Disconnected contribution to hadronic vacuum polarization

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Introduction

- EM current of quarks

$$j_\mu(x) = \sum_f Q_f \, \overline{\psi}^{(f)}(x)\gamma_\mu \psi^{(f)}(x)$$

Q_f: electric charge of flavor f.

$$Q_u = \frac{2}{3}, \quad Q_d = -\frac{1}{3}, \quad Q_s = -\frac{1}{3}, \quad Q_c = \frac{2}{3}$$

- Hadronic vacuum polarization

$$\Pi_{\mu\nu}(Q) = \int \mathrm{d}x \, e^{iQx} \, \langle j_\mu(x) j_\nu(0) \rangle = $$

$$= \frac{1}{TV} \int \mathrm{d}x \int \mathrm{d}y \, e^{iQ(x-y)} \, \langle j_\mu(x) j_\nu(y) \rangle$$

$$\Pi_{\mu\nu}(Q) = (Q^2 \delta_{\mu\nu} - Q_\mu Q_\nu)\, \Pi(Q^2)$$

- $\Pi(Q^2)$ is useful for e.g. anomalous magnetic moment of muon

Outline

Outline

Staggered definition of current-current correlator

- Introduce $U(1)$ phase $c_{x,\mu}$ on links:

$$\left(M^{(f)}(c)\right)^{ab}_{x,y} = \frac{1}{2}\sum_\mu \left(\delta_{y,x+\mu} U^{ab}_{x,\mu}\, e^{iQ_f c_{x,\mu}} - \delta_{y,x-\mu}\left(U^\dagger_{x-\mu,\mu}\right)^{ab} e^{-iQ_f c_{x-\mu,\mu}}\right) + m_f \delta_{x,y}\delta^{ab}$$

  Staggered phase is absorbed into $U_{x,\mu}$.

- Partition function:

$$Z(c) = \int dU\, e^{-S_g(U)} \prod_f \left(\det M^{(f)}(c)\right)^{1/4}$$

- Define current-current correlator as 2$^{\text{nd}}$ derivative:

$$\langle j_\mu(x) j_{\bar\mu}(\bar x)\rangle = i\frac{\partial}{\partial c_{x,\mu}}\, i\frac{\partial}{\partial c_{\bar x,\bar\mu}} \log Z(c)\Big|_{c=0} =$$

$$= \langle j_\mu(x) j_{\bar\mu}(\bar x)\rangle_{\text{connected}} + \langle j_\mu(x) j_{\bar\mu}(\bar x)\rangle_{\text{contact}} + \langle j_\mu(x) j_{\bar\mu}(\bar x)\rangle_{\text{disconnected}}$$

Contributions to correlator

$$\langle j_\mu(x) j_{\bar\mu}(\bar x)\rangle_{\text{connected}} = \qquad \longleftarrow \quad \text{not this talk}$$

$$-\sum_f \frac{Q_f^2}{8} \mathrm{Re}\,\mathrm{tr_c} \left({M^{(f)}}^{-1}_{x+\mu,\bar x} U_{\bar x,\bar\mu} {M^{(f)}}^{-1}_{\bar x+\bar\mu,x} U_{x,\mu} + {M^{(f)}}^{-1}_{x+\mu,\bar x+\bar\mu} U^\dagger_{\bar x,\bar\mu} {M^{(f)}}^{-1}_{\bar x,x} U_{x,\mu} \right)$$

$$\langle j_\mu(x) j_{\bar\mu}(\bar x)\rangle_{\text{contact}} = \delta_{x,\bar x}\delta_{\mu,\bar\mu} \sum_f \frac{Q_f^2}{4} \mathrm{Re}\,\mathrm{tr_c} \left(U_{x,\mu} {M^{(f)}}^{-1}_{x+\mu,x} \right)$$

vanishes due to $q = 0$ subtraction

$$\langle j_\mu(x) j_{\bar\mu}(\bar x)\rangle_{\text{disconnected}} = \qquad \longleftarrow \quad \text{this talk}$$

$$-\left[\sum_f \frac{Q_f}{4} \mathrm{Im}\,\mathrm{tr_c} \left(U_{x,\mu} {M^{(f)}}^{-1}_{x+\mu,x} \right) \right] \times \left[\sum_{\bar f} \frac{Q_{\bar f}}{4} \mathrm{Im}\,\mathrm{tr_c} \left(U_{\bar x,\bar\mu} {M^{(\bar f)}}^{-1}_{\bar x+\bar\mu,\bar x} \right) \right]$$

