Dual representation for massless 1+1-dimensional fermions with chemical potential and U(1) gauge fields

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Solving the complex action problem with dual variables

- In general, field theories with finite chemical potential μ or a topological term have actions S with an imaginary part, and the Boltzmann factor is not a probability:

$$e^{-S[\Phi]} \in \mathbb{C}$$

- There is always a freedom to choose the variables for formulating a quantum field theory.
- Maybe it is possible to identify alternative variables such that the partition sum has only real and positive terms?
- For several systems such an alternative representation without sign problem was found.
- These "dual degrees of freedom" are discretized loops for matter fields and discretized surfaces for gauge fields:

$$Z = \sum_{\{l,s\}} W[l,s] \quad , \quad W[l,s] \in \mathbb{R}_+$$

- The Monte Carlo calculation can be done in terms of the new variables.
- Dual variables sometimes give new insight into physical mechanisms.

Examples of complex action problems solved with dual variables

- $\mathbb{Z}_3$ spin model with chemical potential. (PRL 2011, CPC 2012)
- Polyakov loop model with chemical potential (SU(3) spin model). (NPB 2011 & 2012)
- $\mathbb{Z}_3$ gauge-Higgs model with chemical potential. (PRD 2012, CPC 2013)
- Charged ϕ^4 field at finite density (relativistic Bose gas). (NPB 2013, PLB 2013)
- $U(1)$ gauge-Higgs model with chemical potential. (PRL 2013, CPC 2013)
- Scalar QED_2 with topological term. (PoS 2014, work in preparation)
- O(N) and CP(N-1) models with chemical potential (arXiv:1507:04253 = yesterday)
- Work by S. Chandrasekharan, P. de Forcrand, M. Endres, P. Korzec, Y.D. Mercado, A. Schmidt, U.-J. Wiese, U. Wolff ....
- 1+1-dimensional fermions with topological term and chemical potential. (NPB 2015)
  $\Longrightarrow$ This talk

Problem: There is no generally applicable strategy for dualization.
Each class of models has to be attacked individually.

1+1-dimensional massless fermions on the lattice

- Partition sum and lattice action: ($\mu \neq 0 \Rightarrow$ 2 flavors)

$$Z = \int \mathcal{D}[U]\mathcal{D}[\overline{\psi},\psi]\, e^{-S_F[U,\overline{\psi},\psi] - S_G[U] - i\,\theta\, Q[U]}$$

$$S_F = \frac{1}{2}\sum_{n,\nu} \gamma_\nu(n)\Big[e^{+\mu\,\delta_{\nu,2}}\, U_\nu(n)\,\overline{\psi}(n)\,\psi(n+\hat{\nu}) - e^{-\mu\,\delta_{\nu,2}}\, U_\nu(n)^*\,\overline{\psi}(n+\hat{\nu})\,\psi(n)\Big]$$

$$S_G[U] + i\,\theta\, Q[U] = -\frac{\beta}{2}\sum_n [\, U_p(n) + U_p(n)^*\,] + \frac{\theta}{4\pi}\sum_n [\, U_p(n) - U_p(n)^*\,]$$

- For gauge groups containing $-\mathbb{1}$ the staggered signs $\gamma_\nu(n)$ can be absorbed in the links:

$$U_\nu(n) \to \gamma_\nu(n)\, U_\nu(n) \quad , \qquad U_p(n) \to -U_p(n) \quad , \qquad \mathcal{D}[U] \to \mathcal{D}[U]$$

Staggered $\gamma_\nu(n)$ removed from S_F. Sign of β and θ is flipped. Works also in 4D.

