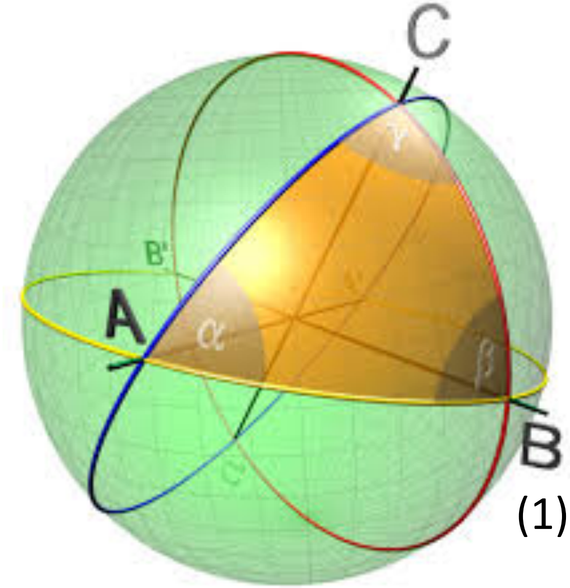
Spherical Finite Elements for Lattice Radial Quantization:

Free Scalar Fields



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Andrew Gasbarro, George Fleming
Yale Univeristy
Richard Brower, Evan Weinberg
Boston University

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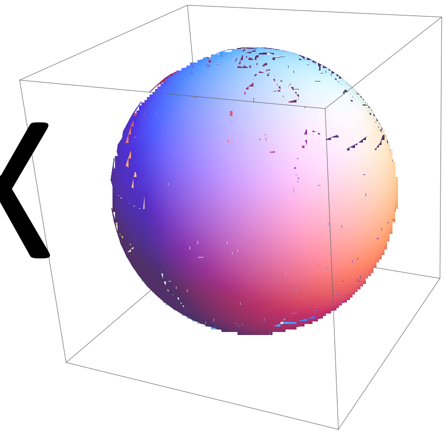
Outline

- Brief introduction to lattice radial quantization
- Brief introduction to linear finite elements
- Spherical finite elements: the idea
- Two roads to computing spherical finite elements
  - Numerical “RG” approach
  - Conformal mapping / Green’s function approach
- The classical theory: solving the generalized eigenvalue problem
- Plots and results

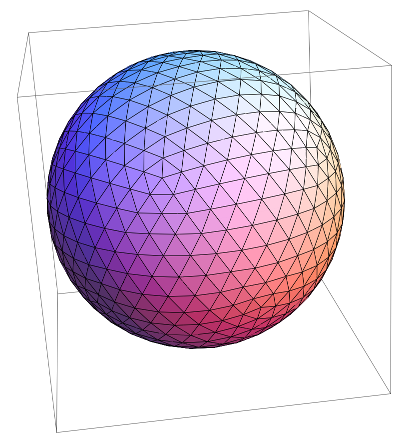
Lattice radial quantization

- In radial quantization, map:
  $t \rightarrow \log(r), \ \vec{x} \rightarrow S^{d-1}$
- The radial (i.e. temporal) part is straightforward to discretize; This work is concerned only with the sphere
- Many ways to construct spherical graphs. We use a scheme based on simplicial refinements of the icosahedron.
- Once one settles on a graph, there is still the question of how to construct the best lattice action. In particular, how to deal with curvature. This is what this work concerns itself with.

$\mathbf{R}$ X



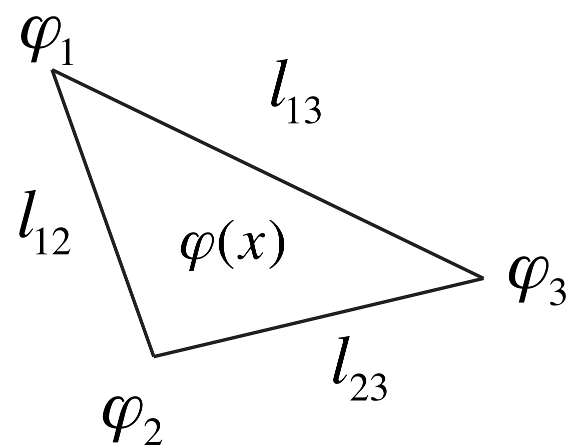
$\mathbf{Z_n}$X



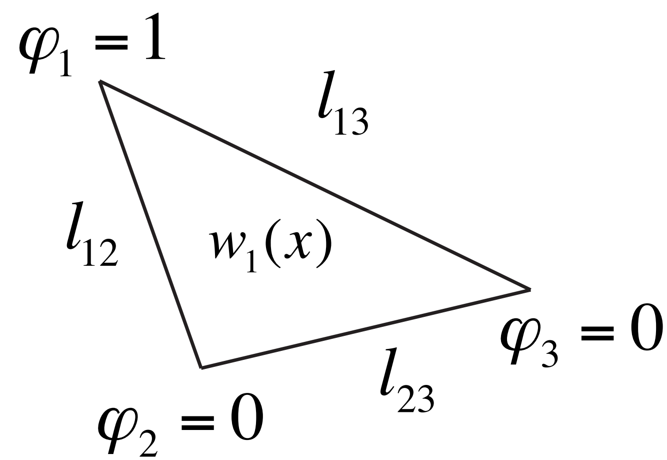
Linear Finite Elements

- Break the volume up into simple elements (triangles):

$$S = \int_V d^2x \sqrt{g(x)} L[\varphi, \nabla\varphi] = \sum_\Delta \int_\Delta d^2x \sqrt{g(x)} L[\varphi, \nabla\varphi]$$

