcal{K}}_i]$, associated with the correlated tunneling-anti-tunneling event

$$\widetilde{n} \longrightarrow \widetilde{n} + \alpha_i - \alpha_i \qquad \alpha_i \in \Gamma_r^\vee$$

Charged bion: Counter-part of magnetic bion in gauge theory on R3 x S1 (generates mass gap for gauge fluctuations), MÜ 2007

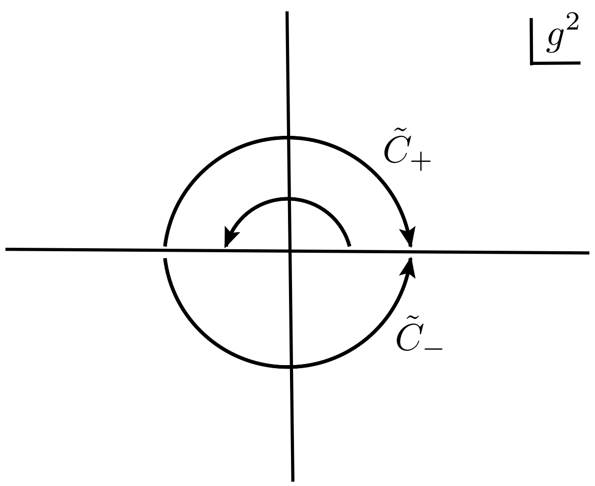
Neutral bion: Same as in the gauge theory on R3 x S1 (generates a center stabilizing potential), w/Poppitz 2011, w/Poppitz-Schäfer, w/Argyres 2012

Misumi, Kanazawa:2014 : Gauge theory with twisted b.c.
Shuryak, Sulejmanpasic 2013+ Shuryak, Zahed, Liu 2015

Nitta, Misumi, Sakai:2014-.. generalizations/universality
Nitta 2014-.. Sigma models, diverse dimensions,

Neutral bion and non-perturbative ambiguity in semi-classical expansion

Naive calculation of neutral bion amplitude, as you may guess following QM example, meaningless at $g^2 > 0$. The quasi-zero mode integral is dominated at small-separations where a molecular event is meaningless. Apply BZJ. Result is two fold-ambiguous!



$$
\begin{aligned}
[\mathcal{K}_i\overline{\mathcal{K}}_i]_{\theta=0^\pm} &= \mathrm{Re}\,[\mathcal{K}_i\overline{\mathcal{K}}_i] + i\,\mathrm{Im}\,[\mathcal{K}_i\overline{\mathcal{K}}_i]_{\theta=0^\pm} \\
&= \left(\log\left(\frac{\lambda}{8\pi}\right) - \gamma\right)\frac{16}{\lambda}e^{-2S_0} \pm i\,\frac{16\pi}{\lambda}e^{-\frac{8\pi}{\lambda}}
\end{aligned}
$$

As it stands, this looks terrible. Is semi-classical expansion at second order void of meaning? This is a general statement valid for many QFTs admitting semi-classical approximation, including the Polyakov model (77)! And it has not been addressed in literature until recently.

In QFT literature, people rarely discussed second or higher order effects in semi-classics, most likely, they thought no new phenomena would occur, and they would only calculate exponentially small subleading effects. The truth is far more subtler!

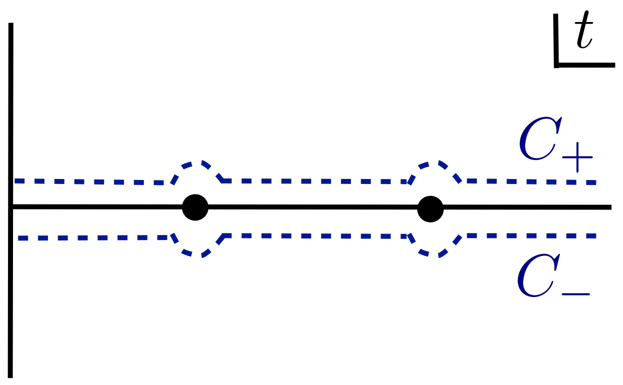
Disaster or blessing in disguise?

Go back to pert. theory, for the compactified center-symmetric CP(N-1) theory. We reduce the long-distance effective theory to simple QM with periodic potentials. Thankfully, the large-order behavior of pert. theory in such QM problems is studied by M. Stone and J. Reeve (78), by using the classic Bender-Wu analysis (69-73).

$$\mathcal{E}(g^2) \equiv E_0 \xi^{-1} = \sum_{q=0}^{\infty} a_q (g^2)^q, \qquad a_q \sim -\frac{2}{\pi}\left(\frac{1}{4\xi}\right)^q q! \left(1 - \frac{5}{2q} + O(q^{-2})\right)$$

Divergent non-alternating series, non-Borel summable, but right and left Borel resummable, with a result:

$$\mathcal{S}_{0\pm}\mathcal{E}(g^2) = \frac{1}{g^2}\int_{C_\pm} dt\; B\mathcal{E}(t)\; e^{-t/g^2} = \mathrm{Re}\mathcal{S}\mathcal{E}(g^2) \mp i\frac{8\xi}{g^2} e^{-\frac{4\xi}{g^2}}$$
$$= \mathrm{Re}\mathbb{B}_0 \mp i\frac{16\pi}{g^2 N} e^{-\frac{8\pi}{g^2 N}}$$



Remarkably,

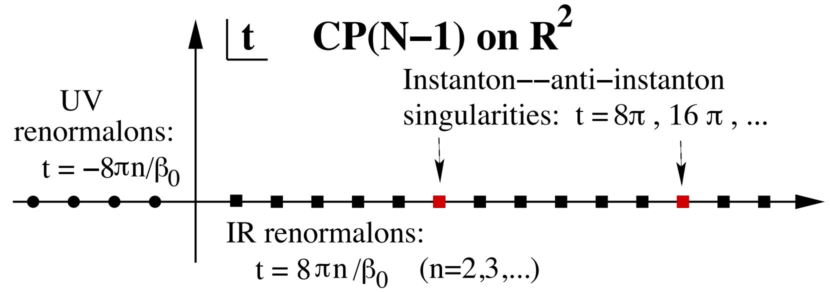
$$\mathrm{Im}\left[\mathcal{S}_\pm \mathcal{E}(g^2) + [\mathcal{K}_i \overline{\mathcal{K}}_i]_{\theta=0^\pm}\right] = 0 \qquad \text{up to } e^{-4S_0} = e^{-4S_I/\beta_0}$$

