The gradient flow in simple field theories

Christopher Monahan

medicine.utah.edu

LATTICE 2015 KOBE

Operator mixing on the lattice

Rotational symmetry broken on the lattice to (hyper)cubic symmetry

1. operators **mix under renormalisation** on the lattice

2. **power divergent mixing** between operators with different dimension

For example: **twist expansion of parton distribution functions** (PDFs)

Twist expansion

PDFs capture longitudinal momentum structure of nucleon constituents

- defined on the light cone: not accessible on a Euclidean lattice
- Mellin moments ⇔ matrix elements of "twist" (dim. - spin) operators

Twist operators organised in powers of momentum transfer (squared)

- twist-2 operators dominate in Bjorken limit
- mix with power-divergent coefficients

Limits lattice calculations to first four moments

Detmold et al, EPJ C3 (2001) 1
Detmold et al, MPLA 18 (2003) 2681

Operator mixing on the lattice

Simple perturbative example (4D φ^4 scalar field theory)

- continuum

$$\langle \Omega | \phi^2(0) \cdot \phi(0) \partial_\mu \partial_\nu \phi(0) | \Omega \rangle = 0$$

- lattice

$$\langle \Omega | \phi^2(0) \cdot \phi(0) \nabla_\mu \nabla_\nu \phi(0) | \Omega \rangle = -\frac{\delta_{\mu\nu}}{48 a^2} + \ldots$$

Smearing

Smearing partially restores rotational symmetry - reduces operator mixing…

e.g. Davoudi & Savage, PRD 86 (2012) 054505

Gradient flow

Narayanan & Neuberger, JHEP 0603 (2006) 064
Lüscher, CMP 293 (2010) 899
Lüscher & Weisz, JHEP 1102 (2011) 51

Gradient flow

- deterministic field evolution in "flow time" toward classical minimum
- renormalised boundary theory requires no further renormalisation

Particularly simple in scalar field theories

Makino & Suzuki, PTEP (2015) 033B08
Makino *et al.*, PTEP (2015) 043B07
Aoki *et al.*, JHEP 1504 (2015) 156

- ideal testing ground

  - no gauge-fixing complications
  - no fermionic complications
  - φ^4 theory has no flow interactions
  - 2D O(3) model still asymptotically free

See also Marco Garofalo's poster and Susanne Ehret's talk at this conference

Gradient flow in φ^4 scalar field theory

$$\frac{\partial \phi(\tau, x)}{\partial \tau} = \partial^2 \phi(\tau, x) \qquad\qquad \phi(\tau = 0, x) = \phi(x)$$

Exact solution possible

- 2D

$$\phi(\tau, x) = \frac{1}{4\pi\tau} \int \mathrm{d}^2 y \, e^{-(x-y)^2/4\tau} \phi(y)$$

- 4D

$$\phi(\tau, x) = \frac{1}{16\pi^2\tau^2} \int \mathrm{d}^4 y \, e^{-(x-y)^2/4\tau} \phi(y)$$

All interactions occur on the boundary, *i.e.* at zero flow time

Makino & Suzuki, PTEP (2015) 033B08
Makino *et al.*, PTEP (2015) 043B07
Aoki *et al.*, JHEP 1504 (2015) 156
Kikuchi & Onogi, JHEP 1411 (2014) 094

Gradient flow in 2D O(3) model

$$\frac{\partial n^i(\tau,x)}{\partial \tau} = \left[\delta^{ij} - n^i(\tau,x) n^j(\tau,x)\right] \partial^2 n^j(\tau,x) \qquad n^i(\tau=0,x) = n^i(x)$$

$$n^i(\tau,x) = \pi^i(\tau,x) \quad \text{for} \quad i=1,2 \qquad n^3(\tau,x) = \sqrt{1-\pi^i(\tau,x)\pi^i(\tau,x)}$$