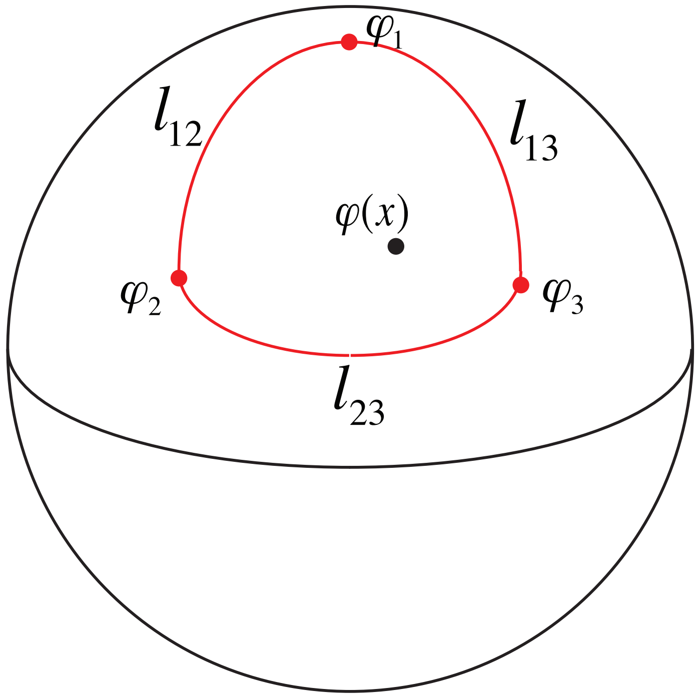


- Inside each element, the field must satisfy Laplace's equation with some boundary conditions.
- **Linear finite elements**:
  - Let elements be **flat triangles**
  - Along edges, linearly interpolate between the values of the field at the corner
  - This determines the boundary value problem
- Integrating over volume of element leaves an action which is only a function of the field values at the corner – ie the lattice d.o.f. Variational wave function is only an intermediary
- For linear finite elements, the coefficients in the action can be computed exactly analytically in terms of invariant lengths



Spherical Finite Elements

- The elements are **spherical triangles**
- Boundary values are linearly interpolated in the great circle arc distance
- Task: compute $\varphi(x)$ inside element that satisfies Laplace's equation and boundary conditions
  - More difficult problem than linear elements
  - Exactly solution may exist in terms of invariant lengths, but we haven't found it (yet?).
  - We compute the solution numerically using two different approaches



Approach #1: Complex Mapping / BEM

- Greens theorem (Dirichlet):

$$\varphi(x) = \int_{\partial V} dl' \tilde{\varphi}(x')(\hat{n}' \bullet \nabla' G(x', x))$$

- Divergence theorem and int by parts:

0

$$S = \int_V d^2x (\nabla'\varphi(x'))^2 = \int_{\partial V} dl'(\varphi(x')\nabla'\varphi(x'))\bullet \hat{n}' - \int_V d^2x(\varphi(x')\nabla'^2\varphi(x'))$$

- Action integral is conformally invariant, so can do integration around unit disk. Use Green's function of unit disk

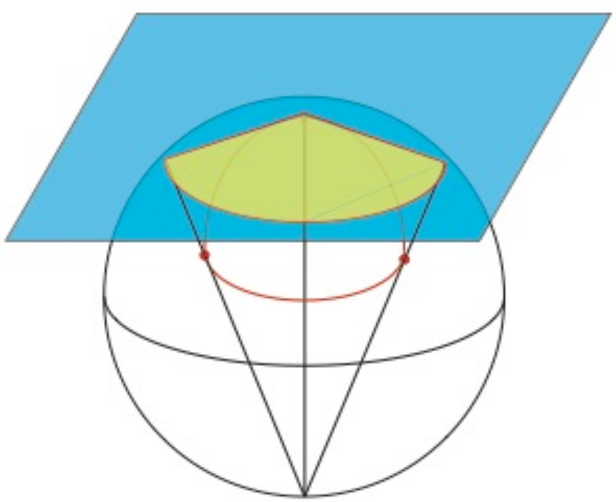
$$S = \frac{1}{4\pi}\int_0^{2\pi}\int_0^{2\pi} d\theta_1\, d\theta_2\, \frac{\left(\tilde{\varphi}(\xi(\theta_1)) - \tilde{\varphi}(\xi(\theta_2))\right)^2}{1-\cos(\theta_1-\theta_2)}$$

