The leading hadronic contribution to γ-Z mixing

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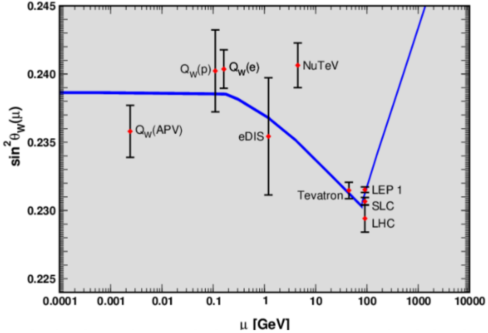
Motivation

- γ-Z mixing determines the running of the electro-weak mixing angle

$$\sin^2\theta_W(Q^2) = \frac{e^2}{g^2} = \sin^2\theta_0 \left(1 + \Delta\sin^2\theta_W(Q^2)\right)$$

[PDG 2014]



- **goal:** estimate the hadronic contribution to $\Delta\sin^2\theta_W(Q^2)$ (for small Q^2) in a lattice calculation

Definitions

- leading hadronic contribution

- Vacuum polarization

$$\Pi^{\gamma Z}_{\mu\nu}(Q^2) \equiv \int d^4x\, e^{iQx} \left\langle j^Z_\mu(x)\big|_{\text{vector}}\, j^\gamma_\nu(0) \right\rangle$$

- The currents are defined as [F. Jegerlehner, Z. Phys.C - Particles and Fields 32, 195-207(1986)]

$$\begin{aligned} j^\gamma_\mu &= \frac{2}{3}\bar{u}\gamma_\mu u - \frac{1}{3}\bar{d}\gamma_\mu d - \frac{1}{3}\bar{s}\gamma_\mu s + \frac{2}{3}\bar{c}\gamma_\mu c \\ j^Z_\mu\big|_{\text{vector}} &= j^3_\mu\big|_{\text{vector}} - \sin^2\theta_W\, j^\gamma_\mu \\ j^3_\mu\big|_{\text{vector}} &= \frac{1}{4}\bar{u}\gamma_\mu u - \frac{1}{4}\bar{d}\gamma_\mu d - \frac{1}{4}\bar{s}\gamma_\mu s + \frac{1}{4}\bar{c}\gamma_\mu c\,. \end{aligned}$$

- Vacuum polarization

$$\Pi^{\gamma Z}(Q^2) = \Pi^{\gamma 3}(Q^2) - \sin^2\theta_W\, \Pi^{\gamma\gamma}(Q^2)$$

$\Pi^{\gamma Z}(Q^2)$ from the time-momentum correlator

- correlator in time-momentum representation

$$G^{\gamma Z}(x_0) = -\int d^3x \left\langle j_k^Z(x)\Big|_{\text{vector}} \; j_k^\gamma(0) \right\rangle \quad \text{with} \quad k = 1, 2, 3$$

- subtracted vacuum polarization $\Pi_R^{\gamma Z}(Q^2)$ [D. Bernecker and H. Meyer, 1107.4388]

$$\Pi_R^{\gamma Z}(Q^2) \equiv \Pi^{\gamma Z}(Q^2) - \Pi^{\gamma Z}(0) = \int_0^\infty dx_0 \, G^{\gamma Z}(x_0) \left[x_0^2 - \frac{4}{Q^2} \sin^2\left(\frac{1}{2} Q x_0\right) \right]$$

- hadronic contribution to the running of the Weinberg angle [F. Jegerlehner, 1107.4683]

$$\Delta_{\text{had}} \sin^2\theta_W(Q^2) = -e^2 \, \Pi_R^{\gamma\gamma}(Q^2) + g^2 \, \Pi_R^{\gamma 3}(Q^2)$$

$$= \frac{e^2}{\sin^2\theta_0} \Pi_R^{\gamma Z}(Q^2)$$

The correlator $G^{\gamma Z}(t)$

- the correlator $G^{\gamma Z}(Q^2)$ for four quark flavors is given by

$$G^{\gamma Z}(x_0) = -\int d^3x \, \langle j_k^Z(x)\big|_{\text{vector}} \; j_k^{\gamma}(0) \rangle$$

$$= \left(\frac{1}{4} - \frac{5}{9}\sin^2\theta_W\right) G^{\ell}(x_0) + \left(\frac{1}{12} - \frac{1}{9}\sin^2\theta_W\right) G^{s}(x_0) + \left(\frac{1}{6} - \frac{4}{9}\sin^2\theta_W\right) G^{c}(x_0)$$

