Pion-pion interaction from $N_f = 2{+}1$ simulations using the stochastic LapH method

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- phase shift calculations from the lattice increasingly elaborate
- can we move on to resonance structure computations?
- (in some sense) simplest resonance form factor: timelike pion form factor

- phenomenological relevance
  - hadronic vacuum polarization contribution to $(g-2)_\mu$ can be related to $R(s) \propto \sigma_{\text{tot}}(e^+e^- \to \text{hadrons})$
  - cross section dominated by two-pion final states at low energies
  - relevant quantity is the timelike pion form factor $F_\pi(s)$

$$R(s) = \frac{1}{4}\left(1 - \frac{4m_\pi^2}{s}\right)^{3/2} |F_\pi(s)|^2$$

e.g. [Jegerlehner, Nyffeler 2009]

Finite-volume methods

- Lüscher method relates infinite-volume (IV) scattering to spectrum of the theory in a finite box (FV)
- single-channel quantization condition

[Lüscher 1986, 1990, 1991; Rummukainen, Gottlieb 1995]

$$\underbrace{\delta(p)}_{\text{infinite-volume phase shift}} + \underbrace{\phi^{\mathbf{P},\Lambda}(q;\gamma)}_{\text{known functions of FV spectrum}} = n\pi$$

$$\sqrt{s} = 2\sqrt{m_\pi^2 + p^2} = \sqrt{E_\mathfrak{n}^2 - \mathbf{P}^2}$$

$$q = \frac{L}{2\pi} p, \qquad \gamma = E_\mathfrak{n}/\sqrt{s}$$

- lab frame energies $E_\mathfrak{n}$ can be extracted on the lattice
- field-theoretic derivations, multi-channel extensions

e.g. [Kim, Sachrajda, Sharpe 2005; Briceno, Hansen, Walker-Loud 2014]

Timelike pion form factor from Lattice QCD

- behavior of that quantization condition under small perturbations encodes even more information [Meyer 2012]
- derivation of Meyer closely related to Lellouch-Lüscher formalism [Lellouch, Lüscher 2001]
- Key formula:

$$|F_\pi(s)|^2 = \frac{3\pi s}{2L^3 p^5} g(\gamma) \left(q\phi'(q) + p\frac{\partial \delta_1(p)}{\partial p} \right) \left| \langle 0 | j^{(\mathbf{P},\Lambda)} | \mathbf{P}, \Lambda, \mathfrak{n} \rangle \right|^2$$

- has been demonstrated to work using overlap fermions [Feng et al. 2014]

Ingredients for self-contained extraction from LQCD

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- extract energy levels for given momentum **P** and irrep Λ
- use all levels across all irreps to map out the phase shift $\delta_1(p)$ and parametrize it
- compute $\phi'(q)$ for each energy level numerically
- extract the *bare* current matrix element

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- stochastic LapH framework: light quark lines are estimated stochastically from N_R random noise vectors ρ

[Morningstar et al. 2011]

$$\mathcal{Q} \approx \frac{1}{N_R} \sum_{r=1}^{N_R} \sum_{b} \varphi^{[b]}(\rho^r) \varrho^{[b]}(\rho^r)^\dagger$$

with source $\varrho^{[b]}(\rho) = V_s P^{(b)} \rho$

sink $\varphi^{[b]}(\rho) = \mathcal{S} \Omega^{-1} V_s P^{(b)} \rho$

$P^{(b)}$ - dilution projector, Ω^{-1} - propagator

$\mathcal{S} = V_s V_s^\dagger$ - LapH smearing (distillation) operator

- natively the sinks φ are smeared immediately after the inversion

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Current Insertions in the stochastic LapH framework

- We need unsmeared sinks for current insertions!
- in the stochastic LapH framework, current insertions behave just like mesons in every other respect
  - factorization of correlators into *"meson" functions* on source and sink timeslices
  - define *current functions* (here: local vector current)

$$\Psi_k^{(V)}(\mathbf{x}, t) = \overline{\varphi}'_{\mathbf{x}t}(\rho_1)^\dagger \gamma_4 \gamma_k \varphi'_{\mathbf{x}t}(\rho_2)$$

$\varphi', \overline{\varphi}'$ - *unsmeared* quark sinks

- minimally invasive hook into code after the inversions to compute *current functions* - rest of stochastic LapH workflow left intact