Where ξ is the mapping function from the disk to the spherical triangle

Approach #1: Complex Mapping / BEM

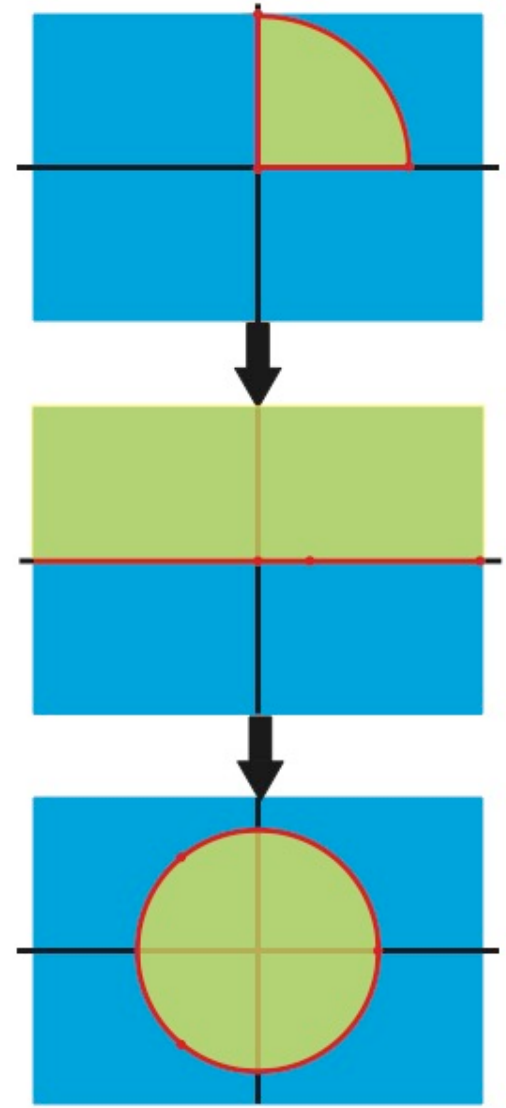
- Want to solve for phi using Green's function
- Map spherical triangle into region where Green's function is known: the unit disk on the complex plane

1.) Stereographic projection takes spherical triangle to circular arc triangle on the plane



2.) Schwartz triangle map takes spherical arc triangle to upper half plane

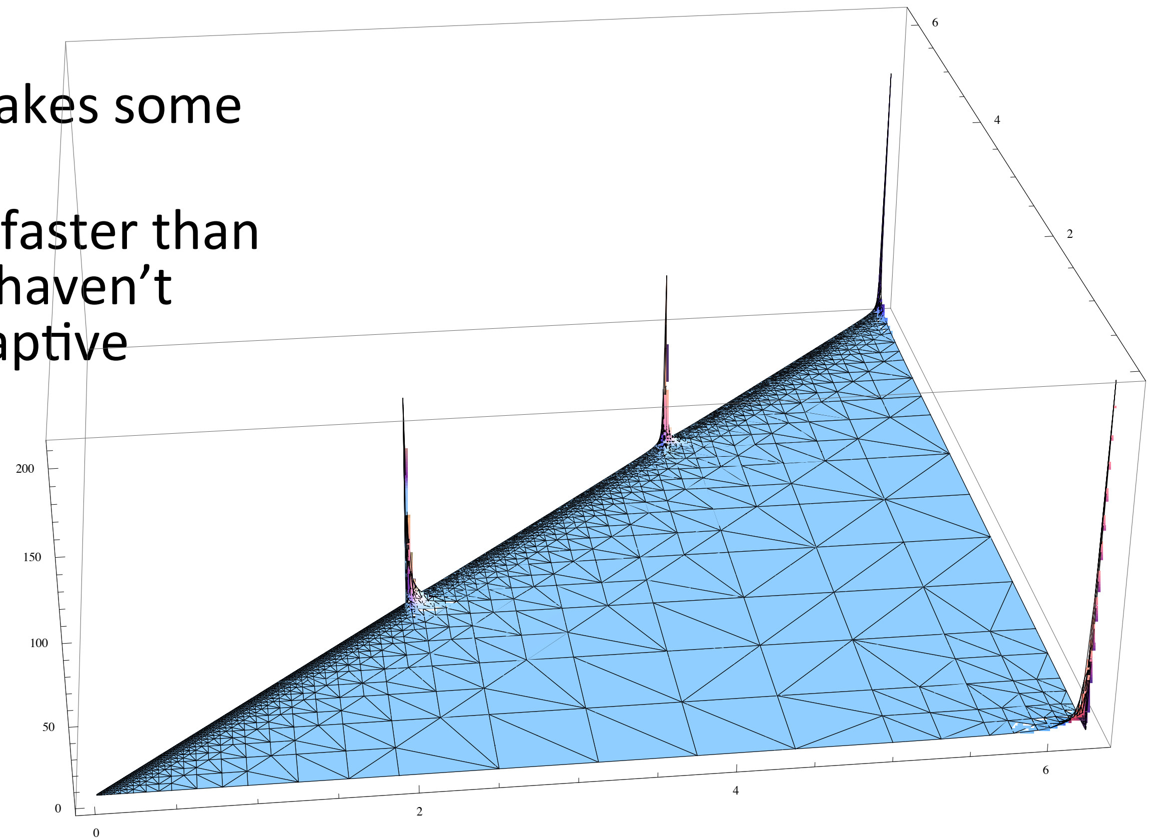
3.) Mobius transformation takes upper half plane to unit disk



Approach #1: Complex Mapping

$$S = \frac{1}{2\pi}\int_0^{2\pi}\int_0^{2\pi} d\theta_1\, d\theta_2 \, \frac{\left(\tilde{\varphi}(\xi(\theta_1)) - \tilde{\varphi}(\xi(\theta_2))\right)^2}{1-\cos(\theta_1-\theta_2)}$$

