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Lattice simulation of the SU(2)-chiral model at zero and non-zero pion density

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Motivation

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  - Toy model for testing methods to obtain mass spectrum from lattice simulations after rotation of vacuum as e.g. in two flavor LQCD with an isospin chemical potential for $\mu > m_{\pi}/2$.
  - Higgs field in technicolor like models.
  - Trivially extendable to $\mathrm{O}(N)$ (or CP^{N-1} in terms of $\mathrm{U}(1)$ gauged $\mathrm{O}(2N-1)$).

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  - Trivially extendable to $\mathrm{O}(N)$ (or CP^{N-1} in terms of $\mathrm{U}(1)$ gauged $\mathrm{O}(2N-1)$).
- Outline
  - Derivation of flux representation partition function for $\mathrm{SU}(2)$-chiral Lagrangian including an isospin chemical potential and source terms.
  - Improved method to measure arbitrary correlators during worm update.
  - Comparison of mass spectrum obtained from lattice simulations and spectrum obtained from saddle point calculation.

Continuum Action

Minkowski space SU(2)-chiral Lagrangian:

$$\mathcal{L}_{eff} = -\frac{f_\pi^2}{4}\,\mathrm{tr}\Big[\big(\partial_\nu\Sigma - \mathrm{i}\mu\delta_{\nu,0}\,(\sigma_3\Sigma - \Sigma\sigma_3)\big)\big(\partial^\nu\Sigma^\dagger - \mathrm{i}\mu\delta^{\nu,0}\,(\sigma_3\Sigma^\dagger - \Sigma^\dagger\sigma_3)\big)\Big] - \frac{f_\pi^2}{4}\,\mathrm{tr}\Big[\Sigma^\dagger S + S^\dagger\Sigma\Big],$$

with:

- $\Sigma = \pm\sqrt{\left|1 - \frac{|\bar{\pi}|^2}{f_\pi^2}\right|}\,\mathbb{1} + \mathrm{i}\,\frac{\bar{\pi}\cdot\bar{\sigma}}{f_\pi} \in \mathrm{SU}(2)$
  - pion fields $\bar{\pi} = (\pi_1, \pi_2, \pi_3)$
  - Pauli matrices $\bar{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$,
- $S = s_4\,\mathbb{1} + \mathrm{i}\,\bar{s}\,\bar{\sigma}$
  - pion source $\bar{s} = (s_1, s_2, s_3)$
  - scalar source s_4 ($\propto (m_u + m_d)$ with quark masses m_u, m_d)
- pion decay constant f_π,
- isospin chemical potential μ.

Discretization

Wick rotate ($x^0 \rightarrow -\mathrm{i}\, x_4$) corresponding action and introduce lattice spacing a :

$$S = \frac{f_\pi^2 a^d}{4} \sum_x \Big\{ -\frac{1}{a^2} \sum_{\nu=1}^{d} \mathrm{tr}\Big[\Sigma_x^\dagger \, e^{a\mu\sigma_3\,\delta_{\nu,d}} \, \Sigma_{x+\widehat{\nu}} \, e^{-a\mu\sigma_3\,\delta_{\nu,d}} + \Sigma_x^\dagger \, e^{-a\mu\sigma_3\,\delta_{\nu,d}} \, \Sigma_{x-\widehat{\nu}} \, e^{a\mu\sigma_3\,\delta_{\nu,d}} \Big] + \frac{2\,d}{a^2} \mathrm{tr}\Big[\Sigma_x^\dagger \Sigma_x \Big] - \mathrm{tr}\Big[\Sigma_x^\dagger \, S_x + S_x^\dagger \, \Sigma_x \Big] \Big\}$$

with the usual substitutions:

- $\int \mathrm{d}^d\, x_E \quad \rightarrow \quad a^d \sum_x$
- $\partial_\nu^E \Sigma - \mu\,\delta_{\nu,d} \big(\sigma_3 \Sigma - \Sigma \sigma_3\big) \quad \rightarrow \quad \frac{1}{a}\big(e^{a\mu\sigma_3\,\delta_{\nu,d}} \, \Sigma_{x+\widehat{\nu}} \, e^{-a\mu\sigma_3\,\delta_{\nu,d}} - \Sigma_x \big)$

(in analogy to covariant derivative in adjoint rep.)

Discretization

Rewrite in terms of dimensionless parameters / fields:

$$S[\Sigma] = -\kappa \sum_{x} \bigg\{ \frac{1}{4} \sum_{\nu=1}^{d} \mathrm{tr}\Big[\Sigma_x^\dagger \, e^{\mu \sigma_3 \delta_{\nu,d}} \, \Sigma_{x+\hat{\nu}} \, e^{-\mu \sigma_3 \delta_{\nu,d}} + \Sigma_x^\dagger \, e^{-\mu \sigma_3 \delta_{\nu,d}} \, \Sigma_{x-\hat{\nu}} \, e^{\mu \sigma_3 \delta_{\nu,d}} \Big] + \mathrm{tr}\Big[\Sigma_x^\dagger \, S_x + S_x^\dagger \, \Sigma_x \Big] \bigg\} + \mathrm{const.},$$

with:

- $f_\pi^2 \, a^{d-2} \rightarrow \kappa$
- $a^2 \, S_x \rightarrow S_x$
- $a\mu \rightarrow \mu$
- $\pi_i / f_\pi \rightarrow \pi_i$

Discretization

Lattice regularized partition function:

$$Z = \int \mathcal{D}[\Sigma]\, \mathrm{e}^{-S[\Sigma]}$$

with:

- $\mathcal{D}[\Sigma] = \prod_x \mu_g(\Sigma_x) = \prod_x \frac{d\pi_x^1 \wedge d\pi_x^2 \wedge d\pi_x^3}{\sqrt{\left|1-|\bar{\pi}_x|^2\right|}}$,
- $\mu_g(\Sigma)$ is SU(2) Haar measure.