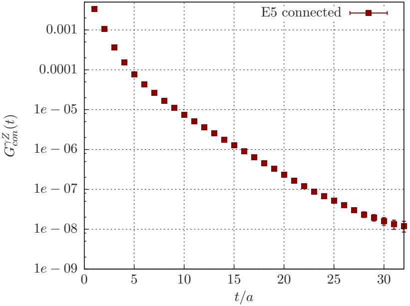
$$+ \textbf{disconnected}$$

- connected correlators

$$G^{\ell}(x_0) \equiv \sum_{\vec{x}} \left\langle \mathrm{Tr}\left[D_{\ell}^{-1}(x,0)\,\gamma_k\, D_{\ell}^{-1}(0,x)\,\gamma_k \right] \right\rangle_G$$

Results connected correlator $\mathbf{G^{\gamma Z}_{con}(t)}$

- E5 ensemble: $\mathbf{64 \times 32^3}$ lattice, $\mathbf{m_\pi \approx 455}$ MeV, $\mathbf{a = 0.063}$ fm, $\mathbf{N_f = 2}$
- $\mathcal{O}$**(a)**-improved Wilson fermions
- local current at source, conserved current at sink



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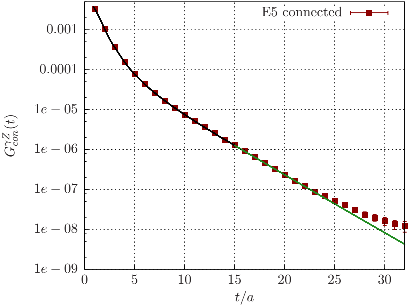
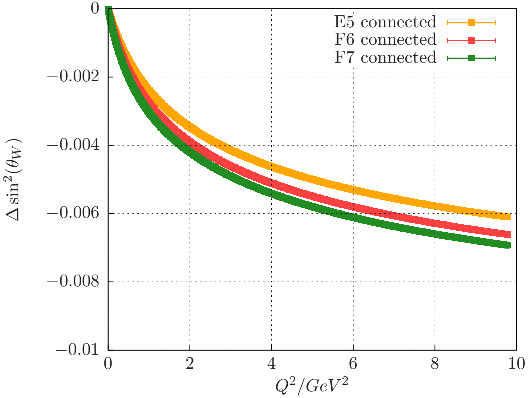


- $\mathbf{\Pi_R^{\gamma Z}(Q^2) = \int_0^\infty dt\, G^{\gamma Z}(t) \left[t^2 - \frac{4}{Q^2}\sin^2\left(\frac{1}{2}Qt\right)\right]}$
- single exponential fit for every flavor for large $\mathbf{t \geq 15 \approx 1}$ fm
- cubic splines for small $\mathbf{t}$

connected contribution to $\Delta_{had}\sin^2\theta_W(Q^2)$

β	a[fm]	lattice	m_π[MeV]	$m_\pi L$	Label	N_{cnfg}
5.3	0.063	64×32^3	455	4.7	E5	1000
5.3	0.063	96×48^3	325	5.0	F6	300
5.3	0.063	96×48^3	280	4.3	F7	250



disconnected contribution to the correlator $G^{\gamma Z}(t)$

- the disconnected contribution to $G^{\gamma Z}(Q^2)$ is given by (neglecting charm)

$$G^{\gamma Z}_{\rm disc}(x_0) \equiv -\int d^3x \left\langle j^Z_k(x)\Big|_{\rm vector}\; j^\gamma_k(0)\right\rangle_{\rm disc}$$
$$= \sin^2\theta_W \,\frac{1}{9}\, G^{(\ell+As),(\ell-s)}_{\rm disc}(x_0)$$