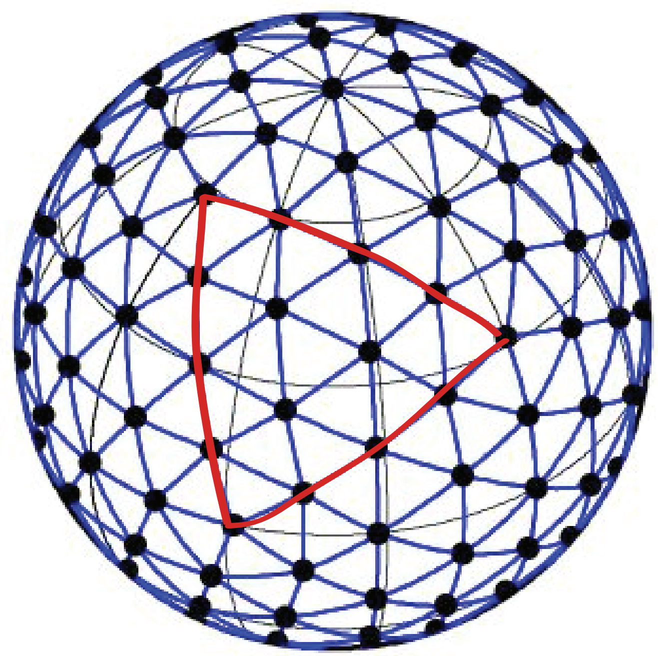
- In practice, integration takes some care.
- Currently, approach 2 is faster than approach 1 because we haven't figured out a perfect adaptive integrator



Approach 2: Numerical Minimization ("RG")

- Spherical Finite elements are a variational method for a wave function **on the manifold**
- Solve the minimization problem for ϕ(x) numerically inside the element using a minimization algorithm (CG)
- How to discretize the boundary value problem inside the element: **linear** finite elements!
- RG is a misnomer because we aren't actually integrating out degrees of freedom. But we do "block" linear elements to make a spherical element
- RG approach currently much faster

$$S_\Delta = \int_\Delta d^2x \sqrt{g(x)}\varphi(x)\nabla^2\varphi(x)$$



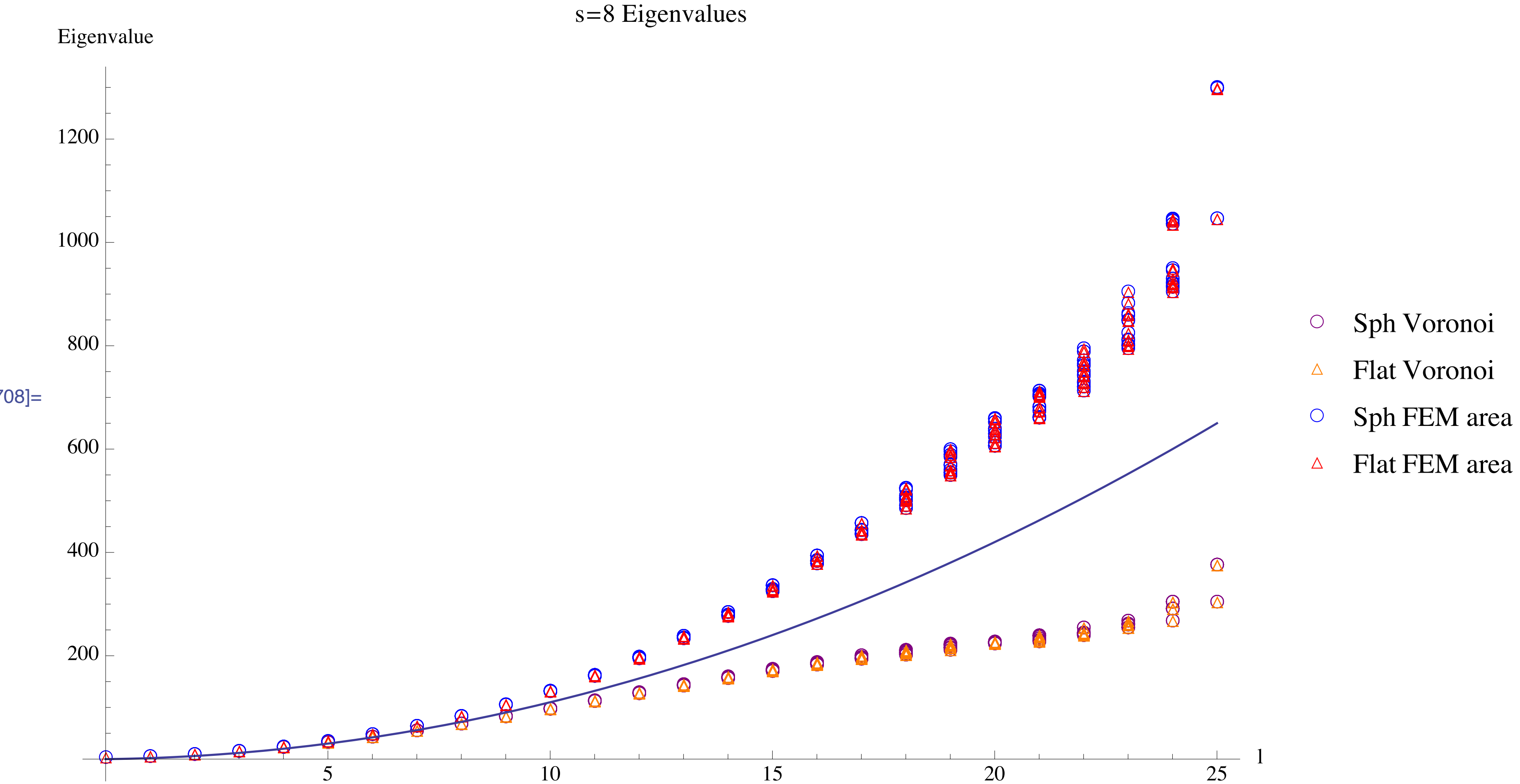
Generalized Eigenvalue Problem

- The study the classical theory, look at spectrum of finite element Laplacian
- In curved space, need a measure in eigenvalue equation

$$\nabla^2 \varphi^{(\lambda)}(x) = \lambda g^{1/2}(x) \varphi^{(\lambda)}(x) \qquad K_{xy} \varphi_y^{(\lambda)} = \lambda A_{xy} \varphi_y^{(\lambda)}$$