Discretization

Lattice regularized partition function:

$$Z = \int \left\{ \prod_x \frac{\mathrm{d}\pi_x^1 \wedge \mathrm{d}\pi_x^2 \wedge \mathrm{d}\pi_x^3}{\sqrt{\left|1 - |\bar{\pi}_x|^2\right|}} \right\} \exp\Bigg(\kappa \sum_x \Bigg\{ \sum_\nu \Bigg(\pm \sqrt{\left|1 - |\bar{\pi}_x|^2\right| \left|1 - |\bar{\pi}_{x+\hat{\nu}}|^2\right|}$$

$$+ \frac{e^{2\mu\delta_{\nu,d}}}{2} \left(\pi_x^1 + i\pi_x^2\right)\left(\pi_{x+\hat{\nu}}^1 - i\pi_{x+\hat{\nu}}^2\right) + \frac{e^{-2\mu\delta_{\nu,d}}}{2} \left(\pi_x^1 - i\pi_x^2\right)\left(\pi_{x+\hat{\nu}}^1 + i\pi_{x+\hat{\nu}}^2\right)$$

$$+ \pi_x^3 \pi_{x+\hat{\nu}}^3 \Bigg) + \bar{s} \cdot \bar{\pi}_x + s_4 \sqrt{\left|1 - |\bar{\pi}_x|^2\right|} \Bigg\} \Bigg)$$

Flux Representation

- write $\bar{s}\cdot\bar{\pi}_x$ in terms of iso-charge eigenbasis:

$$Z = \int \Big\{ \prod_x \frac{d\pi_x^1 \wedge d\pi_x^2 \wedge d\pi_x^3}{\sqrt{\big|1-|\bar{\pi}_x|^2\big|}} \Big\} \exp\Big(\kappa \sum_x \Big\{ \sum_\nu \Big(\pm\sqrt{\big|1-|\bar{\pi}_x|^2\big|\big|1-|\bar{\pi}_{x+\hat{\nu}}|^2\big|} + \frac{e^{2\mu\delta_{\nu,d}}}{2}\big(\pi_x^1 + i\pi_x^2\big)\big(\pi_{x+\hat{\nu}}^1 - i\pi_{x+\hat{\nu}}^2\big) + \frac{e^{-2\mu\delta_{\nu,d}}}{2}\big(\pi_x^1 - i\pi_x^2\big)\big(\pi_{x+\hat{\nu}}^1 + i\pi_{x+\hat{\nu}}^2\big) + \pi_x^3\pi_{x+\hat{\nu}}^3 \Big) + \Big(\frac{s_1 - is_2}{2}\Big)\big(\pi_x^1 + i\pi_x^2\big) + \Big(\frac{s_1 + is_2}{2}\Big)\big(\pi_x^1 - i\pi_x^2\big) + s_3\,\pi_x^3 + s_4\sqrt{\big|1-|\bar{\pi}_x|^2\big|} \Big\}\Big)$$

Flux Representation

- split exponential into product of exponentials.

$$Z = \int \Bigg\{ \prod_x \frac{d\pi_x^1 \wedge d\pi_x^2 \wedge d\pi_x^3}{\sqrt{\left|1 - |\bar{\pi}_x|^2\right|}} \Bigg\{ \prod_\nu \exp\left(\pm\kappa \sqrt{\left|1 - |\bar{\pi}_x|^2\right| \left|1 - |\bar{\pi}_{x+\widehat{\nu}}|^2\right|} \right)$$

$$\exp\left(\frac{\kappa\, e^{2\mu\delta_{\nu,d}}}{2} \left(\pi_x^1 + i\pi_x^2\right)\left(\pi_{x+\widehat{\nu}}^1 - i\pi_{x+\widehat{\nu}}^2\right) \right) \exp\left(\frac{\kappa\, e^{-2\mu\delta_{\nu,d}}}{2} \left(\pi_x^1 - i\pi_x^2\right)\left(\pi_{x+\widehat{\nu}}^1 + i\pi_{x+\widehat{\nu}}^2\right) \right)$$

$$\exp\left(\kappa\pi_x^3\pi_{x+\widehat{\nu}}^3 \right) \Bigg\} \exp\left(\kappa \left(\frac{s_1 - is_2}{2}\right)\left(\pi_x^1 + i\pi_x^2\right) \right) \exp\left(\kappa \left(\frac{s_1 + is_2}{2}\right)\left(\pi_x^1 - i\pi_x^2\right) \right)$$

$$\exp\left(\kappa\, s_3\pi_x^3\right) \exp\left(\kappa\, s_4 \sqrt{\left|1 - |\bar{\pi}_x|^2\right|}\right) \Bigg\}$$

Flux Representation

- write each exponential as a power series

$$Z=\int\bigg\{\prod_x\frac{\mathrm{d}\pi_x^1\wedge\mathrm{d}\pi_x^2\wedge\mathrm{d}\pi_x^3}{\sqrt{\left|1-|\bar{\pi}_x|^2\right|}}\bigg\{\prod_\nu\bigg(\sum_{\xi_{x,\nu}}\frac{1}{\xi_{x,\nu}!}\Big(\pm\kappa\sqrt{\left|1-|\bar{\pi}_x|^2\right|\left|1-|\bar{\pi}_{x+\hat{\nu}}|^2\right|}\Big)^{\xi_{x,\nu}}\bigg)$$

$$\bigg(\sum_{\eta_{x,\nu}}\frac{1}{\eta_{x,\nu}!}\Big(\frac{\kappa\,\mathrm{e}^{2\mu\delta_{\nu,d}}}{2}\big(\pi_x^1+\mathrm{i}\pi_x^2\big)\big(\pi_{x+\hat{\nu}}^1-\mathrm{i}\pi_{x+\hat{\nu}}^2\big)\Big)^{\eta_{x,\nu}}\bigg)$$

$$\bigg(\sum_{\bar{\eta}_{x,\nu}}\frac{1}{\bar{\eta}_{x,\nu}!}\Big(\frac{\kappa\,\mathrm{e}^{-2\mu\delta_{\nu,d}}}{2}\big(\pi_x^1-\mathrm{i}\pi_x^2\big)\big(\pi_{x+\hat{\nu}}^1+\mathrm{i}\pi_{x+\hat{\nu}}^2\big)\Big)^{\bar{\eta}_{x,\nu}}\bigg)\bigg(\sum_{\chi_{x,\nu}}\frac{1}{\chi_{x,\nu}!}\Big(\kappa\pi_x^3\pi_{x+\hat{\nu}}^3\Big)^{\chi_{x,\nu}}\bigg)\bigg\}$$