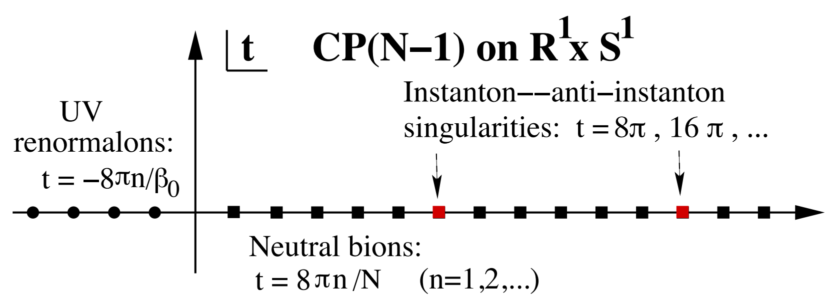
At this stage, in fact a magic happens:

Semi-classical renormalons as neutral bions

The ambiguities which cancel are at $\exp[-2S_I/N]$ order.
Exactly in the IR-renormalon territory ['t Hooft(77), David(81)].

Claim: Neutral bions and neutral topological molecules are semi-classical realization of 't Hooft's elusive renormalons, and it is possible to make sense out of combined perturbative semi-classical expansion.

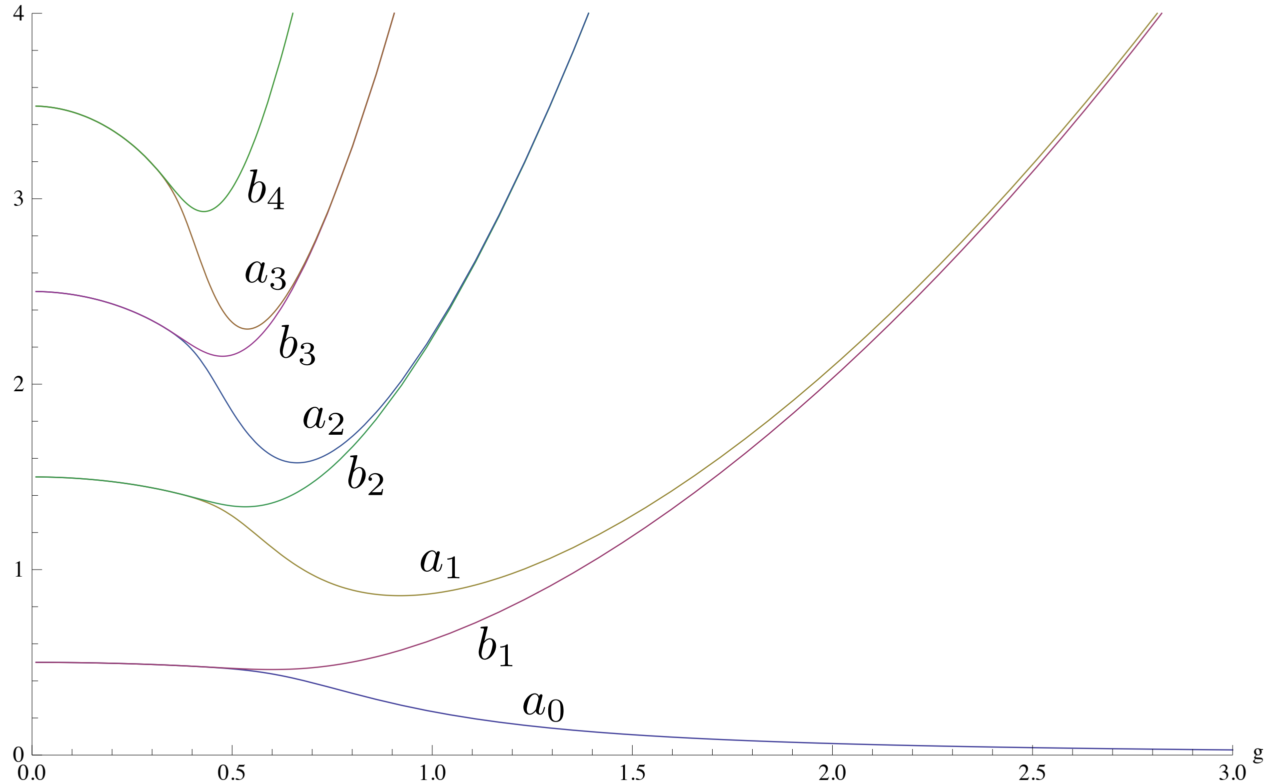




More than three decades ago, 't Hooft gave a famous set of (brilliant) lectures(79): *Can we make sense out of QCD (QFT)?* He was thinking a non-perturbative continuum formulation. It seem plausible to me that, we have a chance, at least, in the semi-classical regime of QFT.

This description was the missing link between 't Hooft renormalons (late 70s) and van Baal's fractional-instantons/caloron constituents (late 90s).

Mass gap in the small-S$_I$ regime



$$m_g = \frac{C}{\sqrt{\lambda}}\left(1 - \frac{7\lambda}{32\pi} + O(\lambda^2)\right) e^{-\frac{4\pi}{\lambda}} \sim e^{-S_I/N} \qquad \text{for } \mathbb{CP}^{N-1}$$

The mass gap at small-S$_I$: Same as large-N solution on R$_2$!
Our small circle still keeps in mind exp[-S$_I$/N] in long distance dynamics! (This is the crucial difference with respect to Bjorken femto-universe, Luscher (98), van Baal 2000 compactifications, QCD on torus).

In the small-S$_I$ regime, this solves the large-N vs. instanton puzzle.
BPST instantons are unimportant, kink-instantons survive large-N limit!