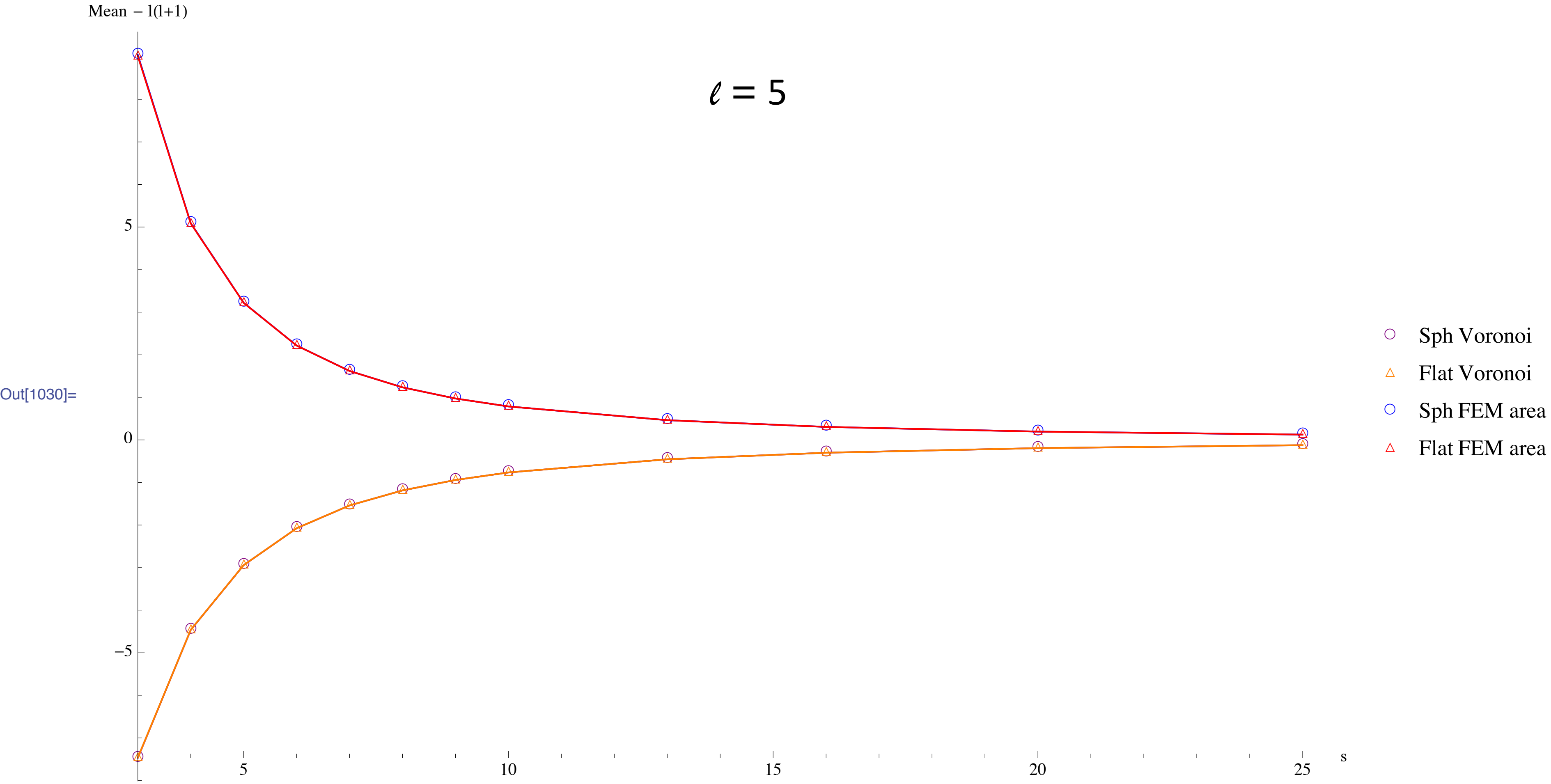
- Different choices for area matrix.
  - Voronoi dual area (ultralocal)
  - FEM area term (point split)

Spectrum


s=8 Eigenvalues
Eigenvalue
1200
1000
800
600
400
200
5
10
15
20
25
l
Sph Voronoi
Flat Voronoi
Sph FEM area
Flat FEM area
708]=