$$\bigg(\sum_{m_x}\frac{1}{m_x!}\Big(\frac{\kappa}{2}\big(s_1-\mathrm{i}s_2\big)\big(\pi_x^1+\mathrm{i}\pi_x^2\big)\Big)^{m_x}\bigg)\bigg(\sum_{\bar{m}_x}\frac{1}{\bar{m}_x!}\Big(\frac{\kappa}{2}\big(s_1+\mathrm{i}s_2\big)\big(\pi_x^1-\mathrm{i}\pi_x^2\big)\Big)^{\bar{m}_x}\bigg)$$

$$\bigg(\sum_{n_x}\frac{1}{n_x!}\big(\kappa\,s_3\pi_x^3\big)^{n_x}\bigg)\bigg(\sum_{o_x}\frac{1}{o_x!}\Big(\kappa\,s_4\sqrt{\left|1-|\bar{\pi}_x|^2\right|}\Big)^{o_x}\bigg)\bigg\}$$

Flux Representation

- change to spherical coordinates on SU(2) :

$$\pi_x^1 = \sin(\alpha_x)\sin(\theta_x)\cos(\phi_x) \quad , \qquad \pi_x^3 = \sin(\alpha_x)\cos(\theta_x)$$
$$\pi_x^2 = \sin(\alpha_x)\sin(\theta_x)\sin(\phi_x) \quad , \qquad \mu_g(\Sigma_x) = \sin^2(\alpha_x)\sin(\theta_x)\mathrm{d}\alpha_x \wedge \mathrm{d}\theta_x \wedge \mathrm{d}\phi_x ,$$

$$Z = \sum_{\{\eta,\bar{\eta},\xi,\chi\, m\bar{m} n,o\}} \left\{ \prod_{x,\nu} \frac{\kappa^{\eta_{x,\nu}+\bar{\eta}_{x,\nu}+\xi_{x,\nu}+\chi_{x,\nu}}}{\eta_{x,\nu}!\,\bar{\eta}_{x,\nu}!\,\xi_{x,\nu}!\,\chi_{x,\nu}!} \right\}$$
$$\left\{ \prod_x \frac{(\kappa(s_1 - i s_2))^{m_x} (\kappa(s_1 + i s_2))^{\bar{m}_x} (\kappa s_3)^{n_x} (\kappa s_4)^{o_x}}{m_x!\,\bar{m}_x!\,n_x!\,o_x!} \right\}$$
$$\left\{ \prod_x \int_0^\pi \mathrm{d}\alpha_x \int_0^\pi \mathrm{d}\theta_x \sin^2(\alpha_x)\sin(\theta_x) \left(\frac{1}{2}\sin(\alpha_x)\sin(\theta_x)\right)^{m_x+\bar{m}_x+\sum_\nu(\eta_{x,\nu}+\bar{\eta}_{x,\nu}+\eta_{x-\hat{\nu},\nu}+\bar{\eta}_{x-\hat{\nu},\nu})}\right.$$
$$\left(\sin(\alpha_x)\cos(\theta_x)\right)^{n_x+\sum_\nu(\chi_{x,\nu}+\chi_{x-\hat{\nu},\nu})}$$
$$\left. \left(\cos(\alpha_x)\right)^{o_x+\sum_\nu(\xi_{x,\nu}+\xi_{x-\hat{\nu},\nu})} e^{2\mu(\eta_{x,4}-\bar{\eta}_{x,4})} \int_0^{2\pi} \mathrm{d}\phi_x\, e^{i\phi_x\left(m_x-\bar{m}_x+\sum_\nu\left(\eta_{x,\nu}-\bar{\eta}_{x,\nu}-\left(\eta_{x-\hat{\nu},\nu}-\bar{\eta}_{x-\hat{\nu},\nu}\right)\right)\right)} \right\}$$

Flux Representation

- define:

$$A_x = \sum_\nu \left(\eta_{x,\nu} + \bar{\eta}_{x,\nu} + \eta_{x-\hat{\nu},\nu} + \bar{\eta}_{x-\hat{\nu},\nu}\right), \quad C_x = \sum_\nu \left(\chi_{x,\nu} + \chi_{x-\hat{\nu},\nu}\right), \quad B_x = \sum_\nu \left(\xi_{x,\nu} + \xi_{x-\hat{\nu},\nu}\right)$$

and:

$$\eta_{x,\nu} - \bar{\eta}_{x,\nu} = k_{x,\nu} \in \mathbb{Z} \quad , \qquad \eta_{x,\nu} + \bar{\eta}_{x,\nu} = |k_{x,\nu}| + 2\,l_{x,\nu} \in \mathbb{N}_0$$

$$m_x - \bar{m}_x = p_x \in \mathbb{Z} \quad , \qquad m_x + \bar{m}_x = |p_x| + 2\,q_x \in \mathbb{N}_0$$