A representation of semi-classical regime of QFT

Graded Resurgence triangle

$f_{(0,0)}$ → (P. vacuum) x pert. fluc.

instanton x (pert. fluc.) ← $e^{-\frac{A}{\lambda}+i\frac{\widetilde{\Theta}_k}{N}} f_{(1,1)}$ $\quad$ $e^{-\frac{A}{\lambda}-i\frac{\widetilde{\Theta}_k}{N}} f_{(1,-1)}$

bions x (pert. fluc.)

$$
\begin{array}{ccccccccc}
 & & & & f_{(0,0)} & & & & \\
 & & & e^{-\frac{A}{\lambda}+i\frac{\widetilde{\Theta}_k}{N}} f_{(1,1)} & & e^{-\frac{A}{\lambda}-i\frac{\widetilde{\Theta}_k}{N}} f_{(1,-1)} & & & \\
 & & e^{-\frac{2A}{\lambda}+2i\frac{\widetilde{\Theta}_k}{N}} f_{(2,2)} & & e^{-\frac{2A}{\lambda}} f_{(2,0)} & & e^{-\frac{2A}{\lambda}-2i\frac{\widetilde{\Theta}_k}{N}} f_{(2,-2)} & & \\
 & e^{-\frac{3A}{\lambda}+3i\frac{\widetilde{\Theta}_k}{N}} f_{(3,3)} & & e^{-\frac{3A}{\lambda}+i\frac{\widetilde{\Theta}_k}{N}} f_{(3,1)} & & e^{-\frac{3A}{\lambda}-i\frac{\widetilde{\Theta}_k}{N}} f_{(3,-1)} & & e^{-\frac{3A}{\lambda}-3i\frac{\widetilde{\Theta}_k}{N}} f_{(3,-3)} & \\
e^{-\frac{4A}{\lambda}+4i\frac{\widetilde{\Theta}_k}{N}} f_{(4,4)} & & e^{-\frac{4A}{\lambda}+2i\frac{\widetilde{\Theta}_k}{N}} f_{(4,2)} & & e^{-\frac{4A}{\lambda}} f_{(4,0)} & & e^{-\frac{4A}{\lambda}-2i\frac{\widetilde{\Theta}_k}{N}} f_{(4,-2)} & & e^{-\frac{4A}{\lambda}-4i\frac{\widetilde{\Theta}_k}{N}} f_{(4,-4)} \\
 & \iddots & & & \vdots & & & \ddots &
\end{array}
$$

⇒Each column is a transseries by itself. (with its own resurgent cancellations)

⇒Each descendant in each column is a singularity in the Borel plane.

⇒Each singularity in the Borel plane is a saddle. (if semi-classically calculable)

⇒Attached to each saddle, there is a Lefschetz thimble.

⇒S.C. QFT as a superposition of infinitely many thimbles or resurgent transseries.

N.P. confluence equations

In order QFT to have a meaningful semi-classical continuum definition, a set of perturbative--non-perturbative confluence equations must hold.

$$0 = \mathrm{Im}\Big(\mathbb{B}_{[0,0],\theta=0^\pm} + \mathbb{B}_{[2,0],\theta=0^\pm}[\mathcal{B}_{ii}]_{\theta=0^\pm} + \mathbb{B}_{[4,0],\theta=0^\pm}[\mathcal{B}_{ij}\mathcal{B}_{ji}]_{\theta=0^\pm} + \mathbb{B}_{[6,0]\theta=0^\pm}[\mathcal{B}_{ij}\mathcal{B}_{jk}\mathcal{B}_{ki}]_{\theta=0^\pm} + \ldots\Big)$$

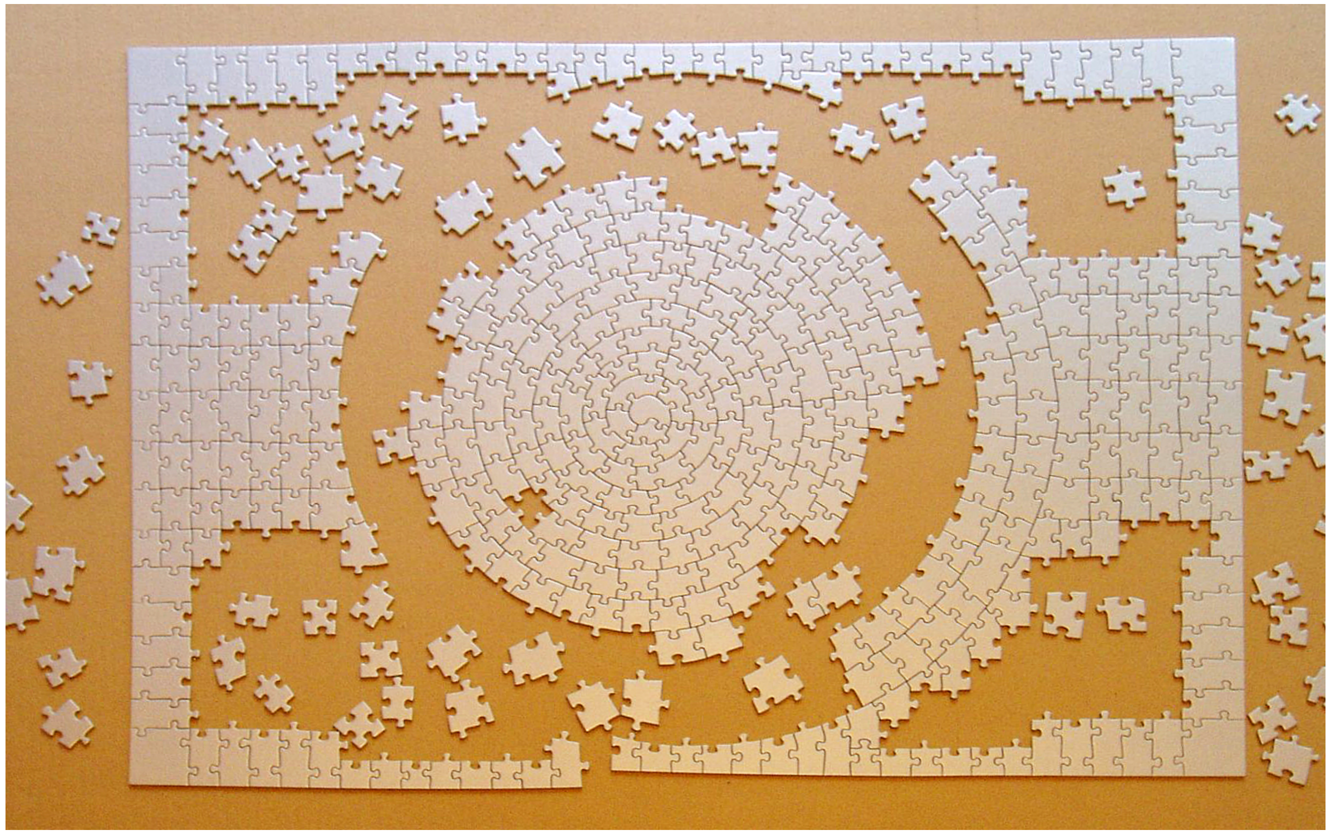
Order by order hierarchical confluence equations:

$$0 = \mathrm{Im}\mathbb{B}_{[0,0]\pm} + \mathrm{Re}\mathbb{B}_{[2,0]}\mathrm{Im}[\mathcal{B}_{ii}]_\pm \,, \qquad (\text{up to } e^{-4S_0})$$

$$0 = \mathrm{Im}\mathbb{B}_{[0,0]\pm} + \mathrm{Re}\mathbb{B}_{[2,0]}\mathrm{Im}[\mathcal{B}_{ii}]_\pm + \mathrm{Im}\mathbb{B}_{[2,0]\pm}\mathrm{Re}[\mathcal{B}_{ii}] + \mathrm{Re}\mathbb{B}_{[4,0]}\mathrm{Im}[\mathcal{B}_{ij}\mathcal{B}_{ji}]_\pm \qquad (\text{up to } e^{-6S_0})$$

$$0 = \ldots$$

Consequence of the median resummation in resurgence theory, Aniceto, Schiappa (13).



Uniform-WKB and resurgent transseries

I will tell you the scientific story of it,
details shown in the back-up pages.

- Gerald Dunne and I wanted to understand the origin of the resurgent transseries better in QM.
- State of the art: Zinn-Justin's work (80s to date). He found a generalization of Bohr-Sommerfeld quantization condition, whose solution was all orders (pert. and non-perturbative) expansion for all levels. (reviewed in ZJ-Jentschura, 2004)
- Pham and Delabaere proved their relation using resurgence. (89) [Pham is also one of the main characters to understand the relations between thimbles and resurgence, and relevance to Feynmann integral.]
- ZJ exact quantization involves two functions. A(E, g), B(E,g), where B(E,g)= N +1/2 where N is level number.
- B(E, g)=N+1/2, solve for E: Rayleigh-Schrodinger perturbation theory for level N
- A(E,g) in their formalism is a nightmare to calculate, and encodes instanton and require multiple appendices and a combination of analytic + numerical calculations (See ZJ-J 2004).
- On the other hand, the result is very beautiful. All multi-instanton sectors + all resurgent cancellations naturally incorporated.
- So, we wanted to rederive it in our way and discovered a pleasant surprise.

Uniform-WKB and resurgent transseries

- We decided to do WKB properly: "uniform-WKB". This is smooth across the classical turning points, better than usual WKB.
- Uniform-WKB uses parabolic cylinder function around the local harmonic minimum, nice at turning points etc.
- We derived the counterpart of exact quantization of Zinn-Justin, and the surprise was that we only had one series entering into the exact quantization condition!
- What happened to the two different function of ZJ? Where did one of them go? We were puzzled briefly. Then, discovered something magical.
- Instead of A(E, g), B(E,g), we wrote A(B, g) and E(B, g) which was more natural, and discovered a one-line relation connecting the two:

$$\frac{\partial E}{\partial B} = -3g^2 \left(2B - g^2 \frac{\partial A}{\partial g^2}\right)$$

Connecting Perturbative and Non-Perturbative Sector

e.g. double-well potential: $B \equiv N + \frac{1}{2}$

$$E(N, g^2) = B - g^2 \left(3B^2 + \frac{1}{4}\right) - g^4 \left(17B^3 + \frac{19}{4}B\right) - g^6 \left(\frac{375}{2}B^4 + \frac{459}{4}B^2 + \frac{131}{32}\right) - \dots$$

- non-perturbative function ($\mathcal{P} \sim (\dots) \exp[-A/2]$):

$$A(N, g^2) = \frac{1}{3g^2} + g^2 \left(17B^2 + \frac{19}{12}\right) + g^4 \left(125B^3 + \frac{153B}{4}\right) + g^6 \left(\frac{17815}{12}B^4 + \frac{23405}{24}B^2 + \frac{22709}{576}\right) +$$

- simple relation:

$$\frac{\partial E}{\partial B} = -3g^2 \left(2B - g^2 \frac{\partial A}{\partial g^2}\right)$$

Uniform-WKB and resurgent transseries

- Solving exact quantization condition iteratively, we were able the all order semi-classical result (all-multiinstantons) by just using perturbative expansion E(B, g)! All expansions around all sectors dictated by the perturbative expansion.

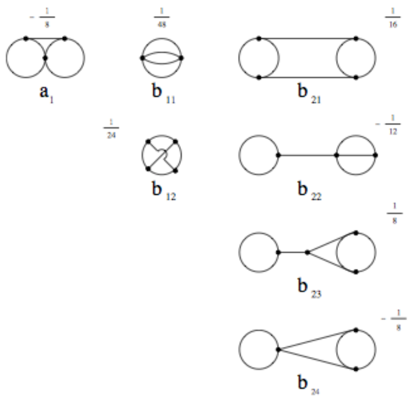
- **Constructive approach:** Unlike traditional resurgence, it is an early term-early term relation. You give me five terms in the WKB pert. expansion, and I tell you the first five term in the expansion around instanton or any other multi-instanton.

- We informed ZJ of our result. At first, he did find it hard to believe. Then, he said, he spend a day with mathematica, and re-proved our relation.

- He said that, retrospectively, it was not the relation, but rather the *simplicity of the relation* that surprised him most.