- disconnected correlator can be written as

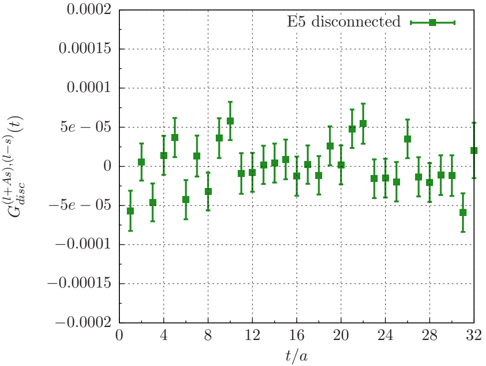
$$G^{(\ell+As),(\ell-s)}_{\rm disc}(x_0-y_0) = \frac{Z_V^2}{L^3}\left\langle\left(\sum_{\vec{x}} \mathrm{Tr}\left[\gamma_k\, D_\ell^{-1}(x,x) + A\,\gamma_k\, D_s^{-1}(x,x)\right]\right)\times \left(\sum_{\vec{y}} \mathrm{Tr}\left[\gamma_k\, D_\ell^{-1}(y,y) - \gamma_k\, D_s^{-1}(y,y)\right]\right)\right\rangle$$

with $A = \frac{3}{4\sin^2\theta_W} - 1$

- difference of light and strange propagator at source y
- **idea:** calculate light- and strange propagator with the same stochastic sources to cancel stochastic noise

disconnected correlator $G^{\gamma Z}_{disc}(t)$

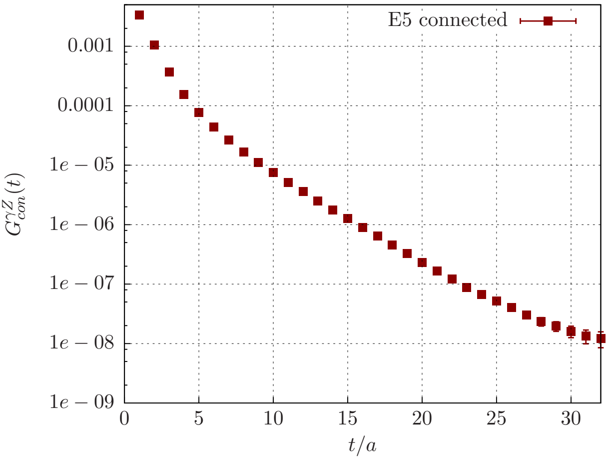
- all-to-all propagator with **3** stochastic sources and generalized hopping parameter expansion [G. Bali et al., 0910.3970; V. G. et al., 1309.2104]
- 64×32^3 lattice with $m_\pi \approx 455$ MeV and $a = 0.063$ fm



- $G^{(\ell+As),(\ell-s)}_{disc}(t)$ consistent with zero

The total correlator $\mathbf{G}^{\gamma Z}(\mathbf{t})$

- $G^{\gamma Z}(t) = G^{\gamma Z}_{con}(t) + \sin^2\theta_W \frac{1}{9} G^{(\ell+As),(\ell-s)}_{disc}(t)$



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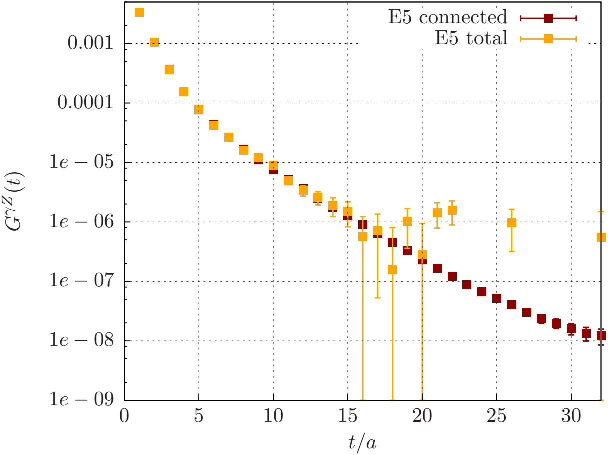
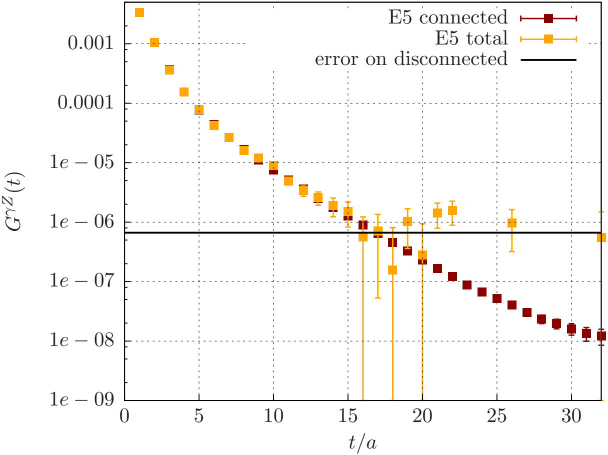