- carry out integration:

$$Z = \sum_{\{k,l,\xi,\chi,p,q,n,o\}} \left\{ \prod_{x,\nu} \frac{\kappa^{|k_{x,\nu}|+2l_{x,\nu}+\xi_{x,\nu}+\chi_{x,\nu}}}{(|k_{x,\nu}|+l_{x,\nu})!\,l_{x,\nu}!\,\xi_{x,\nu}!\,\chi_{x,\nu}!} \right\}$$

$$\left\{ \prod_x \frac{(\kappa s)^{|p_x|+2\,q_x}\, e^{-i\phi_s p_x}\, (\kappa s_3)^{n_x}\, (\kappa s_4)^{o_x}\, e^{2\mu k_{x,4}}}{(|p_x|+q_x)!\,q_x!\,n_x!\,o_x!} \right.$$

$$\left. \delta\Big(p_x + \sum_\nu \left(k_{x,\nu} - k_{x-\hat{\nu},\nu}\right)\Big)\, W\left(A_x + |p_x| + 2q_x, B_x + o_x, C_x + n_x\right) \right\},$$

with

$$W(A,B,C) = \frac{1+(-1)^C}{2}\,\frac{1+(-1)^B}{2}\,\frac{\Gamma\left(\frac{1+C}{2}\right)\Gamma\left(\frac{1+B}{2}\right)\Gamma\left(\frac{2+A}{2}\right)}{2^{(2+A)/2}\,\Gamma\left(\frac{4+A+B+C}{2}\right)},$$

and

$$s = |s_1 \pm i s_2| \quad , \quad \phi_s = \arg(s_1 + i s_2).$$

Flux Representation

- define:

$$A_x = \sum_\nu \left(\eta_{x,\nu} + \bar{\eta}_{x,\nu} + \eta_{x-\hat{\nu},\nu} + \bar{\eta}_{x-\hat{\nu},\nu}\right), \quad C_x = \sum_\nu \left(\chi_{x,\nu} + \chi_{x-\hat{\nu},\nu}\right), \quad B_x = \sum_\nu \left(\xi_{x,\nu} + \xi_{x-\hat{\nu},\nu}\right)$$

and:

$$\eta_{x,\nu} - \bar{\eta}_{x,\nu} = k_{x,\nu} \in \mathbb{Z} \quad , \qquad \eta_{x,\nu} + \bar{\eta}_{x,\nu} = |k_{x,\nu}| + 2\, l_{x,\nu} \in \mathbb{N}_0$$

$$m_x - \bar{m}_x = p_x \in \mathbb{Z} \quad , \qquad m_x + \bar{m}_x = |p_x| + 2\, q_x \in \mathbb{N}_0$$

- carry out integration:

$$Z = \sum_{\{k,l,\xi,\chi,p,q,n,o\}} \left\{ \prod_{x,\nu} \frac{\kappa^{|k_{x,\nu}|+2l_{x,\nu}+\xi_{x,\nu}+\chi_{x,\nu}}}{(|k_{x,\nu}|+l_{x,\nu})!\, l_{x,\nu}!\, \xi_{x,\nu}!\, \chi_{x,\nu}!} \right\}$$

$$\left\{ \prod_x \frac{(\kappa s)^{|p_x|+2\,q_x}\, e^{-i\phi_s p_x}\, (\kappa s_3)^{n_x}\, (\kappa s_4)^{o_x}\, e^{2\mu k_{x,4}}}{(|p_x|+q_x)!\, q_x!\, n_x!\, o_x!} \right.$$

$$\left. \delta\Big(p_x + \sum_\nu \big(k_{x,\nu} - k_{x-\hat{\nu},\nu}\big)\Big)\, W\big(A_x + |p_x| + 2q_x, B_x + o_x, C_x + n_x\big) \right\},$$

- note: due to delta function constraint: $\sum_x p_x = 0 \Rightarrow \prod_x e^{i\phi_s p_x} = 1$ (if $\phi_s =$const.).

Flux Representation

- absorb summation over q, n, o into new weight :

$$Z = \sum_{\{k,l,\xi,\chi,p\}} \left\{ \prod_{x,\nu} \frac{\kappa^{|k_{x,\nu}|+2l_{x,\nu}+\xi_{x,\nu}+\chi_{x,\nu}}}{(|k_{x,\nu}|+l_{x,\nu})!\, l_{x,\nu}!\, \xi_{x,\nu}!\, \chi_{x,\nu}!} \right\}$$
$$\left\{ \prod_x \delta\big(p_x + \sum_\nu (k_{x,\nu} - k_{x-\hat{\nu},\nu})\big)\, e^{2\mu k_{x,4}}\, w(A_x, B_x, C_x, p_x; s, \phi_s, s_3, s_4, \kappa) \right\},$$

with

$$w(A,B,C,p,s,\phi_s,s_3,s_4,\kappa) = \sum_{q,n,o=0}^{\infty} \frac{(\kappa s)^{|p|+2q}\, e^{-i\phi_s p}\, (\kappa s_3)^n\, (\kappa s_4)^o}{(|p|+q)!\, q!\, n!\, o!}$$
$$\frac{1+(-1)^{C+n}}{2}\, \frac{1+(-1)^{B+o}}{2}\, \frac{\Gamma\left(\frac{1+C+n}{2}\right)\Gamma\left(\frac{1+B+o}{2}\right)\Gamma\left(\frac{2+A+|p|+2q}{2}\right)}{2^{(2+A+|p|+2q)/2}\,\Gamma\left(\frac{4+A+|p|+2q+B+o+C+n}{2}\right)}.$$

- $w(A,B,C,p,s,\phi_s,s_3,s_4,\kappa)$ pre-computed at beginning of a simulation.

Simulation Method

- Updates in Monte Carlo simulation are subject to two constraints :

$$Z = \sum_{\{k,l,\xi,\chi,p\}} \left\{ \prod_{x,\nu} \frac{\kappa^{|k_{x,\nu}|+2l_{x,\nu}+\xi_{x,\nu}+\chi_{x,\nu}}}{(|k_{x,\nu}|+l_{x,\nu})!\, l_{x,\nu}!\, \xi_{x,\nu}!\, \chi_{x,\nu}!} \right\}$$
$$\left\{ \prod_x \delta\Big(p_x + \sum_\nu (k_{x,\nu} - k_{x-\hat{\nu},\nu})\Big)\, e^{2\mu k_{x,4}}\, w(A_x, B_x, C_x, p_x; s, \phi_s, s_3, s_4, \kappa) \right\},$$

with

$$w(A,B,C,p,s,\phi_s,s_3,s_4,\kappa) = \sum_{q,n,o=0}^{\infty} \frac{(\kappa s)^{|p|+2q}\, e^{-i\phi_s p}\, (\kappa s_3)^n\, (\kappa s_4)^o}{(|p|+q)!\, q!\, n!\, o!}$$
$$\frac{1+(-1)^{C+n}}{2}\, \frac{1+(-1)^{B+o}}{2}\, \frac{\Gamma\big(\frac{1+C+n}{2}\big)\, \Gamma\big(\frac{1+B+o}{2}\big)\, \Gamma\big(\frac{2+A+|p|+2q}{2}\big)}{2^{(2+A+|p|+2q)/2}\, \Gamma\big(\frac{4+A+|p|+2q+B+o+C+n}{2}\big)}.$$

- *local charge conservation* for the p_x and $k_{x,\nu}$ variables $\Rightarrow$ update by *charged worm* (closed)
- evenness constraint for the combinations $B_x + o_x$ and $C_x + n_x$ $\Rightarrow$ update by *neutral worm* (closed)
- no constraints for $l_{x,\nu}$ variables $\Rightarrow$ sampled by local Metropolis.