Expanding the fermionic partition sum ($\mu = 0$ here)

- Expanding the Boltzmann factors turns Z_F into a sum of configurations of occupation numbers $k_\nu(n), \overline{k}_\nu(n) \in \{0,1\}$:

$$Z_F[U] = \int \mathcal{D}[\overline{\psi},\psi] \prod_{n,\nu} e^{-\frac{1}{2} U_\nu(n)\, \overline{\psi}(n)\, \psi(n+\hat{\nu})} e^{+\frac{1}{2} U_\nu(n)^* \, \overline{\psi}(n+\hat{\nu})\, \psi(n)}$$

$$= \int \mathcal{D}[\overline{\psi},\psi] \prod_{n,\nu} \sum_{k_\nu(n),\overline{k}_\nu(n)=0}^{1} \left[-\frac{U_\nu(n)}{2}\, \overline{\psi}(n)\, \psi(n+\hat{\nu}) \right]^{k_\nu(n)} \left[\frac{U_\nu(n)^*}{2}\, \overline{\psi}(n+\hat{\nu})\, \psi(n) \right]^{\overline{k}_\nu(n)}$$

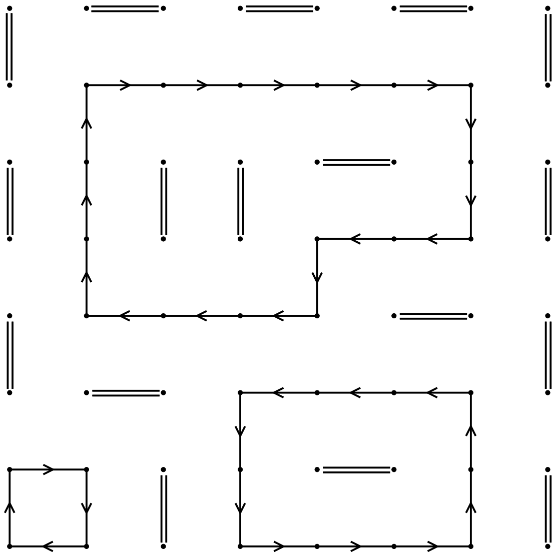
$$= \sum_{\{k,\overline{k}\}} \prod_{n,\nu} \frac{(-1)^{k_\nu(n)}}{2^{\,k_\nu(n)+\overline{k}_\nu(n)}}\, U_\nu(n)^{k_\nu(n)-\overline{k}_\nu(n)}\; S[k,\overline{k}]$$

- Remaining Grassmann integral gives zero or a sign:

$$S[k,\overline{k}] = \int \mathcal{D}[\overline{\psi},\psi] \prod_{n,\nu} \left(\overline{\psi}(n)\, \psi(n+\hat{\nu}) \right)^{k_\nu(n)} \left(\overline{\psi}(n+\hat{\nu})\, \psi(n) \right)^{\overline{k}_\nu(n)}$$

Admissible fermion configurations

Nontrivial $S[k,\overline{k}]$ only for configurations of the $k_\nu(n), \overline{k}_\nu(n)$ where each $\psi(n)$ and $\overline{\psi}(n)$ is activated exactly once. $\Rightarrow$ Complete filling of the lattice with closed loops and dimers.



Fermionic partition sum

- The fermionic partition sum is thus a sum over configurations $\{l, d\}$ of loops and dimers:

$$Z_F[U] \;=\; \left(\frac{1}{2}\right)^V \sum_{\{l,d\}} (-1)^{N_L} \; (-1)^{\frac{1}{2}\sum_l L(l)} \; (-1)^{\sum_l W(l)} \; \prod_l \prod_{(n,\nu)\in l} U_\nu(n)^{s_\nu(n)}$$

N_L ..... number of loops

$L(l)$ ..... length of the loop l

$W(l)$ ..... winding number of the loop l around compact time

$s_\nu(n)$ ..... occupation number of a link in a loop ($s_\nu(n) \in \{-1, +1\}$)

- The loops are dressed with link variables which have to be integrated over. Since ...

$$\int dU_\nu(n) \; [U_\nu(n)]^j \;=\; \delta_{j,0}$$

... the link variables on the loops have to be compensated with plaquettes from the expansion of the Boltzmann factor with the gauge action.