- we have implemented the local vector current, extension straightforward
- we are after matrix elements of the vector current between the vacuum and states belonging to a particular irrep

$$\left| \langle 0 | j^{(\mathbf{P},\Lambda)} | \mathbf{P}, \Lambda, \mathfrak{n} \rangle \right|^2$$

- may use the linear combination of spatial components of

$$j^a_\mu = \bar{\psi}\gamma_\mu \tau^a/2\psi + \mathrm{i}ac_V\partial_\nu \left\{ \bar{\psi}\sigma_{\mu\nu}\tau^a/2\psi \right\}$$

that transforms irreducibly $\Rightarrow j^{(\mathbf{P},\Lambda)}$ [Feng et al. 2014]

- Lattice setup
  - $N_f = 2 + 1$ ensemble $N200$ generated through the CLS effort, $L^3 \times T = 48^3 \times 128$ [Bruno et al. 2015]
  - $\mathcal{O}(a)$-improved Wilson fermions, Lüscher-Weisz gauge action, open temporal BC
  - $m_\pi \approx 280$ MeV, $m_K \approx 460$ MeV, $a \approx 0.064$ fm
  - $m_\pi L \approx 4.4$
- Construction of observables
  - stout-smeared spatial gauge links in Laplacian: $\rho = 0.1, n_\rho = 36$
  - LapH smearing $N_{ev} = 192$
  - one source time t_0 per config
  - dilution scheme: Laplace EV interlace 8, full spin and time dilution - time interlace 8 for relative-time lines
  - we have measured on 856 / 1712 configs

Temporal boundary effects

- boundary effects expected to decay as $\mathrm{e}^{-2m_\pi t}$ near the chiral limit [Bruno et al. 2015]
- we do see large boundary effects in the spectrum of the lattice Laplacian

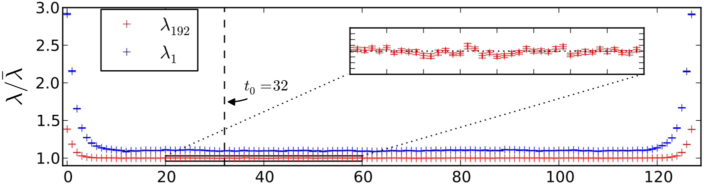


Figure : Smallest and largest retained EV of the lattice Laplacian normalized by their plateau average ($N_{\mathrm{cfg}} = 26$). Lowest EV offset for legibility.

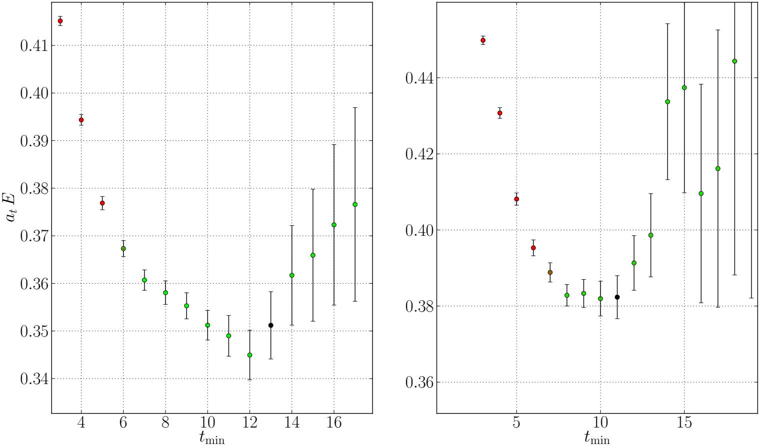


Figure : Energies extracted using correlated single-exponential fits to rotated correlators in [1,1,1] E^+ in the fit window starting at $t_{\rm min}$. Fill color indicates quality of fit with green dots having $\chi^2/\text{d.o.f.} \approx 1$.