Charged Worm

- Idea: update $k_{x,\nu}$ variables without violating constraints by proposing insertion of external source/sink pair and sample their distribution instead of Z itself.

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- Motivation:
  - charged correlator

$$\langle \pi_x^- \pi_y^+ \rangle = \frac{1}{\kappa^2} \frac{\partial^2 \log(Z)}{\partial s_x^- \partial s_y^+} = \left\langle \frac{\frac{1}{2}(|p_x| - p_x) + q_x}{\kappa\, s_x^-} \frac{\frac{1}{2}(|p_y| + p_y) + q_y}{\kappa\, s_y^+} \right\rangle + const.,$$

with $s_x^\pm \propto s_x^1 \mp \mathrm{i}\, s_x^2$ being source for a charged pion $\pi_x^\pm$ at site x.

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  with $s_x^\pm \propto s_x^1 \mp \mathrm{i}\, s_x^2$ being source for a charged pion $\pi_x^\pm$ at site x.
  - shift p_x, p_z such that $\frac{\frac{1}{2}(|p_x|-p_x)+q_x}{\kappa s_x^-}$, $\frac{\frac{1}{2}\left(|p_y|+p_y\right)+q_y}{\kappa s_y^+}$ get absorbed into weight factors at x, y:

$$\langle \pi_x^- \pi_y^+ \rangle = \frac{1}{Z} \sum_{\{k,l,\xi,\chi,p\}} \left\{ \prod_{z,\nu} \frac{\kappa^{|k_{z,\nu}|+2l_{z,\nu}+\xi_{z,\nu}+\chi_{z,\nu}}}{\left(|k_{z,\nu}|+l_{z,\nu}\right)!\, l_{z,\nu}!\, \xi_{z,\nu}!\, \chi_{z,\nu}!} \right\}$$

$$\left\{ \prod_z \delta\Big(p_z - \delta_{z,x} + \delta_{z,y} + \sum_\nu \left(k_{z,\nu} - k_{z-\hat{\nu},\nu}\right)\Big)\, e^{2\mu k_{z,4}}\, w\left(A_z + \delta_{z,x} + \delta_{z,y}, B_z, C_z, p_z; s, \phi_s, s_3, s_4, \kappa\right) \right\} + const.,$$

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$$\left\{ \prod_z \delta\Big(p_z - \delta_{z,x} + \delta_{z,y} + \sum_\nu \big(k_{z,\nu} - k_{z-\hat{\nu},\nu}\big)\Big)\, e^{2\mu k_{z,4}}\, w(A_z + \delta_{z,x} + \delta_{z,y}, B_z, C_z, p_z; s, \phi_s, s_3, s_4, \kappa) \right\} + const.,$$

  - sample y with respect to x by worm algorithm, i.e. propose updates $y \to y + \hat{\nu}$, $k_{y,\nu} \to k_{y,\nu} \pm 1$ until $y \to y + \hat{\nu} = x$ and removal of source/sink pair is accepted.

Charged Worm

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- Motivation:
  - charged correlator

$$\langle \pi_x^- \pi_y^+ \rangle = \frac{1}{\kappa^2} \frac{\partial^2 \log(Z)}{\partial s_x^- \partial s_y^+} = \left\langle \frac{\frac{1}{2}(|p_x| - p_x) + q_x}{\kappa\, s_x^-} \frac{\frac{1}{2}(|p_y| + p_y) + q_y}{\kappa\, s_y^+} \right\rangle + const.,$$

  with $s_x^{\pm} \propto s_x^1 \mp \mathrm{i}\, s_x^2$ being source for a charged pion $\pi_x^{\pm}$ at site x.
  - shift p_x, p_z such that $\frac{\frac{1}{2}(|p_x|-p_x)+q_x}{\kappa s_x^-}$, $\frac{\frac{1}{2}(|p_y|+p_y)+q_y}{\kappa s_y^+}$ get absorbed into weight factors at x, y:

$$\langle \pi_x^- \pi_y^+ \rangle = \frac{1}{Z} \sum_{\{k,l,\xi,\chi,p\}} \left\{ \prod_{z,\nu} \frac{\kappa^{|k_{z,\nu}| + 2 l_{z,\nu} + \xi_{z,\nu} + \chi_{z,\nu}}}{(|k_{z,\nu}| + l_{z,\nu})!\, l_{z,\nu}!\, \xi_{z,\nu}!\, \chi_{z,\nu}!} \right\}$$
$$\left\{ \prod_z \delta\Big(p_z - \delta_{z,x} + \delta_{z,y} + \sum_\nu \big(k_{z,\nu} - k_{z-\hat{\nu},\nu}\big)\Big)\, e^{2\mu k_{z,4}}\, w(A_z + \delta_{z,x} + \delta_{z,y}, B_z, C_z, p_z; s, \phi_s, s_3, s_4, \kappa) \right\} + const.,$$

  - sample y with respect to x by worm algorithm, i.e. propose updates $y \to y + \hat{\nu}$, $k_{y,\nu} \to k_{y,\nu} \pm 1$ until $y \to y + \hat{\nu} = x$ and removal of source/sink pair is accepted.
- $\Rightarrow$ well defined even if source terms are zero.