Current conservation

- Current is conserved in continuum:

$$\frac{\partial}{\partial x_\mu} \langle j_\mu(x) j_{\bar\mu}(\bar x)\rangle = \frac{\partial}{\partial \bar x_{\bar\mu}} \langle j_\mu(x) j_{\bar\mu}(\bar x)\rangle = 0$$

- On lattice, on each configuration:

$$\nabla_\mu^{(b)} \Big(\langle j_\mu(x) j_{\bar\mu}(\bar x)\rangle_{\text{conn.}} + \langle j_\mu(x) j_{\bar\mu}(\bar x)\rangle_{\text{contact}} \Big) = \\ = \bar\nabla_{\bar\mu}^{(b)} \Big(\langle j_\mu(x) j_{\bar\mu}(\bar x)\rangle_{\text{conn.}} + \langle j_\mu(x) j_{\bar\mu}(\bar x)\rangle_{\text{contact}} \Big) = 0$$

$$\nabla_\mu^{(b)} \langle j_\mu(x) j_{\bar\mu}(\bar x)\rangle_{\text{disc.}} = \bar\nabla_{\bar\mu}^{(b)} \langle j_\mu(x) j_{\bar\mu}(\bar x)\rangle_{\text{disc.}} = 0$$

with backward derivatives

$$\nabla_\mu^{(b)} f_{\mu,\bar\mu}(x,\bar x) = \sum_\mu \Big(f_{\mu,\bar\mu}(x,\bar x) - f_{\mu,\bar\mu}(x-\mu,\bar x) \Big)$$

$$\bar\nabla_{\bar\mu}^{(b)} f_{\mu,\bar\mu}(x,\bar x) = \sum_{\bar\mu} \Big(f_{\mu,\bar\mu}(x,\bar x) - f_{\mu,\bar\mu}(x,\bar x-\bar\mu) \Big)$$

- Consistency check of code possible

Disconnected contribution

$$\langle j_\mu(x) j_{\bar\mu}(\bar x)\rangle_{\text{disc.}} = - \left[\sum_f \frac{Q_f}{4} \operatorname{Im}\operatorname{tr}_c \left(U_{x,\mu} {M^{(f)}}^{-1}_{x+\mu,x} \right)\right] \times \left[\sum_{\bar f} \frac{Q_{\bar f}}{4} \operatorname{Im}\operatorname{tr}_c \left(U_{\bar x,\bar\mu} {M^{(\bar f)}}^{-1}_{\bar x+\bar\mu,\bar x} \right)\right]$$

- $\Pi(Q^2)$ depends only on Q^2 $\longrightarrow$ take $Q = (\underline{\mathbf{0}}, q)$ and $\mu = 1, 2, 3$

$$\langle j_\mu(t) j_{\bar\mu}(\bar t)\rangle_{\text{disc.}} = -\frac{1}{V} \times \underbrace{\left[\sum_{\underline{\mathbf{x}}} \sum_f \frac{Q_f}{4} \operatorname{Im}\operatorname{tr}_c \left(U_{\underline{\mathbf{x}},t,\mu} {M^{(f)}}^{-1}_{\underline{\mathbf{x}}+\mu,t\,;\,\underline{\mathbf{x}},t} \right)\right]}_{j_\mu(t)} \times \underbrace{\left[\sum_{\bar{\underline{\mathbf{x}}}} \sum_{\bar f} \frac{Q_{\bar f}}{4} \operatorname{Im}\operatorname{tr}_c \left(U_{\bar{\underline{\mathbf{x}}},\bar t,\bar\mu} {M^{(\bar f)}}^{-1}_{\bar{\underline{\mathbf{x}}}+\bar\mu,\bar t\,;\,\bar{\underline{\mathbf{x}}},\bar t} \right)\right]}_{j_{\bar\mu}(\bar t)}$$

- Way to proceed: calculate $j_\mu(t)$ on each configuration, then correlate with itself.