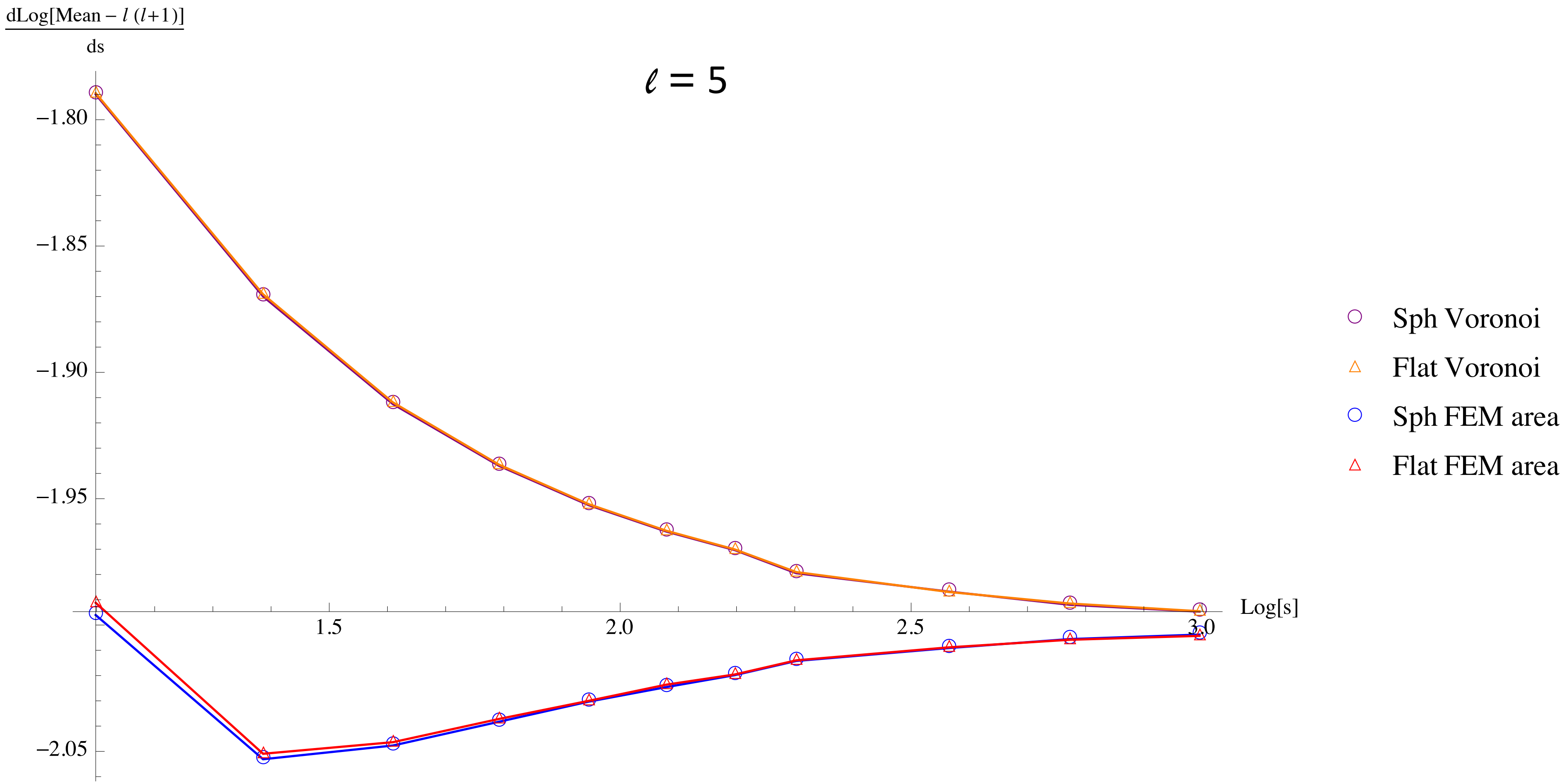
Convergence of the spectrum

- At each level, ℓ, the $2\ell+1$ eigenvalues of that level should become degenerate and take the value $\ell(\ell+1)$
- Test by looking at the statistics for the set of eigenvalues at a given level.