The total correlator $\mathbf{G}^{\gamma Z}(\mathbf{t})$

- $\mathbf{G^{\gamma Z}(t) = G^{\gamma Z}_{con}(t) + \sin^2\theta_W \, \frac{1}{9} \, G^{(\ell+As),(\ell-s)}_{disc}(t)}$



- for $\mathbf{t \gtrsim 16a \approx 1}$ fm the total correlator is dominated by the error on the disconnected contribution

The correlator $G^{\gamma Z}(t)$ for large Euclidean times I

- [A. Francis et al., 1306.2532] split the currents j_μ^γ and j_μ^Z into isoscalar and isovector piece
- correlator (with isospin symmetry)

$$G^{\gamma Z}(t) = G^{I=0}(t) + G^{I=1}(t) \qquad \text{with} \;\; G^{I}(t) = -\int d^3x \left\langle j_k^{Z,I}(x)\; j_k^{\gamma,I}(0) \right\rangle$$

- isoscalar correlator

$$G^{I=0} = -\sin^2\theta_W \frac{1}{18} G^{\ell} + \left(\frac{1}{12} - \sin^2\theta_W \frac{1}{9}\right) G^{s} + \left(\frac{1}{6} - \sin^2\theta_W \frac{4}{9}\right) G^{c} + \sin^2\theta_W \frac{1}{9}\, G_{\text{disc}}^{(\ell+As),(\ell-s)}$$

- spectral representation

$$G^{\gamma Z}(t) = \int_0^\infty d\omega\, \omega^2\; \rho^{\gamma Z}(\omega)\; e^{-\omega|t|} \qquad \Rightarrow \;\; \rho^{\gamma Z}(\omega) = \rho^{I=0}(\omega) + \rho^{I=1}(\omega)$$

  spectral density $\rho^{\gamma Z}(\omega)$
- lowest isovector and isoscalar state

$$\rho^{I=1}(\omega) = 0 \qquad \text{for} \quad \omega < 2m_\pi$$
$$\rho^{I=0}(\omega) = 0 \qquad \text{for} \quad \omega < 3m_\pi$$

The correlator $\mathbf{G}^{\gamma Z}(\mathbf{t})$ for large Euclidean times II

- isoscalar spectral density $\rho^{I=0}(\omega)$ for $\omega < 3m_\pi$

$$0 = -\sin^2\theta_W \frac{1}{18}\rho^\ell + \left(\frac{1}{12} - \sin^2\theta_W \frac{1}{9}\right)\rho^s + \left(\frac{1}{6} - \sin^2\theta_W \frac{4}{9}\right)\rho^c + \sin^2\theta_W \frac{1}{9}\,\rho_{\mathrm{disc}}^{(\ell+As),(\ell-s)}$$

The correlator $\mathbf{G}^{\gamma Z}(\mathbf{t})$ for large Euclidean times II

- isoscalar spectral density $\rho^{I=0}(\omega)$ for $\omega < 3m_\pi$

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$$\Rightarrow \qquad \rho_{\mathrm{disc}}^{(\ell+As),(\ell-s)} = \frac{1}{2}\rho^\ell \qquad \text{for} \quad \omega < 3m_\pi$$