Outline

Reduction to even sites

$$\tilde{j}_\mu(t) = \sum_{\underline{x}} \mathrm{Im}\,\mathrm{tr_c}\left(U_{\underline{x},t,\mu} M^{-1}{}_{\underline{x}+\mu,t\,;\,\underline{x},t}\right) =$$

$$= \sum_{\underline{x}\text{ even}} \mathrm{Im}\,\mathrm{tr_c}\left(U_{\underline{x},t,\mu} M^{-1}{}_{\underline{x}+\mu,t\,;\,\underline{x},t}\right) + \sum_{\underline{x}\text{ odd}} \mathrm{Im}\,\mathrm{tr_c}\left(U_{\underline{x},t,\mu} M^{-1}{}_{\underline{x}+\mu,t\,;\,\underline{x},t}\right)$$

Since $M^{-1}{}_{x+\mu,x} = \epsilon_{x+\mu}\,\epsilon_x \left(M^{-1}\right)^\dagger_{x+\mu,x} = -\left(M^{-1}\right)^\dagger_{x+\mu,x}$,

Odd part can be rewritten:

$$\sum_{\underline{x}\text{ odd}} \mathrm{Im}\,\mathrm{tr_c}\left(U_{\underline{x},t,\mu} M^{-1}{}_{\underline{x}+\mu,t\,;\,\underline{x},t}\right) = -\sum_{\underline{x}\text{ odd}} \mathrm{Im}\,\mathrm{tr_c}\left(\left(M^{-1}\right)^\dagger_{\underline{x}+\mu,t\,;\,\underline{x},t} U_{\underline{x},t,\mu}\right) =$$

$$= \sum_{\underline{x}\text{ odd}} \mathrm{Im}\,\mathrm{tr_c}\left(U^\dagger_{\underline{x},t,\mu} M^{-1}{}_{\underline{x},t\,;\,\underline{x}+\mu,t}\right) = \sum_{\underline{x}\text{ even}} \mathrm{Im}\,\mathrm{tr_c}\left(U^\dagger_{\underline{x}-\mu,t,\mu} M^{-1}{}_{\underline{x}-\mu,t\,;\,\underline{x},t}\right)$$

Combined:

$$\tilde{j}_\mu(t) = \sum_{\underline{x}\text{ even}} \mathrm{Im}\,\mathrm{tr_c}\left(\left(U^{+-}_\mu\, M^{-1}\right)_{\underline{x},t\,;\,\underline{x},t}\right)$$

U^{+-}_μ = (covariant shift in direction μ) + (covariant shift in direction $-\mu$)

Estimation with random vectors

$$\tilde{j}_\mu(t) = \sum_{\underline{\mathbf{x}}\text{ even}} \mathrm{Im}\,\mathrm{tr}_c \left(\left(U_\mu^{+-}\, M^{-1} \right)_{\underline{\mathbf{x}},t\,;\,\underline{\mathbf{x}},t} \right)$$

- N random vectors $\xi^{(r)}_{\underline{\mathbf{x}},t,a}$ on even sites, $\left\langle \xi^{(r)}_{\underline{\mathbf{x}},t,a} \left(\xi^{(r)}_{\bar{\underline{\mathbf{x}}},\bar{t},\bar{a}} \right)^* \right\rangle = \delta_{\underline{\mathbf{x}},\bar{\underline{\mathbf{x}}}}\, \delta_{t,\bar{t}}\, \delta_{a,\bar{a}}$

$$\tilde{j}_\mu(t) = \left\langle \frac{1}{N} \sum_{r=1}^{N} \mathrm{Im} \left(\sum_{\underline{\mathbf{x}}\text{ even}} \sum_{\bar{\underline{\mathbf{x}}}\text{ even}} \sum_{\bar{t}} \sum_{a,\bar{a}} \left(\xi^{(r)}_{\underline{\mathbf{x}},t,a} \right)^* \left(U_\mu^{+-}\, M^{-1} \right)_{\underline{\mathbf{x}},t,a\,;\,\bar{\underline{\mathbf{x}}},\bar{t},\bar{a}} \xi^{(r)}_{\bar{\underline{\mathbf{x}}},\bar{t},\bar{a}} \right) \right\rangle =$$

$$= \left\langle \frac{1}{N} \sum_{r=1}^{N} \mathrm{Im} \left\langle \xi^{(r)} \middle| U_\mu^{+-}\, M^{-1}\, \xi^{(r)} \right\rangle_t \right\rangle = \left\langle \frac{1}{N} \sum_{r=1}^{N} j_\mu^{(r)}(t) \right\rangle$$

Our choice: Z_2 random sources.