Exact solution no longer possible: generate iterative tree-level expansion

$$n^i(\tau,x) = \int \mathrm{d}^2 y \int \frac{\mathrm{d}^2 p}{(2\pi)^2} e^{ip\cdot(x-y)} \left[e^{-\tau p^2} n^i(y) - \int_0^\tau \mathrm{d}s\, e^{-sp^2} R^i(s,y)\right]$$

$$R^i(s,y) = n^i(s,y) n^j(s,x) \partial^2 n^j(s,x)$$

Interactions occur in the bulk, *i.e.* at non-zero flow time, but no closed loops

See also Sinya Aoki's and Kengo Kikuchi's talks at this conference

Operator mixing on the lattice

Simple perturbative example (4D φ^4 scalar field theory)

- continuum

$$\langle\Omega|\phi^2(0)\cdot\phi(0)\partial_\mu\partial_\nu\phi(0)|\Omega\rangle = 0$$

- lattice

$$\langle\Omega|\phi^2(0)\cdot\phi(0)\nabla_\mu\nabla_\nu\phi(0)|\Omega\rangle = -\frac{\delta_{\mu\nu}}{48a^2} + \dots$$

- smeared via gradient flow

$$\langle\Omega|\phi^2(\tau,0)\cdot\phi(\tau,0)\nabla_\mu\nabla_\nu\phi(\tau,0)|\Omega\rangle = \delta_{\mu\nu}\frac{e^{-4\pi^2\tau/a^2}-1}{256\pi^2\tau} + \dots$$

Simple perturbative example (2D O(3) model)

- continuum

$$\langle \Omega | \pi^2(0) \cdot \pi(0) \partial^2 \partial_\mu \partial_\nu \pi(0) | \Omega \rangle = 0$$

- lattice

$$\langle \Omega | \pi^2(0) \cdot \pi(0) \nabla^2 \nabla_\mu \nabla_\nu \pi(0) | \Omega \rangle = \delta_{\mu\nu} \frac{\pi}{8a} + \ldots$$

- smeared via gradient flow

$$\langle \Omega | \pi^2(\tau, 0) \cdot \pi(\tau, 0) \nabla^2 \nabla_\mu \nabla_\nu \pi(\tau, 0) | \Omega \rangle = \delta_{\mu\nu} \frac{1 - e^{-4\pi^2 \tau / a^2}}{32 \pi \tau} + \ldots$$

Smeared operator product expansion

Replace local operators with “smeared” operator basis

CJM & Orginos, PRD (2015) 033B08

- constructed from fields at fixed physical flow time
- nonperturbative matrix elements and perturbative coefficients
  - flow time
  - renormalisation scale
- reproduces leading space-time behaviour of OPE
- choose flow times in suitable window
  - large enough to damp ultraviolet fluctuations
  - small enough not to distort hadronic physics
- for small flow times, one can construct renormalisation group equations
  - relate low energy hadronic matrix elements to high energy coefficients

Example calculation: perturbative coefficients

Simple example (4D φ^4 scalar field theory):

- Wilson's operator product expansion

$$\phi(x)\phi(0) = \frac{c_{\mathbb{I}}}{4\pi x^2}\mathbb{I} + c_{\phi^2}\phi^2(0) + c_{\partial_\mu} x^\mu \partial_\mu \phi^2(0) + \ldots$$

- smeared operator product expansion

$$\phi(x)\phi(0) = \frac{d_{\mathbb{I}}}{4\pi x^2}\mathbb{I} + d_{\phi^2}\phi^2(\tau, 0) + d_{\partial_\mu} x^\mu \partial_\mu \phi^2(\tau, 0) + \ldots$$

Recall our simple example

$$\phi(x)\phi(0) = \frac{d_{\mathbb{I}}}{4\pi x^2}\mathbb{I} + d_{\phi^2}\phi^2(\tau,0) + d_{\partial_\mu} x^\mu \partial_\mu \phi^2(\tau,0) + \ldots$$