Resurgence at work

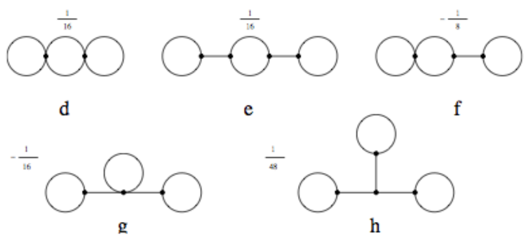
- fluctuations about $\mathcal{I}$ (or $\bar{\mathcal{I}}$) saddle are determined by those about the vacuum saddle, to all fluctuation orders



- "QFT computation": fluctuation about $\mathcal{I}$ for double-well:

2-loop (Shuryak/Wöhler, 1994); 3-loop

(Escobar-Ruiz/Shuryak/Turbiner, arXiv:1501.03993)



$$e^{-\frac{S_0}{g}}\left[1-\frac{71}{72}\,g-0.607535\,g^2-\dots\right]$$

$$\text{resurgence}:\, e^{-\frac{S_0}{g}}\left[1+\frac{1}{72}\,g\left(-102N^2-174N-71\right)\right.$$

$$\left.+\frac{1}{10368}g^2\left(10404N^4+17496N^3-2112N^2-14172N-6299\right)+\dots\right]$$

- known for all N and to essentially any loop order, directly from perturbation theory !

Multi-instantons and uniform-WKB

- Uniform-WKB gives an indirect way to calculate all multi-instanton amplitudes.
- Can we check/confirm this by the usual instanton methods?
- Yes: by two methods.
- One is proper improvement of BZJ-prescription by Misumi, Nitta, Sakai(2015).
  "Resurgence in sine-Gordon quantum mechanics: Exact agreement between multi-instantons and uniform WKB"
- The other is to apply Lefschetz thimble/complex gradient flow to multi-instanton quasi-zero mode direction.
- Two papers, Behthash, Sulejmanpasic, et.al.(2015)) **"The curious incident of multi-instantons and the necessity of Lefschetz thimbles"**
- BZJ- is partly black box, and tricky. To get things right is an art.
- Lefschetz thimble is a machine. No need to think hard.

Resurgence theory in path integrals

Key step is in the analytic continuation of path integral in field space (cf. Pham, and recent papers by Witten in phase space formulation, recent talks by Kontsevich "Resurgence from the path integral perspective", Perimeter Institute, August, 2012, Simons Center 2014), to make sense of steepest descent and Stokes phenomenon in path integrals.

Ongoing work: Complexification of path integral in configuration space formulation. Once the path integral is complexified, it is a new world that we need to build new intuitions!

Subtle differences between two underlined ideas, but technical. I will be happy to discuss in person.

To my mind, earlier works on the subject are a bit formal, and the richness and implications of the construction is not yet realized.

Recent works by Y. Tanizaki, T. Kanazawa, 2014,
M.U., Cherman, 2014

A CONTROVERSIAL ISSUE

Can complex, multi-valued, singular configurations contribute to a physical path integral (physical: with Hilbert space interpretation)?

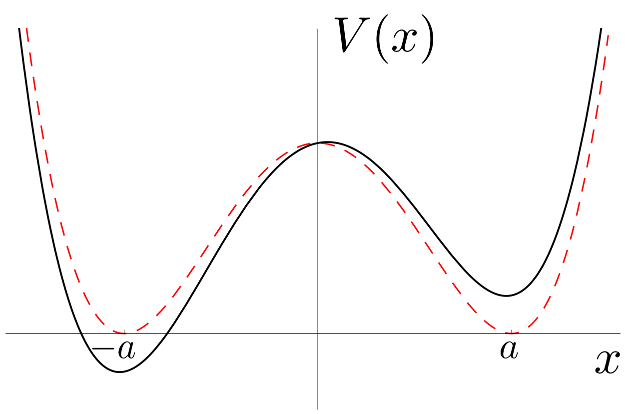
Many times rejected in the past, (usually deemed non-sense)& smoothness of the instantons is always presented as a virtue!

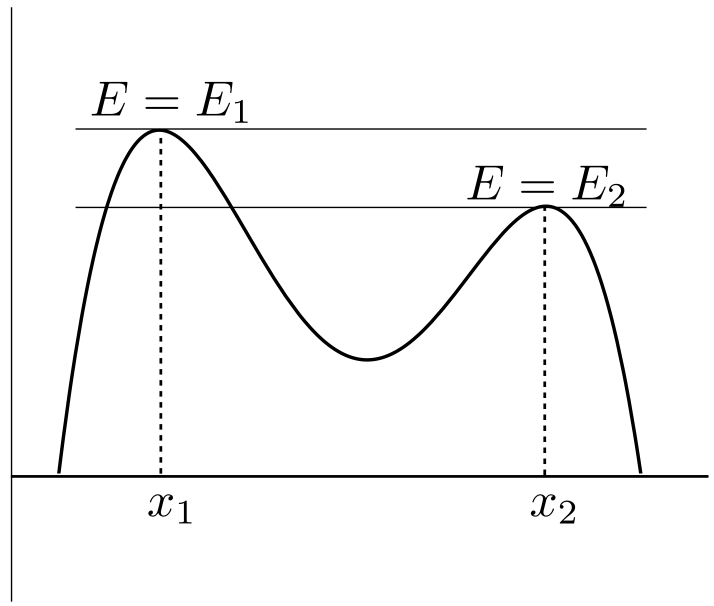
But the truth is that no one (*either in favor or opposition*) had the <u>proper formalism</u> to even address this question in path integrals! So, forget the discussions on this around 80s. In fact, forget almost everything until 2010.

A recent paper gives a serious deliberation on the issue (in the context of analytic continuation of Liouville theory, Harlow, Maltz, and Witten, 2011). But remain undecided, quote: "We do not have a clear rationale for why this (inclusion of multi-valeud "solutions") is allowed."

Supersymmetric QM and necessity of complex saddles!

Take Double-well susy QM. This system breaks susy spontaneously. (Witten, 81)
Quantize fermions and reduce the system to Bose-Fermi pair of Hamiltonians with tilted potential.





$$V_\pm = \frac{1}{2}(z^2 - 1)^2 \pm gz$$

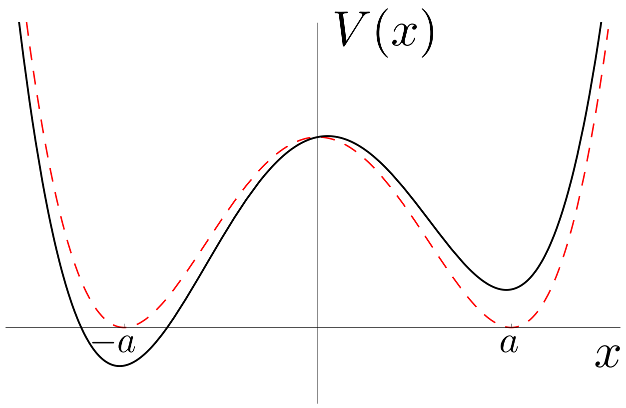
Ground state energy is zero to all orders in P.T. But is known to be lifted non-perturbatively. What causes it?