- We use isospin symmetric masses: $m_u = m_d = m_l$

$$j^{(u)} + j^{(d)} + j^{(s)} = \frac{2}{3}\tilde{j}^{(l)} - \frac{1}{3}\tilde{j}^{(l)} - \frac{1}{3}\tilde{j}^{(s)} = \frac{1}{3} \cdot \left(\tilde{j}^{(l)} - \tilde{j}^{(s)} \right) = \frac{1}{3}\tilde{j}^{(l-s)}$$

Noise reduction: Use same random vectors for l and s.
see e.g. [Gülpers, Lattice2014]

Truncated solver method

[Collins *et al.*, PoS LATTICE (2007) 141]

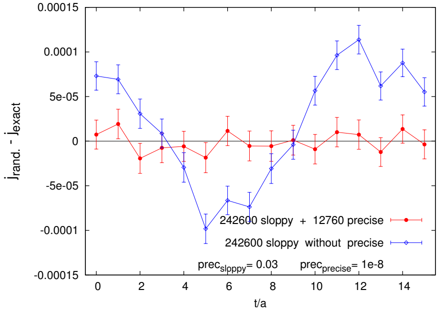
- Need to compute $M^{-1}\xi$ on many random vectors.
- Reduce work: $N_p \ll N_s, \quad N_p + N_s = N$

  For $r = 1, 2, \ldots, N_s$, compute $M^{-1}\xi^{(r)}$ with low precision $\longrightarrow \; j_s^{(r)}$

  For $r = N_s + 1, \ldots, N$, both low and high precision $\longrightarrow \; j_s^{(r)}, j_p^{(r)}$

$$j = \Big\langle (\text{sloppy}) \Big\rangle + \Big\langle (\text{precise}) - (\text{sloppy}) \Big\rangle =$$

$$= \left\langle \frac{1}{N_s} \sum_{r=1}^{N_s} j_s^{(r)} \right\rangle + \left\langle \frac{1}{N_p} \sum_{r=N_s+1}^{N} \left(j_p^{(r)} - j_s^{(r)} \right) \right\rangle$$



Hopping parameter expansion

[Thron *et al.*, Phys.Rev. D57 (1998) 1642]
[Bali *et al.*, Comput.Phys.Commun. 181 (2010) 1570]

- $M = m + D, \qquad M^\dagger = m - D, \qquad M^\dagger M = m^2 - D^2$

$$M^{-1} = M^\dagger \left(M^\dagger M\right)^{-1} = \frac{m - D}{m^2 - D^2}$$

- Then the current:

$$\tilde{j}_\mu(t) = \sum_{\underline{\mathbf{x}}} \mathrm{Im}\,\mathrm{tr_c}\left(\left(U_\mu \frac{m-D}{m^2-D^2}\right)_{\underline{\mathbf{x}},t\,;\,\underline{\mathbf{x}},t}\right) =$$

$$= \sum_{\underline{\mathbf{x}}} \mathrm{Im}\,\mathrm{tr_c}\left(\left(U_\mu(-D)\,\frac{1}{m^2-D^2}\right)_{\underline{\mathbf{x}},t\,;\,\underline{\mathbf{x}},t}\right)$$

- Rewrite the inverse using n^{th} degree polynomial $\quad a_0 + a_1 x + \cdots + a_n x^n$

$$\frac{1}{m^2 - D^2} = \sum_{k=0}^{n} a_k \left(D^2\right)^k + \frac{1}{m^2-D^2} \sum_{k=0}^{n+1} \underbrace{\left(a_{k-1} - m^2\, a_k\right)}_{b_k} \cdot \left(D^2\right)^k$$

with $\quad a_{-1} = 1 \quad$ and $\quad a_{n+1} = 0.$

Hopping parameter expansion

- Current consists of two parts:

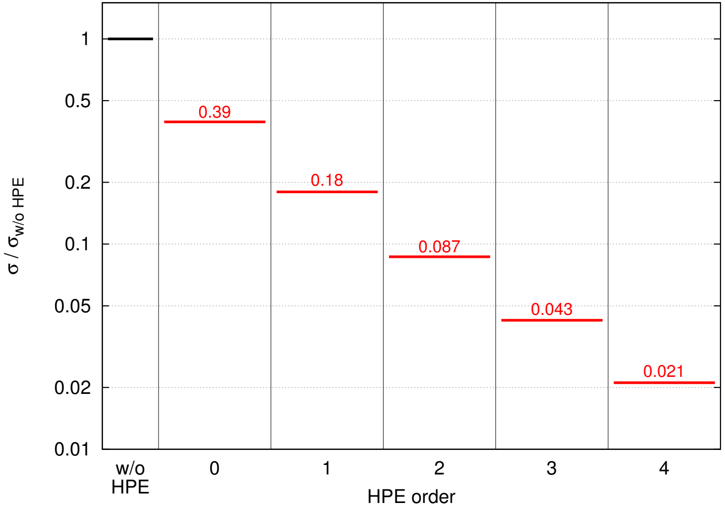
$$\tilde{j}_\mu(t) = \sum_{k=0}^{n} a_k \overbrace{\sum_{\underline{\mathbf{x}}} \mathrm{Im}\,\mathrm{tr_c}\left(U_\mu(-D)\,(D^2)^k\right)_{\underline{\mathbf{x}},t\,;\,\underline{\mathbf{x}},t}}^{K_\mu^{(k)}(t)} + \\ + \sum_{k=0}^{n+1} b_k \underbrace{\sum_{\underline{\mathbf{x}}} \mathrm{Im}\,\mathrm{tr_c}\left(U_\mu(-D)\,\frac{(D^2)^k}{m^2 - D^2}\right)_{\underline{\mathbf{x}},t\,;\,\underline{\mathbf{x}},t}}_{R_\mu^{(k)}(t)}$$

- Recipe:
  - Calculate $K_\mu^{(k)}(t)$ exactly.
  - Calculate $R_\mu^{(k)}(t)$ using random vectors:

$$R_\mu^{(k)}(t) = \frac{1}{N}\sum_{r=1}^{N} \mathrm{Im}\left\langle \xi^{(r)} \middle| U_\mu\,(-D)\,\frac{(D^2)^k}{m^2 - D^2}\,\xi^{(r)} \right\rangle_t$$

    Use same random vector set for all $k = 0, 1, \ldots, n+1$.
  - Choose coefficients a_k such that the noise of $\sum_k b_k R_\mu^{(k)}(t)$ is minimal.

Hopping parameter expansion



Hopping parameter expansion

$$K_\mu^{(k)}(t) = \sum_{\underline{\mathbf{x}}} \mathrm{Im}\,\mathrm{tr_c}\left(U_\mu(-D)\left(D^2\right)^k\right)_{\underline{\mathbf{x}},t\,;\,\underline{\mathbf{x}},t}$$

- Computing $K_\mu^{(k)}(t)$ $\longrightarrow$ calculate loops
  - $k=0$ $K_\mu^{(0)}(t) = 0$
  - $k=1$ 1 loop of length 4 $\longrightarrow$ plaquette
  - $k=2$ 8 additional loops of length 6
  - $k=3$ 167 additional loops of length 8
  - $k=4$ 4402 additional loops of length 10
  - ...