Neutral Worm

- Similarly: update $\xi_{x,\nu}$, $\chi_{x,\nu}$ variables by insertion of pairs of external sources and sample their distribution:
  - for $\chi_{x,\nu}$ update use:

$$\langle \pi_x^3 \pi_y^3 \rangle = \frac{1}{Z} \sum_{\{k,l,\xi,\chi,p\}} \left\{ \prod_{z,\nu} \frac{\kappa^{|k_{z,\nu}|+2l_{z,\nu}+\xi_{z,\nu}+\chi_{z,\nu}}}{(|k_{z,\nu}|+l_{z,\nu})!\, l_{z,\nu}!\, \xi_{z,\nu}!\, \chi_{z,\nu}!} \right\}$$

$$\left\{ \prod_z \delta\big(p_z + \sum_\nu (k_{z,\nu} - k_{z-\hat{\nu},\nu})\big)\, e^{2\mu k_{z,4}}\, w(A_z, B_z, C_z + \delta_{z,x} + \delta_{z,y}, p_z; s, \phi_s, s_3, s_4, \kappa) \right\} + const.,$$

  - for $\xi_{x,\nu}$ update use:

$$\langle \pi_x^4 \pi_y^4 \rangle = \frac{1}{Z} \sum_{\{k,l,\xi,\chi,p\}} \left\{ \prod_{z,\nu} \frac{\kappa^{|k_{z,\nu}|+2l_{z,\nu}+\xi_{z,\nu}+\chi_{z,\nu}}}{(|k_{z,\nu}|+l_{z,\nu})!\, l_{z,\nu}!\, \xi_{z,\nu}!\, \chi_{z,\nu}!} \right\}$$

$$\left\{ \prod_z \delta\big(p_z + \sum_\nu (k_{z,\nu} - k_{z-\hat{\nu},\nu})\big)\, e^{2\mu k_{z,4}}\, w(A_z, B_z + \delta_{z,x} + \delta_{z,y}, C_z, p_z; s, \phi_s, s_3, s_4, \kappa) \right\} + const..$$

  - sample y with respect to x by worm algorithm.

Neutral Worm

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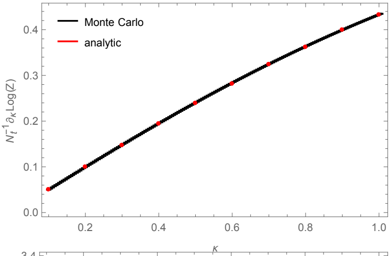
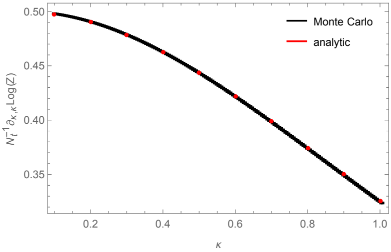
$$\left\{ \prod_z \delta\Big(p_z + \sum_\nu \big(k_{z,\nu} - k_{z-\hat{\nu},\nu}\big)\Big)\, e^{2\mu k_{z,4}}\, w(A_z, B_z + \delta_{z,x} + \delta_{z,y}, C_z, p_z; s, \phi_s, s_3, s_4, \kappa) \right\} + const..$$

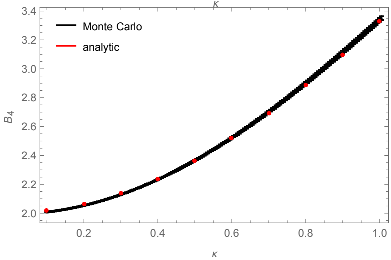
  - sample y with respect to x by worm algorithm.
- again: well defined even if corresponding source terms are zero.

Crosscheck of Code

- For $d = 1$, compare result from Monte Carlo simulation with exact result:



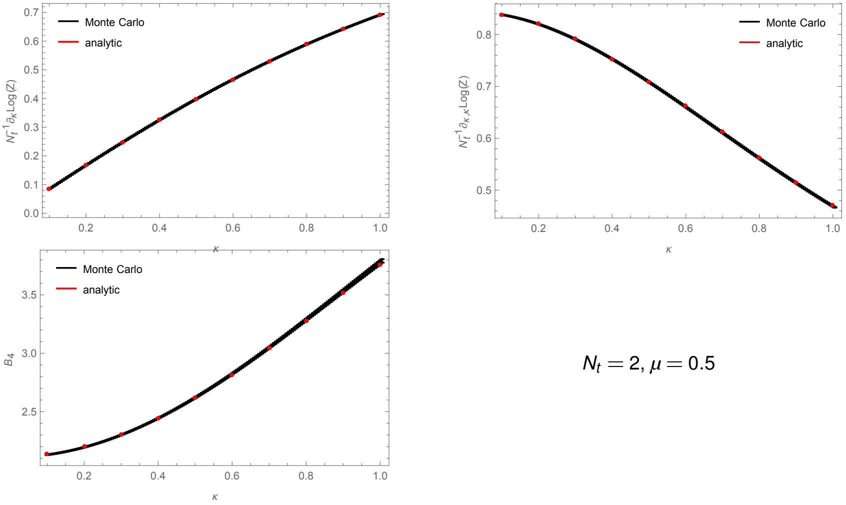




$N_t = 2, \mu = 0.0$

Crosscheck of Code

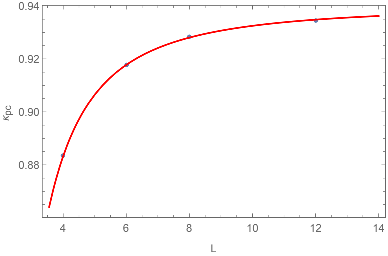
- For $d = 1$, compare result from Monte Carlo simulation with exact result:



$N_t = 2, \mu = 0.5$

Crosscheck of Code

- For $d = 3$ scaling analysis at $\mu = 0$ consistent with known results $\nu = 0.7377$, $\kappa_c = 0.93590$ [e.g. Engels & Karsch, arXiv:1105.0584]:



$$\kappa_{pcr}(L) = \kappa_{cr} + \frac{a}{L^{1/\nu}}\left(1 + \frac{b}{L^{1/\nu}}\right)$$

$$\Rightarrow \quad \kappa_{cr} \approx 0.94 \text{ for } \nu = 0.7377$$

(rather small lattices)

Improved Estimators for General Correlators

- Consider e.g. "radial correltaor" (radial in π_1-π_2-plane):

$$\langle \pi_x^r \pi_y^r \rangle = \frac{1}{\kappa^2} \frac{\partial^2 \log(Z)}{\partial s_x \partial s_y} = \left\langle \frac{|p_x| + 2q_x}{\kappa s_x} \frac{|p_y| + 2q_y}{\kappa s_y} \right\rangle + const.,$$

where $s_x = s_y = s = |s_1 \pm \mathrm{i} s_2|$, as before.