Expansion of the Boltzmann factor of the gauge action:

- Factorization of the gauge action Boltzmann factor as product over plaquettes:

$$e^{-S_G[U]\,-\,i\theta Q[U]} \;=\; e^{-\sum_n \left[\eta\, U_p(n)\,+\,\overline{\eta}\, U_p(n)^*\right]} \;=\; \prod_n \, e^{-\,\eta\, U_p(n)}\, e^{-\,\overline{\eta}\, U_p(n)^{-1}}$$

$$\eta \;\equiv\; \frac{\beta}{2} - \frac{\theta}{4\pi} \;\;,\;\; \overline{\eta} \;\equiv\; \frac{\beta}{2} + \frac{\theta}{4\pi}$$

- Expansion of the individual exponentials:

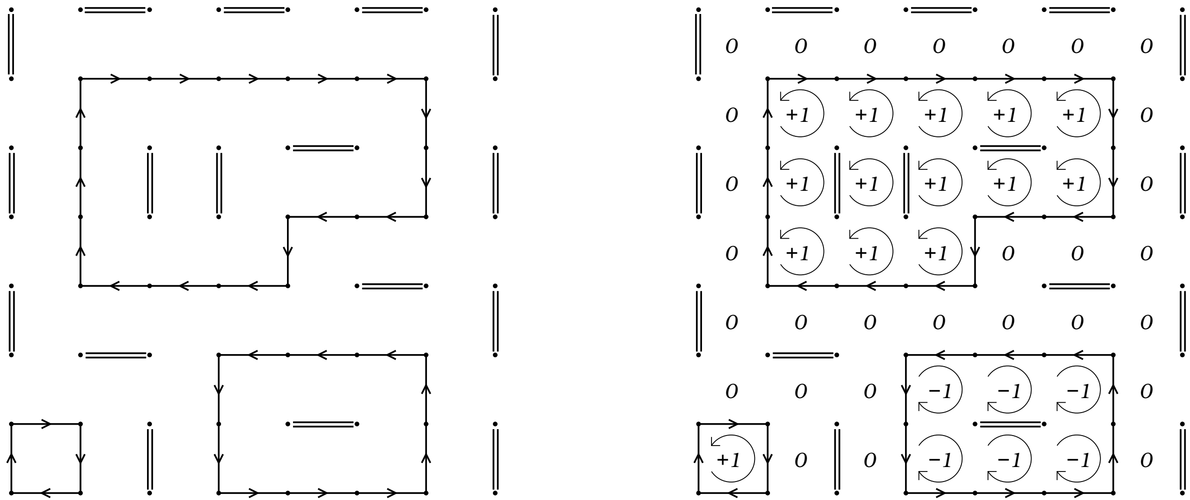
$$e^{-\,\eta\, U_p(n)}\, e^{-\,\overline{\eta}\, U_p(n)^{-1}} \;=\; \sum_{p(n)\in\mathbb{Z}} (-1)^{p(n)}\, I_{|p(n)|}\Big(\sqrt{\eta\overline{\eta}}\Big) \left(\sqrt{\frac{\eta}{\overline{\eta}}}\right)^{p(n)} U_p(n)^{\,p(n)}$$

$p(n) \in \mathbb{Z}$ .... plaquette occupation numbers

$I_{|p(n)|}(\sqrt{\eta\overline{\eta}}\,)$ .... modified Bessel functions

Saturation of the link variables:

Example how link variables along the loops are saturated with flux from the plaquettes:



Dual form of the partition sum:

- The partition function is a sum over all admissible configurations of loops l, dimers d and plaquette occupation numbers p:

$$Z = \left(\frac{1}{2}\right)^V \sum_{\{l,d,p\}} (-1)^{N_L + N_P + \frac{1}{2}\sum_l L(l)} \prod_n I_{|p(n)|}\left(\sqrt{\eta\overline{\eta}}\right) \left(\sqrt{\frac{\eta}{\overline{\eta}}}\right)^{p(n)}$$

  Here we have introduced the total plaquette occupation number: $N_P = \sum_n p(n)$

- The dual representation is an exact mapping. Left to show:

$$(-1)^{N_L + N_P + \frac{1}{2}\sum_l L(l)} = 1 \qquad \forall \quad \text{admissible configurations}$$

- Proof in two steps: (NPB 897 (2015) 732 [arXiv:1503.05479])

  1. Factorization theorem: $\rightarrow$ Sign is product of signs for configs with a single loop.
  2. Admissible configs with a single loop are analyzed with a recursive construction.