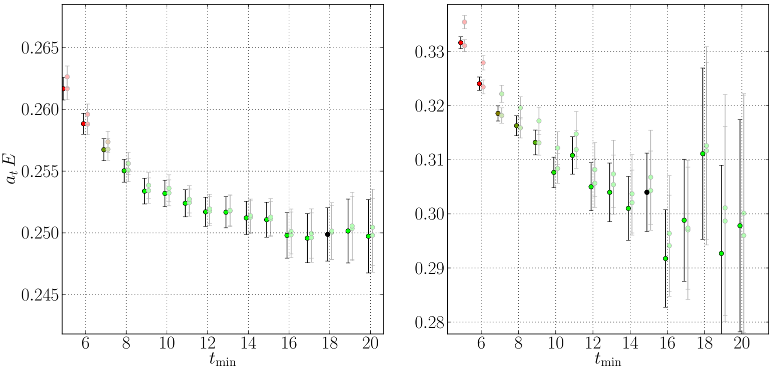
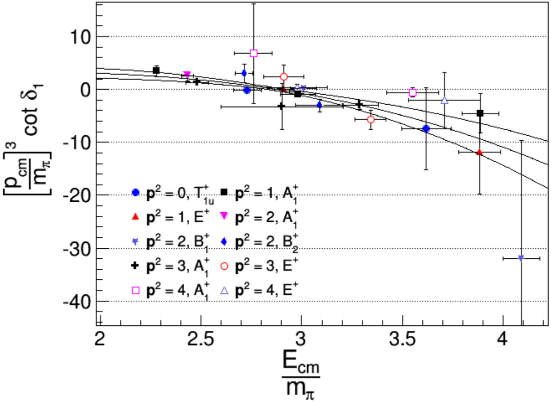


Figure : Same for [001] A_1^+. Grayed-out curves are checks of GEVP systematics with different numbers of operators and diagonalization times.



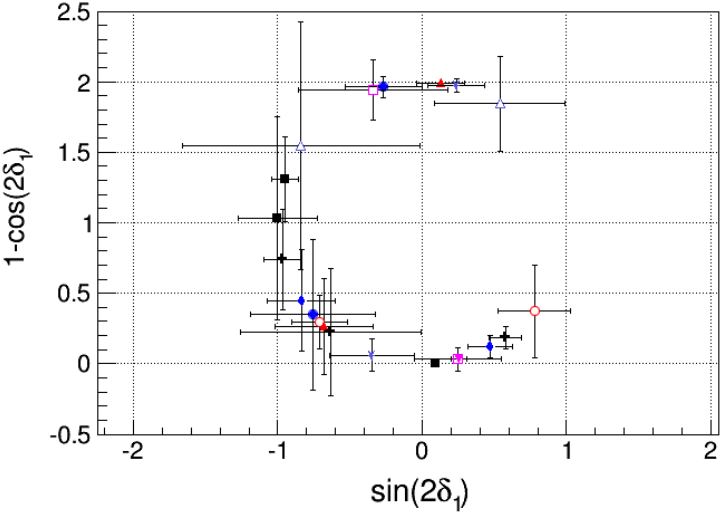
$$\frac{2m_K}{m_\pi} \approx 3.4$$

resonance parametrization:

$$p^3 \cot \delta_1 = \left(m_R^2 - s\right) \frac{6\pi}{g_R^2} \sqrt{s}$$

$$\frac{m_\rho}{m_\pi} = 2.88(4), \qquad g_{\rho\pi\pi} = 7.3(1.1), \qquad \frac{\chi^2}{\text{d.o.f.}} = 0.81$$

Argand diagram



- On the lattice, for a given total momentum and irrep, we compute

$$C_{ij} = \langle 0|\mathrm{T}\ O_i(t+t_0)\bar{O}_j(t_0)|0\rangle$$
$$\tilde{C}_j = \langle 0|\mathrm{T}\ J(t+t_0)\bar{O}_j(t_0)|0\rangle$$

  for the corresponding vector current J and a set of operators creating the states of interest at time t_0.
- GEVP to extract excited states from $C_{ij} \Rightarrow C^{(\mathrm{rot})}_{\mathfrak{n}\mathfrak{n}}$
- use GEVP eigenvectors to obtain

$$\tilde{C}^{(\mathrm{rot})}_{\mathfrak{n}} \overset{t\to\infty}{\to} \langle 0|J|\mathfrak{n}\rangle\,\langle \mathfrak{n}|\bar{O}^{(\mathrm{rot})}_{\mathfrak{n}}|0\rangle\,\mathrm{e}^{-E_{\mathfrak{n}}(t-t_0)}$$

$$\tilde{C}_{\mathfrak{n}}^{(\mathrm{rot})} \overset{t\to\infty}{\to} \langle 0|J|\mathfrak{n}\rangle \langle \mathfrak{n}|\bar{O}_{\mathfrak{n}}^{(\mathrm{rot})}|0\rangle \, \mathrm{e}^{-E_{\mathfrak{n}}(t-t_0)}$$

Now form appropriate ratios to cancel remaining terms:

1. $R_1 = \dfrac{\tilde{C}_{\mathfrak{n}}^{(\mathrm{rot})}}{\sqrt{C_{\mathfrak{n}\mathfrak{n}}^{(\mathrm{rot})}}\,\mathrm{e}^{-\frac{1}{2}E_{\mathfrak{n}}(t-t_0)}}$ using correlators and extr. energies
2. $R_2 = \dfrac{\tilde{C}_{\mathfrak{n}}^{(\mathrm{rot})}\sqrt{\left|\langle \mathfrak{n}|\bar{O}_{\mathfrak{n}}^{(\mathrm{rot})}|0\rangle\right|}}{C_{\mathfrak{n}\mathfrak{n}}^{(\mathrm{rot})}}$ using correlators and extr. overlaps
3. $R_3 = \dfrac{\tilde{C}_{\mathfrak{n}}^{(\mathrm{rot})}}{\left|\langle \mathfrak{n}|\bar{O}_{\mathfrak{n}}^{(\mathrm{rot})}|0\rangle\right|\mathrm{e}^{-E_{\mathfrak{n}}(t-t_0)}}$ using extr. overlaps and energies

$\Rightarrow$ all $|R_i| \overset{t\to\infty}{\to} |\langle 0|J|\mathfrak{n}\rangle|$

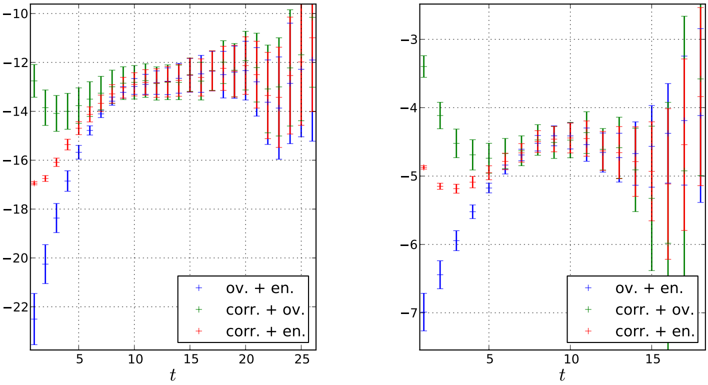


Figure : Bare current matrix element for first and second excited state in [0,0,1] A_1^+ from R_i.

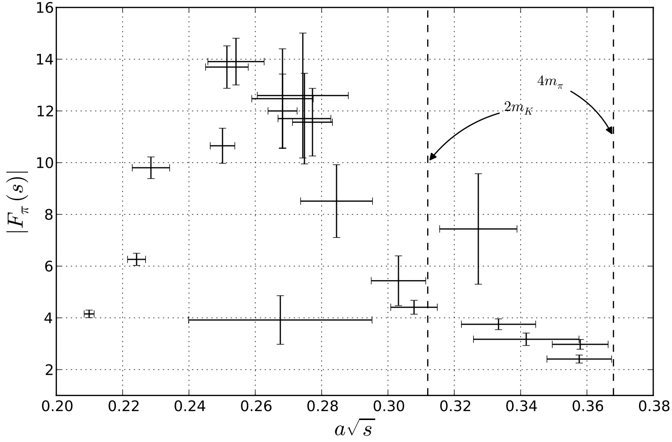
$\mathcal{O}(a)$-improvement and renormalization

The $\mathcal{O}(a)$-improved and renormalized vector current reads

$$V_\mu^{(\text{imp, ren})} = Z_V\,(1 + b_V am)\left(\bar{\psi}\gamma_\mu\psi + \mathrm{i}ac_V\partial_\nu\left\{\bar{\psi}\sigma_{\mu\nu}\psi\right\}\right)$$

- perturbative b_V and c_V (1-loop) [Aoki, Frezzotti, Weisz 1998]
- $Z \cdot r_m \cdot am = am_{\text{PCAC}}$ [M. Bruno, private communication]
- nonperturbative Z_V from χSF (preliminary) [M. Dalla Brida, private communication]

Extraction of Current Matrix Elements



PRELIMINARY

- boundary effects from simulations with open temporal BC seemingly uncritical in spectroscopy applications

- self-contained computation on the timelike pion form factor feasible

- plan to obtain significantly more statistics: twice the number of configs and another source time per config

- *might* improve uncertainties beyond the naive MC estimates
  - better resolution of GEVP
  - stabilization of correlated fits
  - finite volume methods highly nonlinear