Outline

Estimator for the correlator

$$\langle j_\mu(t) j_{\bar\mu}(\bar t)\rangle_{\text{disc.}} = -\frac{1}{V} \times$$

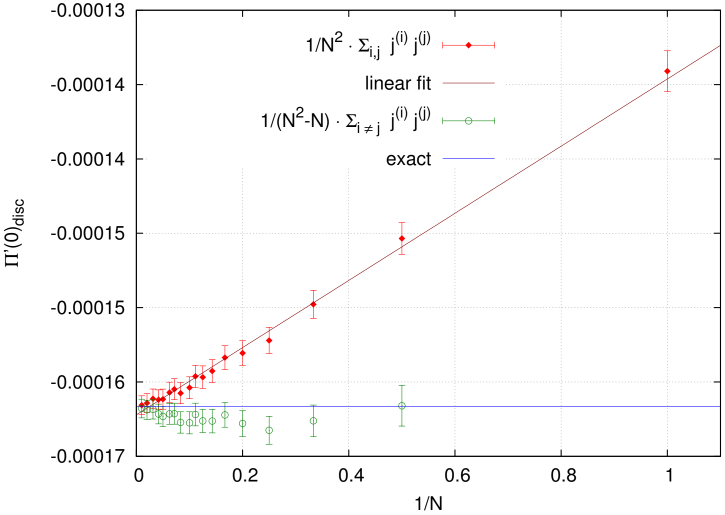
$$\underbrace{\left[\sum_{\underline{\mathbf{x}}}\sum_{f}\frac{Q_f}{4}\,\mathrm{Im}\,\mathrm{tr}_c\left(U_{\underline{\mathbf{x}},t,\mu} {M^{(f)}}^{-1}_{\underline{\mathbf{x}}+\mu,t\,;\,\underline{\mathbf{x}},t}\right)\right]}_{j_\mu(t)} \times \underbrace{\left[\sum_{\bar{\underline{\mathbf{x}}}}\sum_{\bar f}\frac{Q_{\bar f}}{4}\,\mathrm{Im}\,\mathrm{tr}_c\left(U_{\bar{\underline{\mathbf{x}}},\bar t,\bar\mu} {M^{(\bar f)}}^{-1}_{\bar{\underline{\mathbf{x}}}+\bar\mu,\bar t\,;\,\bar{\underline{\mathbf{x}}},\bar t}\right)\right]}_{j_{\bar\mu}(\bar t)}$$

- $\displaystyle\frac{1}{N}\sum_{r=1}^{N} j_\mu^{(r)}(t)$ is unbiased estimator for $\tilde j_\mu(t)$.
- But $\displaystyle\left(\frac{1}{N}\sum_{r=1}^{N} j_\mu^{(r)}(t)\right)\left(\frac{1}{N}\sum_{\bar r=1}^{N} j_{\bar\mu}^{(\bar r)}(\bar t)\right)$ is biased estimator for $\tilde j_\mu(t)\tilde j_{\bar\mu}(\bar t)$:

$$\tilde j_\mu(t)\tilde j_{\bar\mu}(\bar t) - \left\langle \left(\frac{1}{N}\sum_{r=1}^{N} j_\mu^{(r)}(t)\right)\left(\frac{1}{N}\sum_{\bar r=1}^{N} j_{\bar\mu}^{(\bar r)}(\bar t)\right)\right\rangle = \mathcal{O}\left(\frac{1}{N}\right)$$

- Unbiased estimator: $\displaystyle\frac{1}{N^2-N}\sum_{r\neq\bar r} j_\mu^{(r)}(t) j_{\bar\mu}^{(\bar r)}(\bar t)$

Unbiasing


-0.00013
-0.00014
-0.00014
-0.00015
-0.00015
-0.00016
-0.00017
0
0.2
0.4
0.6
0.8
1
1/N
Π'(0)disc
1/N2 · Σi,j j(i) j(j)
linear fit
1/(N2-N) · Σi ≠ j j(i) j(j)
exact

Outline

Observable & Ensembles

- For $q^2 > 0$

$$\Pi(q^2) = \sum_t \frac{\cos(qt) - 1}{\hat{q}^2} \langle j_\mu(t) j_\mu(0) \rangle$$