- Bad observable in this form:
  - source term required to be well defined
  - can only be measured in between worm updates
- Obtain improved estimator by splitting: $|p_x| + 2q_x = \left(\frac{1}{2}(|p_x| + p_x) + q_x\right) + \left(\frac{1}{2}(|p_x| - p_x) + q_x\right)$ and shifting variables (as has been done for the charged correlator), such that

$$\langle \pi_x^r \pi_y^r \rangle = \frac{1}{Z} \sum_{\sigma_1, \sigma_2 \in \{\pm\}} \sum_{\{k,l,\xi,\chi,p\}} \left\{ \prod_{z,\nu} \frac{\kappa^{|k_{z,\nu}| + 2l_{z,\nu} + \xi_{z,\nu} + \chi_{z,\nu}}}{(|k_{z,\nu}| + l_{z,\nu})!\, l_{z,\nu}!\, \xi_{z,\nu}!\, \chi_{z,\nu}!} \right\}$$
$$\left\{ \prod_z \delta\Big(p_z + \sigma_1 \delta_{z,x} + \sigma_2 \delta_{z,y} + \sum_\nu (k_{z,\nu} - k_{z-\hat{\nu},\nu})\Big)\, e^{2\mu k_{z,4}} \right.$$
$$\left. w(A_z + \delta_{z,x} + \delta_{z,y}, B_z, C_z, p_z; s, \phi_s, s_3, s_4, \kappa) \right\} + const.,$$

Improved Estimators for General Correlators

- Improved estimator:

$$\langle \pi^r_x \pi^r_y \rangle = \frac{1}{Z} \sum_{\sigma_1,\sigma_2 \in \{\pm\}} \sum_{\{k,l,\xi,\chi,p\}} \left\{ \prod_{z,\nu} \frac{\kappa^{|k_{z,\nu}|+2l_{z,\nu}+\xi_{z,\nu}+\chi_{z,\nu}}}{(|k_{z,\nu}|+l_{z,\nu})!\, l_{z,\nu}!\, \xi_{z,\nu}!\, \chi_{z,\nu}!} \right\}$$

$$\left\{ \prod_z \delta\big(p_z + \sigma_1 \delta_{z,x} + \sigma_2 \delta_{z,y} + \sum_\nu (k_{z,\nu} - k_{z-\widehat{\nu},\nu})\big)\, e^{2\mu k_{z,4}} \right.$$

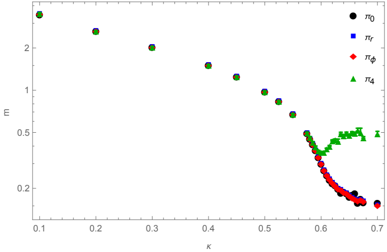
$$\left. w(A_z + \delta_{z,x} + \delta_{z,y}, B_z, C_z, p_z; s, \phi_s, s_3, s_4, \kappa) \right\} + const.,$$

- two pieces with $\sigma_1 \neq \sigma_2$ can be sampled during charged worm.
- the pieces with $\sigma_1 = \sigma_2$ contribute only if source s is non-zero; can be sampled during worm after insertion of two equally charged monomers:
  - instead of proposing worm update $y \to y + \widehat{\nu}, k_{y,\nu} \to k_{y,\nu} \pm 1$, propose a jump $y \to z$, $p_y \to p_y \pm 1$, $p_z \to p_z \pm 1$, with the same sign for the shift ± 1 at y and z, such that in order to satisfy the delta-function constraint at z, the charge of the external source would have to change (opposite sign ± 1 for the shifts at y and z lead to disconnected piece for the charged correlator itself).
- same trick works for arbitrary cross correlators, such as $\langle \pi^r_x \pi^3_y \rangle$ or $\langle \pi^4_x \pi^3_y \rangle$. They can be measured by starting with $\langle \pi^r_x \pi^r_y \rangle$ or $\langle \pi^4_x \pi^4_y \rangle$, then letting the worm jump and change its head by inserting appropriate monomers.

Chiral Symmetry breaking in 4D

- Symmetry breaking transition also exists in 4D.

pion masses vs κ at $\mu = 0$, $s_4 = 0.01$

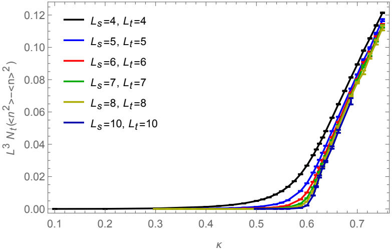


- mass splitting at $\kappa = \kappa_{pcr} \approx 0.605$

Chiral Symmetry breaking in 4D

- Symmetry breaking transition also exists in 4D.

isospin susceptibility vs κ at $\mu = 0$, $s_4 = 0$

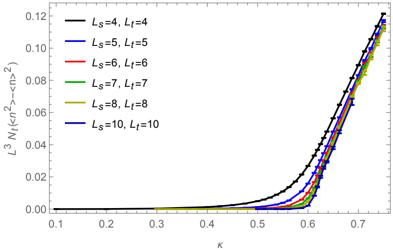


- isospin density itself is zero at $\mu = 0$ for all κ
- isospin susceptibilitiy becomes non-zero for $\kappa > \kappa_{pcr}$
- scaling analysis consistent with $\nu = 0.5$, $\kappa_c = 0.615$

Chiral Symmetry breaking in 4D

- Symmetry breaking transition also exists in 4D.

isospin susceptibility vs κ at $\mu = 0$, $s_4 = 0$



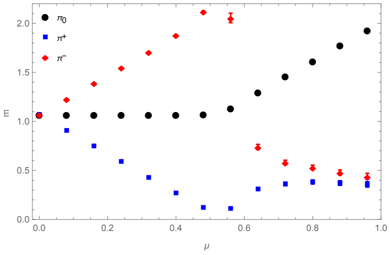
- isospin density itself is zero at $\mu = 0$ for all κ
- isospin susceptibilitiy becomes non-zero for $\kappa > \kappa_{pcr}$
- scaling analysis consistent with $\nu = 0.5$, $\kappa_c = 0.615$

- Possibility to define continuum limit although perturbatively non-renormalizable theory in 4D.