The correlator $\mathbf{G}^{\gamma Z}(\mathbf{t})$ for large Euclidean times II

- isoscalar spectral density $\rho^{I=0}(\omega)$ for $\omega < 3m_\pi$

$$0 = -\sin^2\theta_W \frac{1}{18}\rho^\ell + \cancel{\left(\frac{1}{12} - \sin^2\theta_W \frac{1}{9}\right)\rho^s} + \cancel{\left(\frac{1}{6} - \sin^2\theta_W \frac{4}{9}\right)\rho^c} + \sin^2\theta_W \frac{1}{9}\,\rho_{\text{disc}}^{(\ell+As),(\ell-s)}$$

$$\Rightarrow \qquad \rho_{\text{disc}}^{(\ell+As),(\ell-s)} = \frac{1}{2}\rho^\ell \qquad \text{for} \quad \omega < 3m_\pi$$

- for large times we find for the disconnected correlator

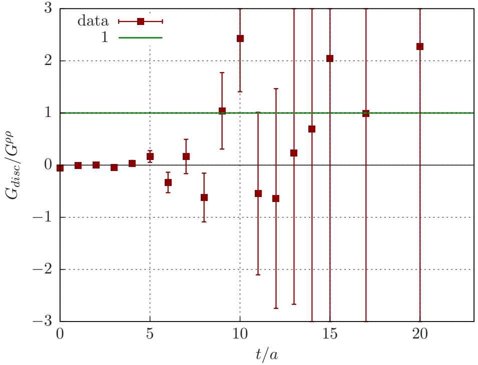
$$G_{\text{disc}}^{(\ell+As),(\ell-s)}(t) \longrightarrow \frac{1}{2}G^\ell(t) = G^{\rho\rho}(t) \qquad \text{for} \quad t \to \infty$$

- ratio of disconnected and ρ correlator

$$\frac{G_{\text{disc}}^{(\ell+As),(\ell-s)}(t)}{G^{\rho\rho}(t)} \longrightarrow 1 \qquad \text{for} \quad t \to \infty$$

The ratio of $G_{disc}^{(\ell+As),(\ell-s)}$ and $G^{\rho\rho}(t)$

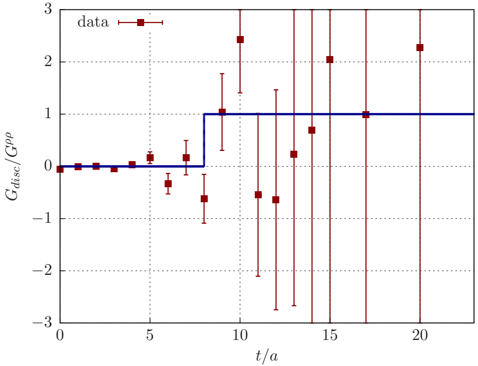
- $\frac{G_{disc}^{(\ell+As),(\ell-s)}(t)}{G^{\rho\rho}(t)} \longrightarrow 1$ for $t \to \infty$



- up to $t \approx 8a$ we can distinguish the ratio from 1

The ratio of $\mathbf{G}_{\text{disc}}^{(\ell+\mathbf{As}),(\ell-\mathbf{s})}$ and $\mathbf{G}^{\rho\rho}(\mathbf{t})$

- $\dfrac{\mathbf{G}_{\text{disc}}^{(\ell+\mathbf{As}),(\ell-\mathbf{s})}(\mathbf{t})}{\mathbf{G}^{\rho\rho}(\mathbf{t})} \longrightarrow \mathbf{1}$ for $\mathbf{t} \to \infty$



- up to $\mathbf{t} \approx \mathbf{8a}$ we can distinguish the ratio from $\mathbf{1}$
- **idea:** use $\mathbf{G}_{\text{disc}}^{(\ell+\mathbf{As}),(\ell-\mathbf{s})}(\mathbf{t})/\mathbf{G}^{\rho\rho}(\mathbf{t}) = \mathbf{1}$ for $\mathbf{t} > \mathbf{8}$ to give an upper bound for the magnitude of the disconnected contribution

$\Pi^{\gamma Z}(Q^2)$ with disconnected estimate

$$\Pi_R^{\gamma Z}(Q^2) \equiv \Pi^{\gamma Z}(Q^2) - \Pi^{\gamma Z}(0) = \int_0^\infty dt\, G^{\gamma Z}(t) \left[t^2 - \frac{4}{Q^2}\sin^2\left(\frac{1}{2}Qt\right)\right]$$