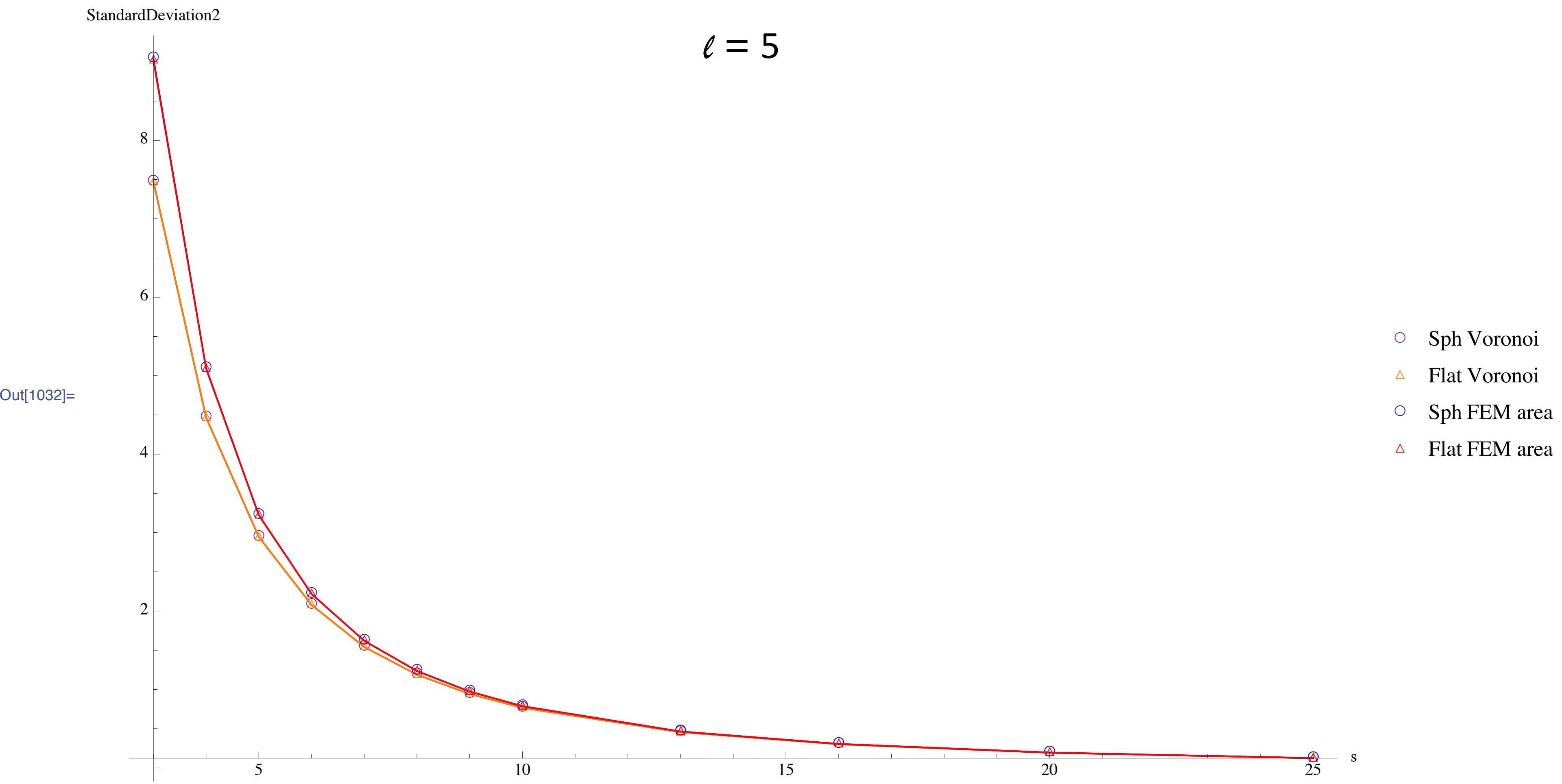


Mean − l(l+1)
ℓ = 5
5
0
−5
Out[1030]=
5
10
15
20
25
s
Sph Voronoi
Flat Voronoi
Sph FEM area
Flat FEM area

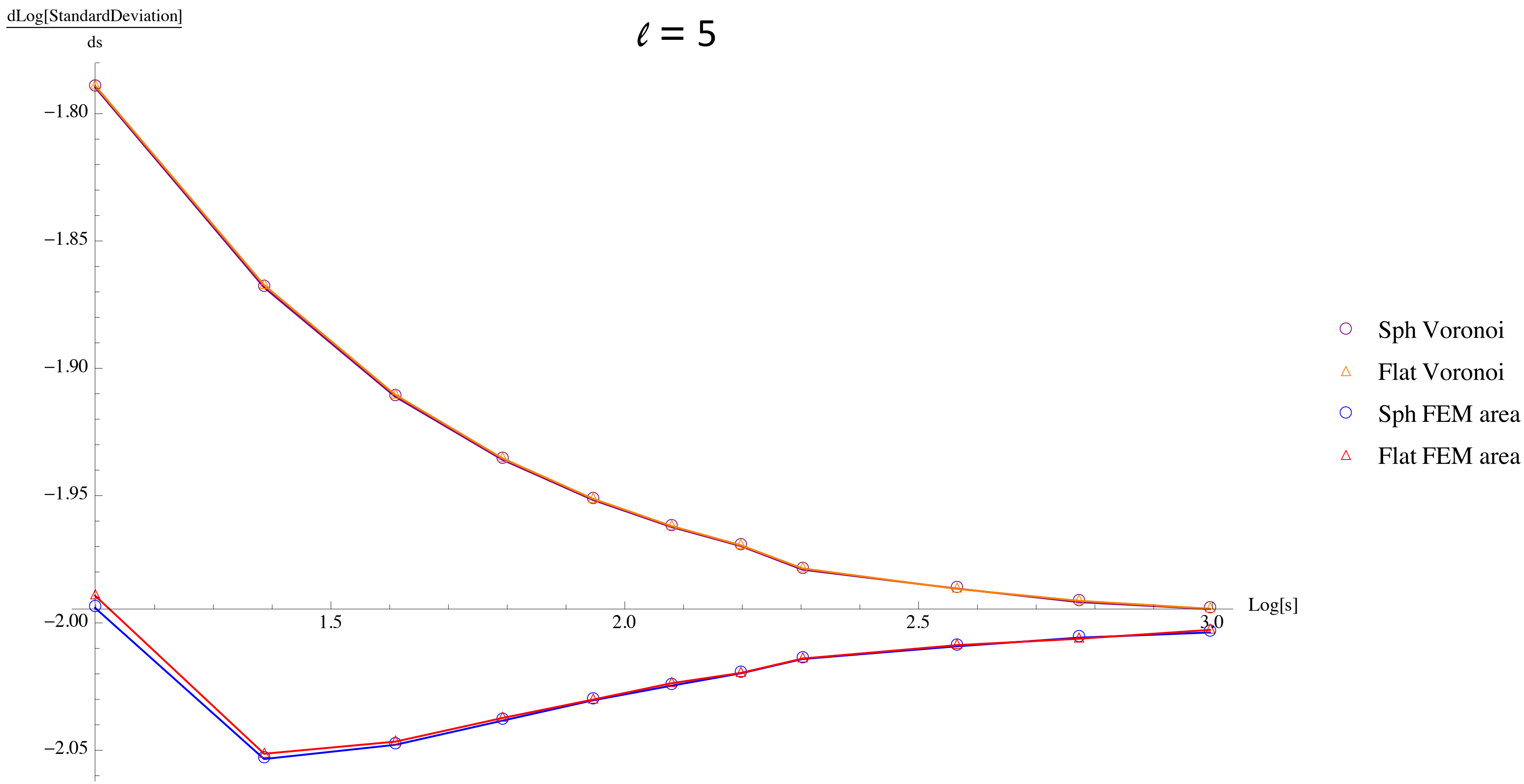
Out[1108]=



dLog[Mean − l (l+1)] / ds
ℓ = 5
−1.80
−1.85
−1.90
−1.95
−2.05
1.5
2.0
2.5
3.0
Log[s]
Sph Voronoi
Flat Voronoi
Sph FEM area
Flat FEM area