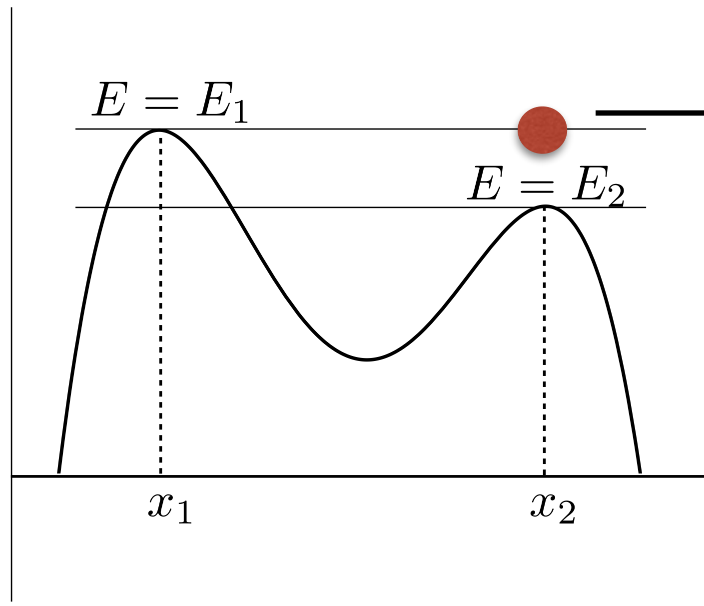
In the inverted potential, there is an obvious real bounce solution, but this is not related to ground state properties.

At level E1, the classical particle will fly of to infinity, infinite action, irrelevant. So, what causes the non-zero ground state energy in bosonized description?

Supersymmetric QM and necessity of complex saddles!

Take Double-well susy QM. This system breaks susy spontaneously. (Witten, 81)
Quantize fermions and reduce the system to Bose-Fermi pair of Hamiltonians with tilted potential.



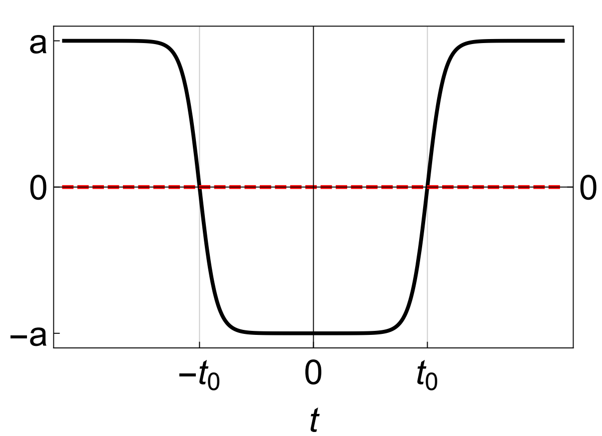


Complex conjugate turning points

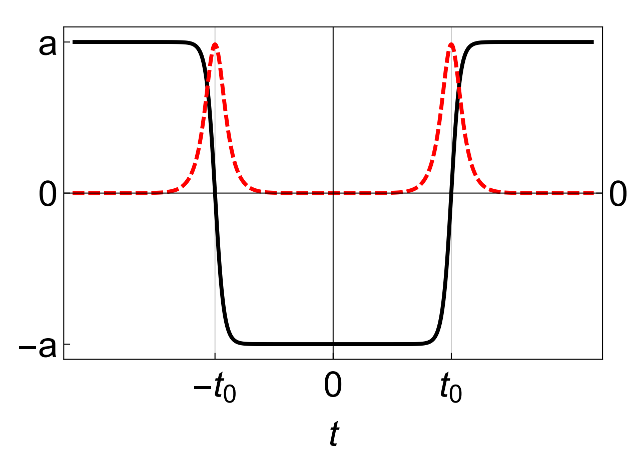
$$V_{\pm} = \frac{1}{2}(z^2 - 1)^2 \pm gz$$

If complex saddle is not included, we would conclude Susy is unbroken. Contradiction!

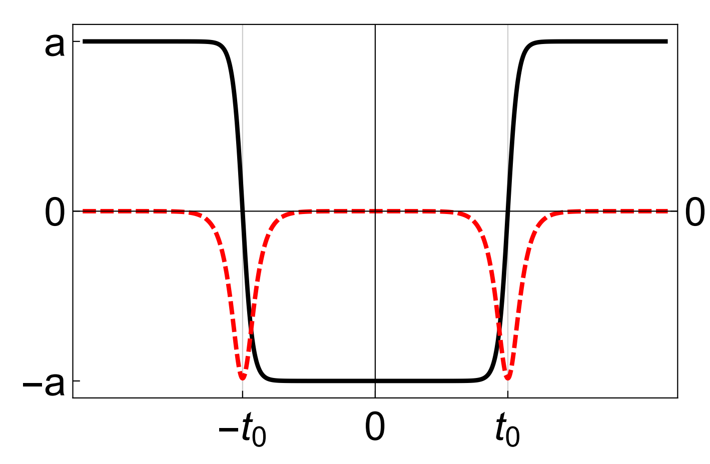
Exact bounce



Exact complex bion-1

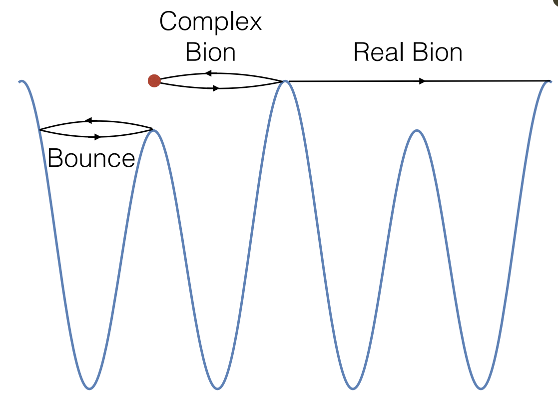


Exact complex bion-2



A far more dramatic case

Take particle on a circle, base space compact, periodic potential. This system has Witten index zero but susy is known to be unbroken. Two ground states, Bose-Fermi paired.