Leading disconnected coefficient

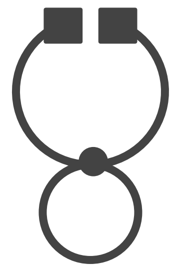
$$d_{\mathbb{I}}(x^2/\tau) = 4\pi^2 x^2 \Big\{ \langle \Omega|\phi(x)\phi(0)|\Omega\rangle - \langle \Omega|\phi^2(\tau,0)|\Omega\rangle \Big\}_{\mathcal{O}(m^2)}$$

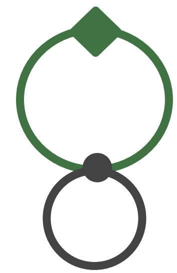
Or, graphically: at one loop

$$d_{\mathbb{I}}(x^2/\tau) = 4\pi^2 x^2 \Big\{ \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{e^{ik\cdot x} - e^{-2k^2\tau}}{k^2+m^2} \Big\}_{\mathcal{O}(m^2)} = 1 - \frac{x^2}{8\tau} + \frac{m^2 x^2}{4}\left[\gamma_{\mathrm{E}} - 1 + \log\left(\frac{x^2}{8\tau}\right)\right]$$

Note that $$c_{\mathbb{I}}^{\overline{MS}}(\mu^2 x^2) = 1 + \frac{m^2 x^2}{4}\left[1 + 2\gamma_{\mathrm{E}} + \log\left(\frac{\mu^2 x^2}{16}\right)\right]$$

At two loops $d_{\mathbb{I}}(x^2/\tau) = 4\pi^2 x^2 \Big\{ \langle \Omega | \phi(x)\phi(0) | \Omega \rangle - \langle \Omega | \phi^2(\tau, 0) | \Omega \rangle \Big\}_{\mathcal{O}(m^2)}$





$$d_{\mathbb{I}}(x^2/\tau, \mu^2 x^2) = 4\pi^2 x^2 \left\{ \int \frac{\mathrm{d}^4 k_1}{(2\pi)^4} \frac{e^{ik_1\cdot x} - e^{-k_1^2 \tau}}{k_1^2 + m^2} \left[1 - \frac{\lambda}{2} \int \frac{\mathrm{d}^4 k_2}{(2\pi)^4} \frac{1}{k_2^2 + m^2} \right] \right\}_{\mathcal{O}(m^2)}$$

$$= 1 - \frac{x^2}{8\tau} + \frac{m_{\mathrm{R}}^2 x^2}{4} \left[\gamma_{\mathrm{E}} - 1 + \log\left(\frac{x^2}{8\tau} \right) \right]$$

Renormalised boundary theory remains renormalised at non-zero flow time

Recall our simple example

$$\phi(x)\phi(0) = \frac{d_{\mathbb{I}}}{4\pi x^2}\mathbb{I} + d_{\phi^2}\phi^2(\tau,0) + d_{\partial_\mu} x^\mu \partial_\mu \phi^2(\tau,0) + \ldots$$

Leading connected coefficient

$$d_{\phi^2}(x^2/\tau) = \left\{ \langle \Omega|\phi(x)\phi(0)\tilde{\phi}(p_1)\tilde{\phi}(p_2)|\Omega\rangle - \langle \Omega|\phi^2(\tau,0)\tilde{\phi}(p_1)\tilde{\phi}(p_2)|\Omega\rangle \right\}_{\mathcal{O}(m^0)}$$