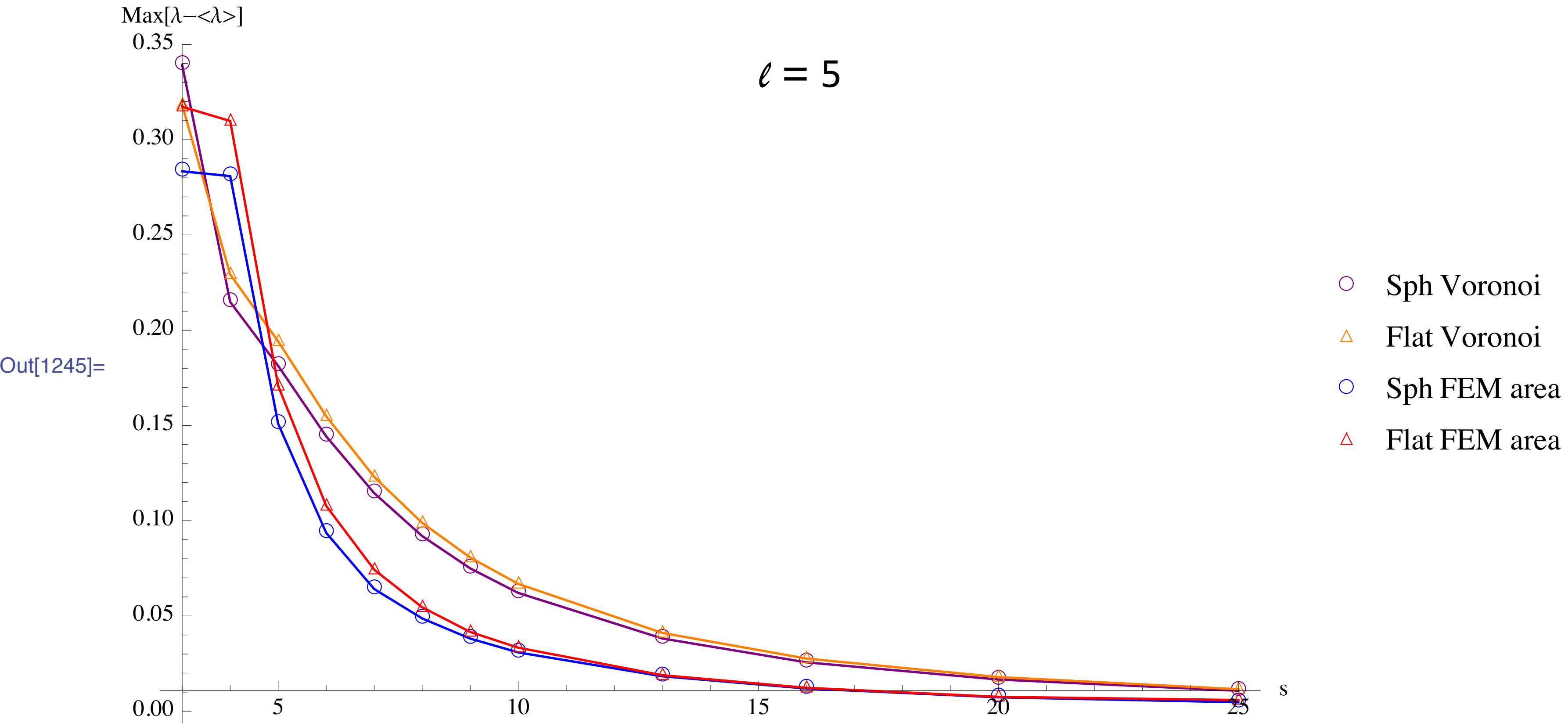

StandardDeviation2
ℓ = 5
8
6
4
2
5
10
15
20
25
s
Sph Voronoi
Flat Voronoi
Sph FEM area
Flat FEM area
Out[1032]=

Out[1121]=



$\ell = 5$
$\frac{dLog[StandardDeviation]}{ds}$
−1.80
−1.85
−1.90
−1.95
−2.00
−2.05
1.5
2.0
2.5
3.0
Log[s]
Sph Voronoi
Flat Voronoi
Sph FEM area
Flat FEM area

Deviation from flat spectrum at low s



Conclusions

- Developed elements that have no curvature singularities and well defined tangent spaces
- Found that the spectrum converges well (1/s^2) and that the spherical elements can be seen as having a slight edge over the flat elements at the classical level

Questions

- Can we find an exact solution in terms of invariant lengths?
- Does a corresponding Regge calculus exist?
- Spherical elements to fermions and gauge fields

References

- Brower, R. C., G. T. Fleming, and Herbert Neuberger. "Lattice radial quantization: 3D Ising." *Physics Letters B* 721.4 (2013): 299-305.
- Brower, Richard C., Michael Cheng, and George T. Fleming. "Improved Lattice Radial Quantization." *arXiv preprint arXiv:1407.7597* (2014).
- Brower, Richard C., Michael Cheng, and George T. Fleming. "Quantum Finite Elements: 2D Ising CFT on a Spherical Manifold." Lattice, 2014.

Image sources

1. https://upload.wikimedia.org/wikipedia/commons/9/93/Spherical_triangle_3d_opti.png
2. http://www1.gly.bris.ac.uk/~teanby/figures/subdivision.jpg


Thank you!