Inverted potential

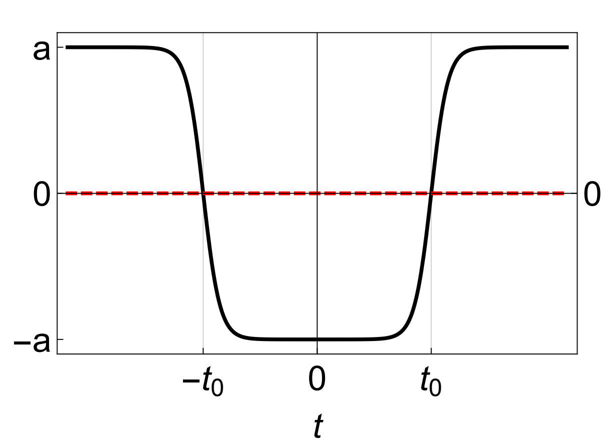
⇒If complex saddle is not included, real saddle gives negative ground state energy! In violation of Susy algebra. (a would-be genuine disaster)!

⇒Complex saddle is strictly necessary. But it is not only multi-valued, but also singular. Yet, its action is finite. Imaginary part of action $i\pi$. This is the hidden topological angle (HTA) (Behtash et.al.2015)

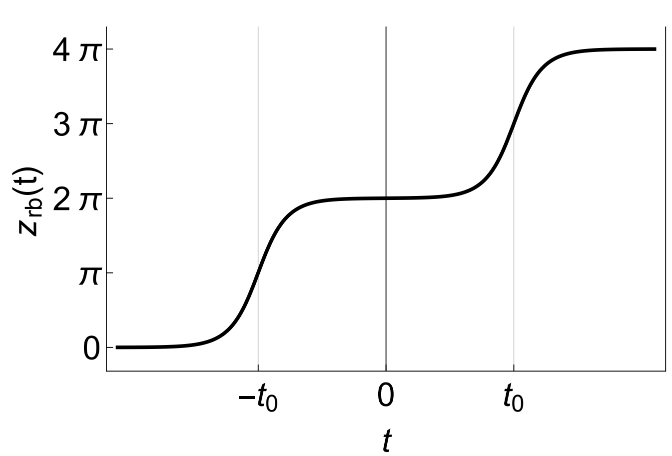
⇒ Contradiction with Susy algebra is prevented thanks to multi-valued singular solution!

I believe this is the sense in which we have to go through a change of perspective in path integrals! We will soon see that usual smooth instanton saddles are actually rare, and a new world is out there.

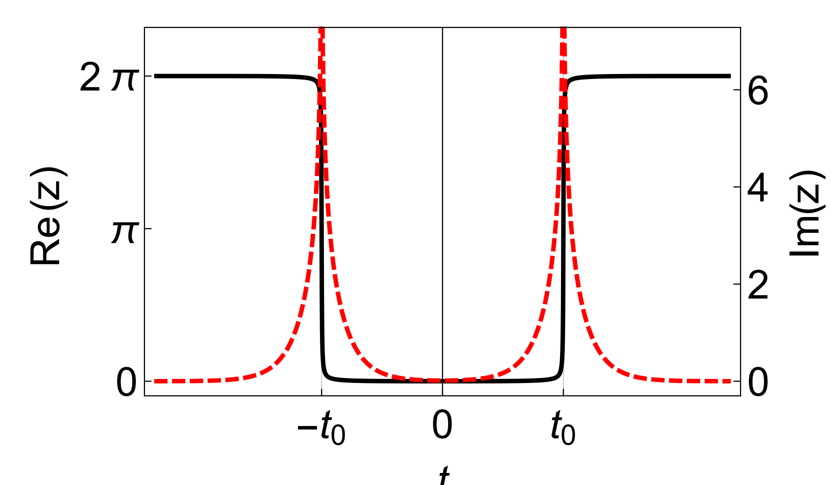
Exact bounce



Real bion



Exact complex bion



Lefschetz thimbles vs. resurgence

- Both heavily depend on the behavior of the theory upon analytic continuation, where asymptotic expansions are consistent with analytic continuation properties. (both taking into account Stokes phenomena)

- All systems that I know of that admits a Lefschetz thimbles decomposition can be expressed as a resurgent transseries.

- **However, the reverse is not always true.** It is not clear that all systems that have a resurgent transseries representation can be given a Lefschetz thimble decomposition. OPE in strongly coupled QFT may be in this category. (If OPE is resurgent expansion, we should still feel lucky and proceed. Many open questions here.)

- In thimble decomposition, the complex multi-valued saddles/solutions must also be taken into account.

Conclusions and prospects

It seems plausible that continuity and resurgence theory can be used in combination to provide a non-perturbative continuum definition of asymptotically free theories, and more general QFTs.

The construction may have practical utility and region of overlap with lattice field theory. One can check predictions of the formalism numerically. (Stochastic pert. theory?)

Resurgence provides a more refined classification of non-perturbative saddles wrt topological classification, e.g., as shown in resurgence triangle.

Back-up material

Uniform-WKB

what is the origin of resurgent behavior in QM and QFT ?

- QM: uniform WKB 1306.4405, 1401.5202) $\Rightarrow$

(i) trans-series structure is generic
(ii) all multi-instanton effects encoded in perturbation theory

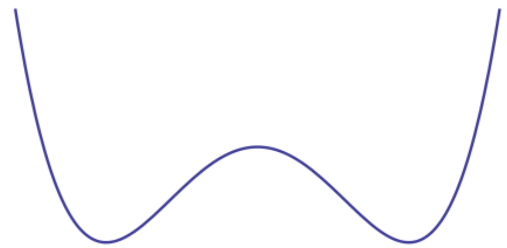
Provides relations between different saddles
which seems to be not captured by standard resurgence analysis.