Mass Spectrum

- Mass spectrum as a function of μ as obtained from our simulations ($\kappa = 1.0 > \kappa_{pcr}$, $s_4 = 0.5$):

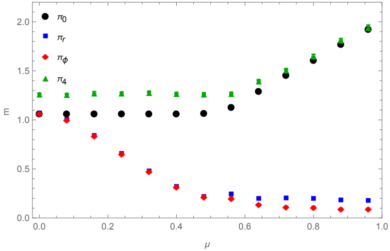
  - isocharge eigenbasis:
    $\pi_0 = \pi_3$,
    $\pi^{\pm} = \pi_1 \mp i\pi_2$
  - preferred basis for $\mu < m_\pi/2$ but no longer tangential to SU(2) after vacuum has rotated away from $\mathbb{1}$.



Mass Spectrum

- Mass spectrum as a function of μ as obtained from our simulations ($\kappa = 1.0 > \kappa_{pcr}$, $s_4 = 0.5$):

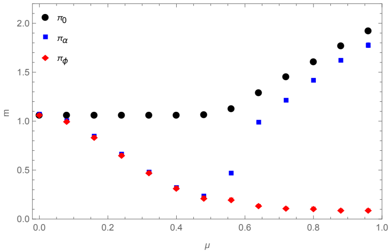
  - cylindrical coordinates:
    $\pi_0 = \pi_3$,
    $\pi_1 = r\cos(\phi)$,
    $\pi_2 = r\sin(\phi)$,
    π_4
  - π_4 not tangential to SU(2) for $\mu < m_\pi/2$,
  - π_r not tangential to SU(2) for $\mu > m_\pi/2$



Mass Spectrum

- Mass spectrum as a function of μ as obtained from our simulations ($\kappa = 1.0 > \kappa_{pcr}$, $s_4 = 0.5$):

  - spherical coordinates:
    $\pi_0 = \pi_3$,
    $\pi_1 = \sin(\alpha)\cos(\phi)$,
    $\pi_2 = \sin(\alpha)\sin(\phi)$,
    $\pi_4 = \cos(\alpha)$
  - all excitations tangential to SU(2)



- polar coordinates for sources:

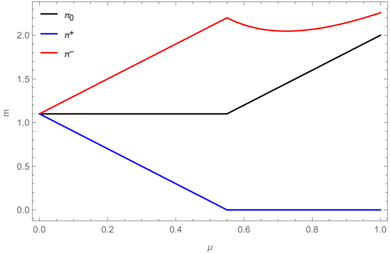
$$s_{x,4} = \tilde{s}_x \cos(\alpha_{s,x}), \quad s_{x,3} = \tilde{s}_x \sin(\alpha_{s,x}) \cos(\theta_{s,x}), \quad s_x = \tilde{s}_x \sin(\alpha_{s,x}) \sin(\theta_{s,x}).$$

$$\Rightarrow \frac{1}{\kappa \tilde{s}_x} \frac{\partial Z}{\partial \alpha_{s,x}} = \cos(\alpha_{s,x}) \left(\sin(\theta_{s,x}) \frac{1}{\kappa} \frac{\partial Z}{\partial s_x} + \cos(\theta_{s,x}) \frac{1}{\kappa} \frac{\partial Z}{\partial s_{x,3}} \right) - \sin(\alpha_{s,x}) \frac{1}{\kappa} \frac{\partial Z}{\partial s_{x,4}},$$

where the appropriate values of θ_s and α_s to be used in the "π_α correlator" are determined from the ratios of the condensates measured along with the full correlator.

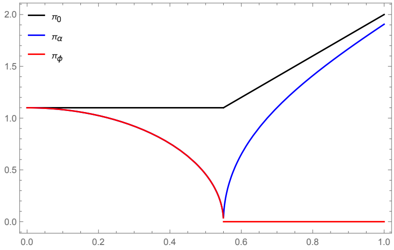
Mass Spectrum

- Compare with mass spectrum obtained by solving equations of motion at minimum of effective potential [Son & Stephanov, hep-ph/0011365]:
  - for $\mu < m_\pi/2$: isocharge eigenstates
  - for $\mu > m_\pi/2$:
    - $\pi^+ \to \pi_\phi$,
    - continuation of π^- has no fixed orientation in isospin space but rotates in π_1-π_2-plane as function of time.



Mass Spectrum

- Mass spectrum from Hessian of effective potential :
  - spherical coordinates:
    $\pi_0 = \pi_3$,
    $\pi_1 = \sin(\alpha)\cos(\phi)$,
    $\pi_2 = \sin(\alpha)\sin(\phi)$,
    $\pi_4 = \cos(\alpha)$
  - looks similar to corresponding plot obtained from our simulations



Conclusion & Outlook

- Conclusion
  - Possible to simulate $SU(2)$ (or $O(4)$) σ-model (so far the non-linear case, but linear case is also possible) with a chemical potential and various source terms.
  - Improved observable to measure full (cross) correlator.
  - Measurement of mass spectrum in symmetric and broken phase + difficulties / ambiguities in the latter case.

- Outlook
  - Generalization to $O(N)$ σ-model (linear and non-linear) straight forward.
  - New lattice formulation for CP^{N-1} models in terms of $\mathrm{U}(1)$ gauged $O(2N-1)$ models.
  - Including Higher order terms from χ-PT expansion.
  - Other symmetry groups?