- for $t \leq 8a$, the correlator is well described by the connected part

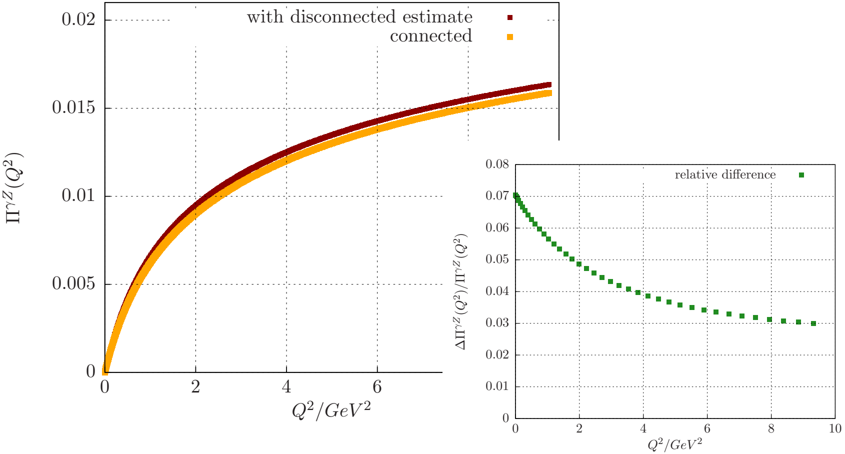
$$G^{\gamma Z}(t) = \left(\frac{1}{4} - \frac{5}{9}\sin^2\theta_W\right) G^{\ell}(t) + \left(\frac{1}{12} - \frac{1}{9}\sin^2\theta_W\right) G^{s}(t) + \left(\frac{1}{6} - \frac{4}{9}\sin^2\theta_W\right) G^{c}(t)$$

- for $t > 8a$ we use $G_{\rm disc}^{(\ell+As),(\ell-s)}(t)/G^{\rho\rho}(t) = 1$ as upper bound for disconnected part

$$G^{\gamma Z}(t) = G_{\rm con}^{\gamma Z}(t) + \sin^2\theta_W \frac{1}{9} G^{\rho\rho}(t)$$

- give an upper bound for the magnitude of the disconnected contribution to $\Delta_{\rm had}\sin^2\theta_W(Q^2)$

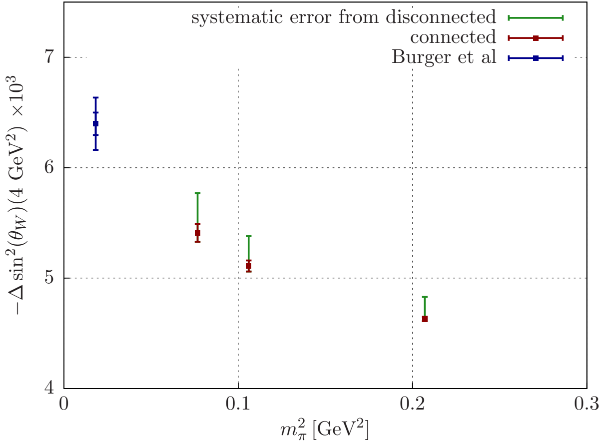
$\Pi^{\gamma Z}(Q^2)$ with disconnected estimate



- difference can be used as an conservative estimate for systematic error from neglecting the disconnected contribution
- at $\mathbf{Q^2} \approx \mathbf{4}$ GeV^2 maximum disconnected contribution $\approx$ **4%**

chiral behavior of $\Delta_{\text{had}} \sin^2\theta_W(Q^2)$

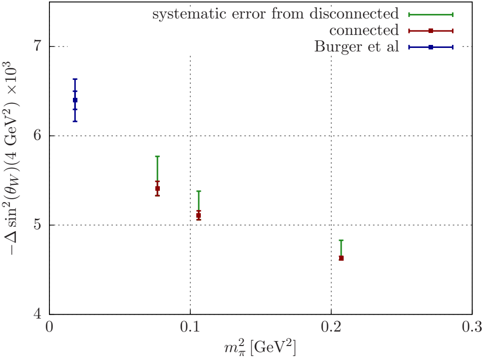
- three different pion masses
- $\Delta_{\text{had}} \sin^2\theta_W(Q^2)$ at $Q^2 = 4$ GeV^2