Or, graphically: at one loop

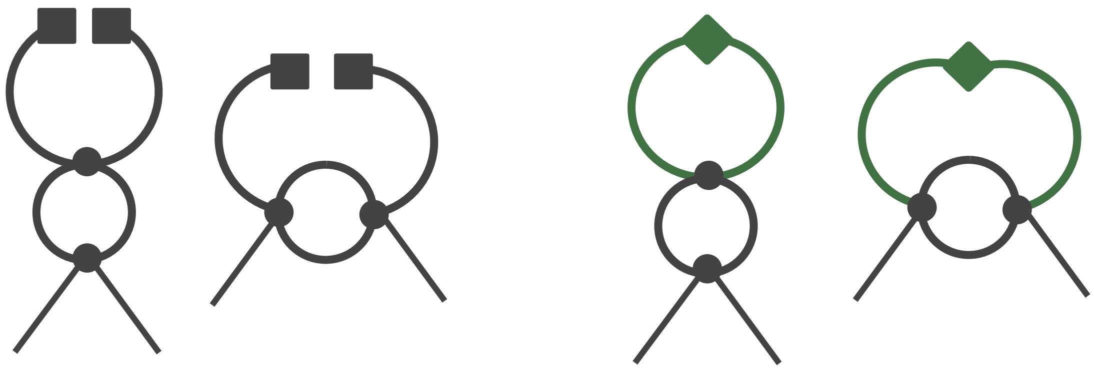
$$d_{\phi^2}(x^2/\tau) = \frac{1}{(p_1^2+m^2)(p_2^2+m^2)}\left\{1 - \frac{\lambda}{2}\int \frac{\mathrm{d}^4 k}{(2\pi)^4}\frac{e^{ik\cdot x} - e^{-(k^2+(k-p_1-p_2)^2)\tau}}{(k^2+m^2)((k-p_1-p_2)^2+m^2)}\right\}_{\mathcal{O}(m^0)}$$

$$= 1 + \frac{\lambda}{32\pi^2}\left[\gamma_{\mathrm{E}} - 1 + \log\left(\frac{x^2}{8\tau}\right)\right]$$

Note that $d_{\phi^2}\langle \Omega|\phi^2(\tau,0)\tilde{\phi}(p_1)\tilde{\phi}(p_2)|\Omega\rangle = 1 + \frac{\lambda}{32\pi^2}\left[2\gamma_{\mathrm{E}} + \log\left(\frac{m^2x^2}{4}\right)\right]$

At two loops

$$d_{\phi^2}(x^2/\tau) = \left\{ \langle\Omega|\phi(x)\phi(0)\tilde{\phi}(p_1)\tilde{\phi}(p_2)|\Omega\rangle - \langle\Omega|\phi^2(\tau,0)\tilde{\phi}(p_1)\tilde{\phi}(p_2)|\Omega\rangle \right\}_{\mathcal{O}(m^0)}$$



Leading to

$$d_{\phi^2}(x^2/\tau, \mu^2 x^2) = 1 + \frac{\lambda_{\mathrm{R}}}{32\pi^2}\left[\gamma_{\mathrm{E}} - 1 + \log\left(\frac{x^2}{8\tau}\right)\right]$$

Renormalised boundary theory remains renormalised at non-zero flow time

Consider

$$\pi(x)\pi(0) = \frac{b_{\mathbb{I}}}{4\pi}\mathbb{I} + b_{\pi^2}\pi^2(\tau,0) + b_{\partial_\mu} x^\mu \partial_\mu \pi^2(\tau,0) + \ldots$$

One loop calculations (almost) as straightforward as 4D φ^4 scalar field theory

$$b_{\mathbb{I}}(x^2/\tau) = -g_0^2\delta^{ij}\left[\gamma_{\mathrm{E}} + \log\left(\frac{x^2}{8\tau}\right)\right]$$

$$b_{\pi^2}(x^2/\tau) = 1 - \frac{g_0^2}{4\pi}\delta^{ik}\delta^{jl}\left[\gamma_{\mathrm{E}} + \log\left(\frac{x^2}{8\tau}\right)\right]$$

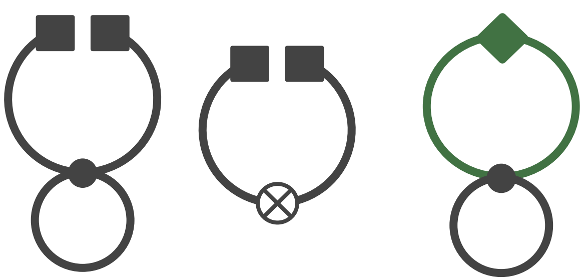
Consider

$$\pi(x)\pi(0) = \frac{b_{\mathbb{I}}}{4\pi}\mathbb{I} + b_{\pi^2}\pi^2(\tau,0) + b_{\partial_\mu} x^\mu \partial_\mu \pi^2(\tau,0) + \ldots$$