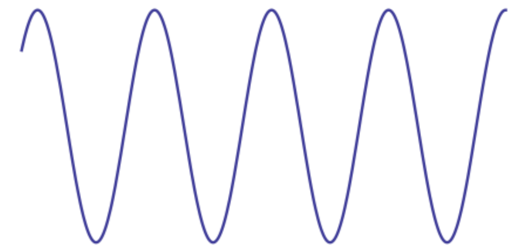
It is also constructive approach, in the sense that if you know the first K order of (WKB) perturbation theory around the trivial saddle, you can produce the first K order of the perturbation theory around, say, instanton saddle.

Unlike the usual resurgence where early terms are related to late terms.

Uniform WKB & Resurgent Trans-series (GD/MÜ:1306.4405, 1401.5202)

$$-\frac{d^2}{dx^2}\psi + \frac{V(g\,x)}{g^2}\psi = E\,\psi \to -g^4\frac{d^2}{dy^2}\psi(y) + V(y)\psi(y) = g^2\,E\,\psi(y)$$





- weak coupling: degenerate harmonic classical vacua
- non-perturbative effects: $g^2 \leftrightarrow \hbar \quad \Rightarrow \quad \exp\left(-\frac{c}{g^2}\right)$
- approximately harmonic

$\Rightarrow$ uniform WKB with parabolic cylinder functions

- ansatz (with parameter ν): $\psi(y) = \frac{D_\nu\left(\frac{1}{g}u(y)\right)}{\sqrt{u'(y)}}$

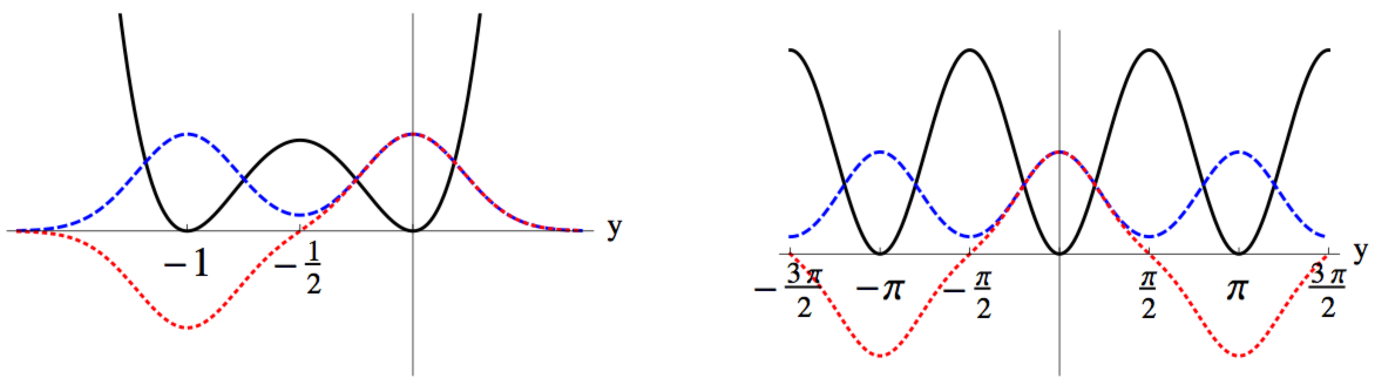
We decided to do WKB more properly: “uniform-WKB”. This is smooth across the classical turning points, better than usual WKB.

Uniform WKB & Resurgent Trans-Series

- perturbative expansion for E and $u(y)$:

$$E = E(\nu, g^2) = \sum_{k=0}^{\infty} g^{2k} E_k(\nu)$$

- $\nu = N$: usual perturbation theory (not Borel summable)
- global analysis $\Rightarrow$ boundary conditions:



- midpoint $\sim \frac{1}{g}$; non-Borel summability $\Rightarrow \quad g^2 \to e^{\pm i\,\epsilon}\, g^2$
- trans-series encodes analytic properties of D_ν
  $\Rightarrow$ generic and universal

Uniform WKB & Resurgent Trans-Series

$$D_\nu(z) \sim z^\nu\, e^{-z^2/4}\,(1+\dots) + e^{\pm i\pi\nu}\frac{\sqrt{2\pi}}{\Gamma(-\nu)}z^{-1-\nu}\,e^{z^2/4}\,(1+\dots)$$

$\longrightarrow$ exact quantization condition

$$\frac{1}{\Gamma(-\nu)}\left(\frac{e^{\pm i\pi}\,2}{g^2}\right)^{-\nu} = \frac{e^{-S/g^2}}{\sqrt{\pi\, g^2}}\mathcal{P}(\nu, g^2)$$

$\Rightarrow$ ν is only exponentially close to N (here $\xi \equiv \frac{e^{-S/g^2}}{\sqrt{\pi\, g^2}}$):

$$\nu = N + \frac{\left(\frac{2}{g^2}\right)^N \mathcal{P}(N, g^2)}{N!}\xi$$

$$-\frac{\left(\frac{2}{g^2}\right)^{2N}}{(N!)^2}\left[\mathcal{P}\frac{\partial\mathcal{P}}{\partial N} + \left(\ln\left(\frac{e^{\pm i\pi}\,2}{g^2}\right) - \psi(N+1)\right)\mathcal{P}^2\right]\xi^2 + O(\xi^3)$$

- insert: $E = E(\nu, g^2) = \sum_{k=0}^\infty g^{2k} E_k(\nu)$ $\Rightarrow$ trans-series

Connecting Perturbative and Non-Perturbative Sector

Zinn-Justin/Jentschura conjecture:
generate *entire trans-series* from just two functions:

(i) perturbative expansion $E = E(N, g^2)$
(ii) single-instanton fluctuation function $\mathcal{P}(N, g^2)$
(iii) rule connecting neighbouring vacua (parity, Bloch, ...)

in fact ... 1306.4405, 1401.5202)

$$\mathcal{P}(N, g^2) = \exp\left[S \int_0^{g^2} \frac{dg^2}{g^4} \left(\frac{\partial E(N, g^2)}{\partial N} - 1 + \frac{\left(N+\frac{1}{2}\right)\, g^2}{S} \right)\right]$$

$\Rightarrow$ perturbation theory $E(N, g^2)$ encodes everything !

All multi-instanton sectors + all resurgent cancellations:
the full trans-series