- [Burger et al. arXiv:1505.03283] connected diagrams only

Summary

- leading hadronic contribution to the running of $\sin^2\theta_W$ using the time-momentum correlator
- connected and disconnected contribution to the correlator
- disconnected contribution consistent with zero
- use asymptotic behavior of correlator to give a conservative upper bound for the disconnected contribution



Backup

the mixed time-momentum representation method

- hadronic vacuum polarization

$$\Pi_{kk}(\omega, \mathbf{q}=0) = \int d^4x e^{i\,Q\cdot x} \langle j_k^\gamma(x)\; j_k^\gamma(0)\rangle = -\int dt\; e^{i\omega t}\, G^{\gamma\gamma}(t)$$

- vector correlator

$$G^{\gamma\gamma}(t) = -\int d^3x\, \langle j_k^\gamma(x) j_k^\gamma(0)\rangle \qquad \text{with} \quad j_k^\gamma = \frac{2}{3}\bar{u}\gamma_k u - \frac{1}{3}\bar{d}\gamma_k d + \ldots$$

- tensor structure of the vacuum polarization

$$\Pi_{kk}(\omega, \mathbf{q}=0) = (Q_k Q_k - \delta_{kk} Q^2)\,\Pi(Q^2) \overset{Q^2=\omega^2}{=} -\omega^2\,\Pi(\omega^2)$$

- subtracted vacuum polarization after Taylor expansion at $Q^2 = 0$

$$\hat{\Pi}(\omega^2) = 4\,\pi^2 \left[\Pi(\omega^2) - \Pi(0)\right] = 4\,\pi^2 \int_{-\infty}^{\infty} dt\, G^{\gamma\gamma}(t) \left[\frac{e^{-i\omega t} - 1}{\omega^2} + \frac{t^2}{2}\right]$$

$$= 4\pi^2 \int_0^{\infty} dt\, G^{\gamma\gamma}(t) \left[t^2 - \frac{4}{\omega^2} \sin^2\left(\frac{1}{2}\omega t\right)\right]$$

generalized Hopping Parameter Expansion

cf. [Bali et al. arXiv:0910.3970]

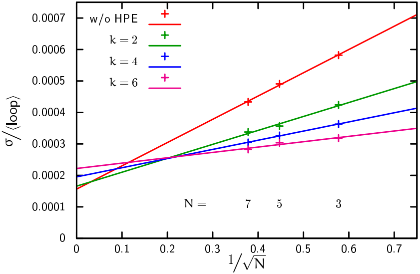
- $\mathcal{O}(\mathbf{a})$-improved Wilson-Dirac operator

$$\mathbf{D_{sw}} = \frac{\mathbf{1}}{\mathbf{2\kappa}}\mathbb{1} + \mathbf{c_{sw}B} - \frac{\mathbf{1}}{\mathbf{2}}\mathbf{H} = \mathbf{A} - \frac{\mathbf{1}}{\mathbf{2}}\mathbf{H} = \mathbf{A}\left(\mathbb{1} - \frac{\mathbf{1}}{\mathbf{2}}\mathbf{A^{-1}H}\right)$$

- generalized hopping parameter expansion

$$\mathbf{D_{sw}^{-1}} = \sum_{\mathbf{i=0}}^{\mathbf{k-1}} \left(\frac{\mathbf{1}}{\mathbf{2}}\mathbf{A^{-1}H}\right)^{\mathbf{i}} \mathbf{A^{-1}} + \left(\frac{\mathbf{1}}{\mathbf{2}}\mathbf{A^{-1}H}\right)^{\mathbf{k}} \mathbf{D_{sw}^{-1}}$$

- $\mathbf{D_{sw}^{-1}}$ on the right hand side estimated using stochastic sources
- $\langle \text{loop} \rangle = \left\langle \sum_{\vec{\mathbf{x}}} \text{Tr}\left(\mathbf{D^{-1}(x,x)}\right)\right\rangle_{\mathbf{G}}$
- choose $\mathbf{N = 3}$ sources with order $\mathbf{k = 6}$ of the generalized HPE