with $\hat{q}^2 = 4\sin^2\left(\frac{qa}{2}\right)$

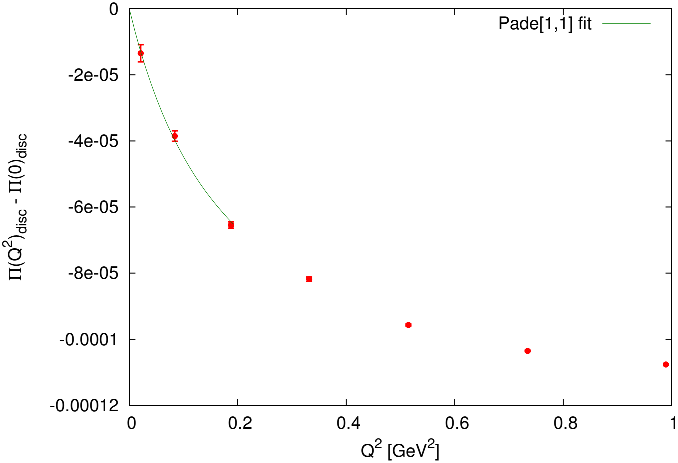
- For $q^2 = 0$

$$\Pi(0) = \lim_{q\to 0} \sum_t \frac{\cos(qt) - 1}{\hat{q}^2} \langle j_\mu(t) j_\mu(0) \rangle = \sum_t -\frac{t^2}{2} \langle j_\mu(t) j_\mu(0) \rangle$$

- Ensembles
  - $a = 0.134\,\text{fm}$, physical quark masses, $48^3 \times 64$
  - $a = 0.118\,\text{fm}$, physical quark masses, $56^3 \times 96$
  - $a = 0.095\,\text{fm}$, physical quark masses, $64^3 \times 96$
  - $a = 0.078\,\text{fm}$, physical quark masses, $80^3 \times 128$

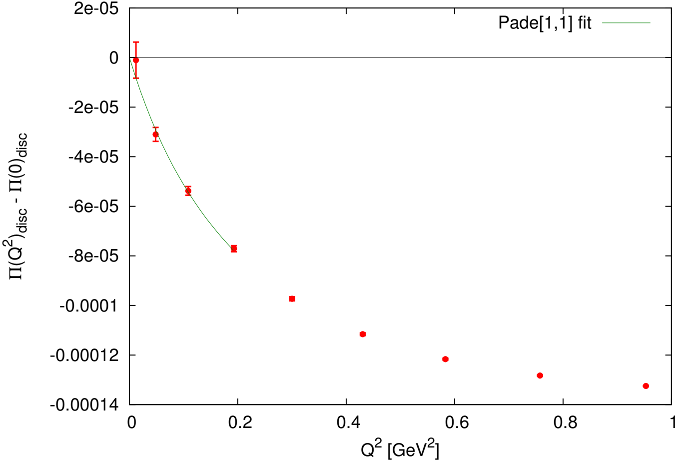
Coarsest lattice (preliminary)

$a = 0.134\,\text{fm}$, physical quark masses, $48^3 \times 64$



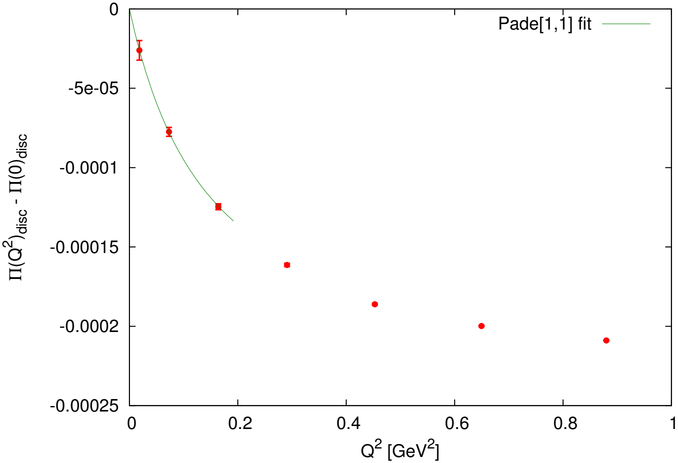
Medium lattice (preliminary)

$a = 0.118\,\text{fm}$, physical quark masses, $56^3 \times 96$



Fine lattice (preliminary)

$a = 0.095\,\text{fm}, \quad \text{physical quark masses}, \quad 64^3 \times 96$



Outline

Conclusions & Outlook

- Conclusions
  - Noise reduction techniques to compute j_μ
  - First results at 3 lattice spacings
- Outlook
  - Look for further noise reduction techniques
  - Evaluate on finer lattice
  - Study systematics (e.g. moments vs. Padé fits)