Dual representation of massless 1+1-dimensional fermions with θ-term:

- Partition function is a sum over admissible configurations of loops, dimers and plaquette occupation numbers:

$$Z \;=\; \left(\frac{1}{2}\right)^V \sum_{\{l,d,p\}} \prod_n I_{|p(n)|}\left(\sqrt{\eta\overline{\eta}}\right) \left(\sqrt{\frac{\eta}{\overline{\eta}}}\right)^{p(n)}$$

- Fermion loops that wind forward (backward) in time are weighted with $e^{+\mu N_t}$ ($e^{-\mu N_t}$)

- The partition sum has only real and positive terms for positive $\eta = \frac{\beta}{2} - \frac{\theta}{4\pi}$ and $\overline{\eta} = \frac{\beta}{2} + \frac{\theta}{4\pi}$.

- Monte Carlo calculations can be done directly in terms of the dual variables.

- Bulk observables are obtained as derivatives with respect to the parameters. For correlators one can introduce local sources.

Mass $\neq 0 \;\Rightarrow$ signs are back!!

Generalization: Nanowires in a 3+1-dimensional electromagnetic field:

- The dual mapping can be generalized to a system of straight nanowires of fermions with a relativistic dispersion relation. (work in preparation)

- Each nanowire gives rise to effectively 1+1-dimensional fermions in the t-x_j plane, with the other two spatial coordinates $x_k, k \neq j$ fixed.

- The steps of the dualization for each t-x_j plane are identical to the 1+1 case:
  - Absorb the staggered signs and flip the gauge coupling β.
  - Integrate out the Grassmann fields in the corresponding plane.
    $\Rightarrow$ Same algebra and thus the same signs from fermions.

- Integrating out the 4-dimensional gauge fields can be done in 2 steps:
  - In a first step, in each plane one uses only the plaquettes that fall into this plane. This gives rise to a positive overall sign (same proof as in 2d).
  - All other contributions from gauge fields do not change the sign since all deformations of surfaces contribute an even number of plaquettes.
  - Overall positive sign for the dual representation.

Nanowires in a 3+1-dimensional electromagnetic field:

Partition function has an additional product over all wires and uses 4-d plaquette surfaces:

$$Z \;=\; \prod_{j=1}^{N_w} \sum_{\{l_j,\, d_j\}} e^{\,N_t \sum_j \mu_j W_j} \sum_{\{p\}} \prod_{n} \prod_{\sigma<\tau} I_{|p_{\sigma,\tau}(n)|}(\beta/2)$$

W_j ..... total winding winding number of the loops l_j around compact time

Real and positive dual representation at arbitrary chemical potentials.

Summary

- Rewriting lattice field theories to dual variables can solve the complex action problem.
- Two sources of complex action: chemical potential and topological term.
- For massless 1+1-dimensional fermions the partition sum can be mapped to a dual representation with only real and positive terms.
- The dual variables are fermion loops and dimers and the corresponding plaquette occupation numbers.
- Positivity of the weights comes from interplay of fermion properties and the gauge interaction.
- First example of a complete (large β, $\mu \neq 0$, $\theta \neq 0$) positive dualization of fermions interacting with a gauge field. Mass $\neq 0$ $\Rightarrow$ signs are back!!
- The proof of positivity generalizes to nanowires interacting with 4-d U(1) gauge fields.
- Dualized models provide excellent reference data for testing your own favorite approach to complex action problems.