Isoscalar and Isovector part of the currents

- split the currents into $\mathrm{I} = \mathbf{1}$ and $\mathrm{I} = \mathbf{0}$ part
- $j_k^Z(x)\big|_{\text{vector}} = j_k^3(x)\big|_{\text{vector}} - \sin^2\theta_W\, j_k^\gamma(x)$

$$j_k^3(x)\big|_{\text{vector}} = \underbrace{\frac{1}{4}\left(\bar{u}\gamma_\mu u - \bar{d}\gamma_\mu d\right)}_{\mathrm{I}=1} \underbrace{-\frac{1}{4}\bar{s}\gamma_\mu s + \frac{1}{4}\bar{c}\gamma_\mu c}_{\mathrm{I}=0}$$

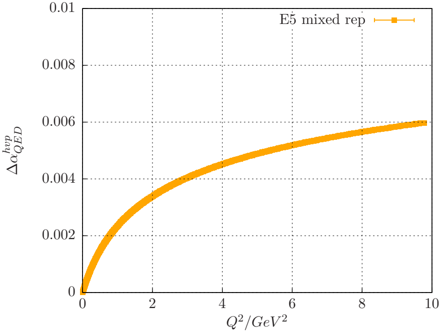
$$j_k^\gamma(x) = \underbrace{\frac{1}{2}\left(\bar{u}\gamma_\mu u - \bar{d}\gamma_\mu d\right)}_{\mathrm{I}=1,\;\; j_k^\rho} + \underbrace{\frac{1}{6}\bar{u}\gamma_\mu u + \frac{1}{6}\bar{d}\gamma_\mu d - \frac{1}{3}\bar{s}\gamma_\mu s + \frac{2}{3}\bar{c}\gamma_\mu c}_{\mathrm{I}=0}$$

- ρ-correlator

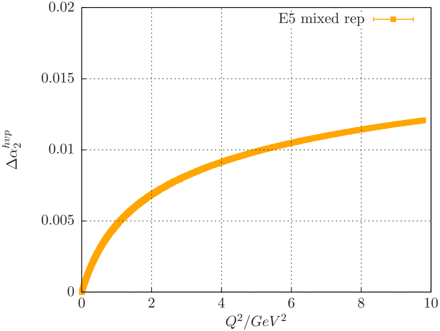
$$G^{\rho\rho}(x_0) = -\int d^3x\, \langle j_k^\rho(x)\;\; j_k^\rho(0)\rangle$$

connected contribution to $\Delta\alpha_{QED}$ and $\Delta\alpha_2$

- $\Delta_{had}\alpha_{QED} = -e^2\,\Pi_R^{\gamma\gamma}(Q^2)$



- $\Delta_{had}\alpha_2 = -g^2\,\Pi_R^{\gamma 3}(Q^2)$



- $\Delta_{had}\sin^2\theta_W(Q^2) = \Delta_{had}\alpha_{QED} - \Delta_{had}\alpha_2$

disconnected contribution to $\Delta\alpha_{QED}$ and $\Delta\alpha_2$

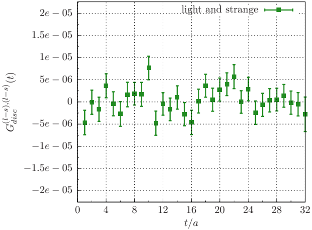
- separate correlators for α_Q**ED** and α_2

$$G^{\gamma Z}_{disc}(x_0) = G^{\gamma 3}_{disc}(x_0) - \sin^2\theta_W \, G^{\gamma\gamma}_{disc}(x_0)$$

- disconnected correlator for $\Delta\alpha_{QED}$

$$G^{\gamma\gamma}_{disc}(x_0) = \frac{1}{9}\, G^{(\ell-s),(\ell-s)}_{disc}(x_0)$$

$\rightarrow$ cancellation on both sides



- disconnected correlator for $\Delta\alpha_2$

$$G^{\gamma 3}_{disc}(x_0) = \frac{1}{12}\, G^{s,(\ell-s)}_{disc}(x_0)$$

$\rightarrow$ no light-light disconnected contribution