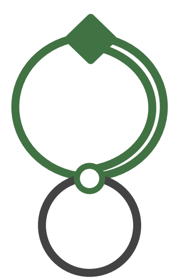
One loop calculations (almost) as straightforward as 4D φ^4 scalar field theory

Two loops - interactions complicate the picture

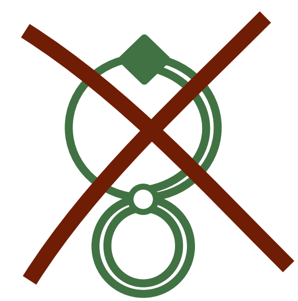
1. quantum interactions



2. tree interactions



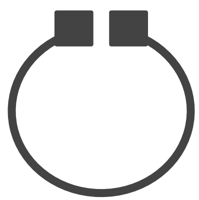
but still no flow loops

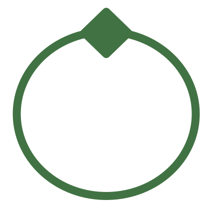


Consider

$$\pi(x)\pi(0) = \frac{b_{\mathbb{I}}}{4\pi}\mathbb{I} + b_{\pi^2}\pi^2(\tau,0) + b_{\partial_\mu}x^\mu\partial_\mu\pi^2(\tau,0) + \ldots$$

One loop calculations (almost) as straightforward as 4D φ^4 scalar field theory





Two loops - interactions complicate the picture

I. quantum interactions

2. tree interactions

but still no flow loops

- all divergences can be absorbed by original renormalisation parameters

Example calculation: renormalisation group equations

Renormalisation group equation (4D φ^4 scalar field theory):

$$\mu\frac{\mathrm{d}}{\mathrm{d}\mu} \to \mu\frac{\mathrm{d}}{\mathrm{d}\mu} - 2\tau\frac{\mathrm{d}}{\mathrm{d}\tau}$$

For sufficiently small flow times

$$[\phi^2(0)]_\mathrm{R} = \mathcal{Z}_{\phi^2}(\tau,\mu)\phi^2(\tau,0) \qquad \mu\frac{\mathrm{d}}{\mathrm{d}\mu}\log\left[\mathcal{Z}_{\phi^2}(\tau,\mu^2)\right] = 2\gamma_m$$

Perturbative coefficient obeys

$$\left[\mu\frac{\mathrm{d}}{\mathrm{d}\mu} - 2\tau\frac{\mathrm{d}}{\mathrm{d}\tau} + 2(\zeta_{\phi^2} - \gamma)\right] d_{\phi^2} = 0 \qquad \zeta_{\phi^2} = \tau\frac{\mathrm{d}}{\mathrm{d}\tau}\log\left[\mathcal{Z}_{\phi^2}(\tau,\mu^2)\right]$$

Corresponding nonperturbative matrix elements satisfy

$$\left[\mu\frac{\mathrm{d}}{\mathrm{d}\mu} - 2\tau\frac{\mathrm{d}}{\mathrm{d}\tau} + 2(\zeta_{\phi^2} + \gamma)\right]\langle\Omega|\phi^2(\tau,0)\tilde{\phi}(p_1)\tilde{\phi}(p_2)|\Omega\rangle = 0$$

Following a line of constant physics

$$\left[\mu\frac{\mathrm{d}}{\mathrm{d}\mu} + \zeta_{\phi^2} + \gamma\right]\langle\Omega|\phi^2(1/\mu^2,0)\tilde{\phi}(p_1)\tilde{\phi}(p_2)|\Omega\rangle = 0$$

Perturbation theory and nonperturbative results

perturbation nonperturbative

In theory, there is no difference between theory and practice;
in practice, there is.

Savitch, 1984

(A very few) Nonperturbative results

Nonperturbative calculation in 2D φ^4 scalar field theory

- implemented in C++

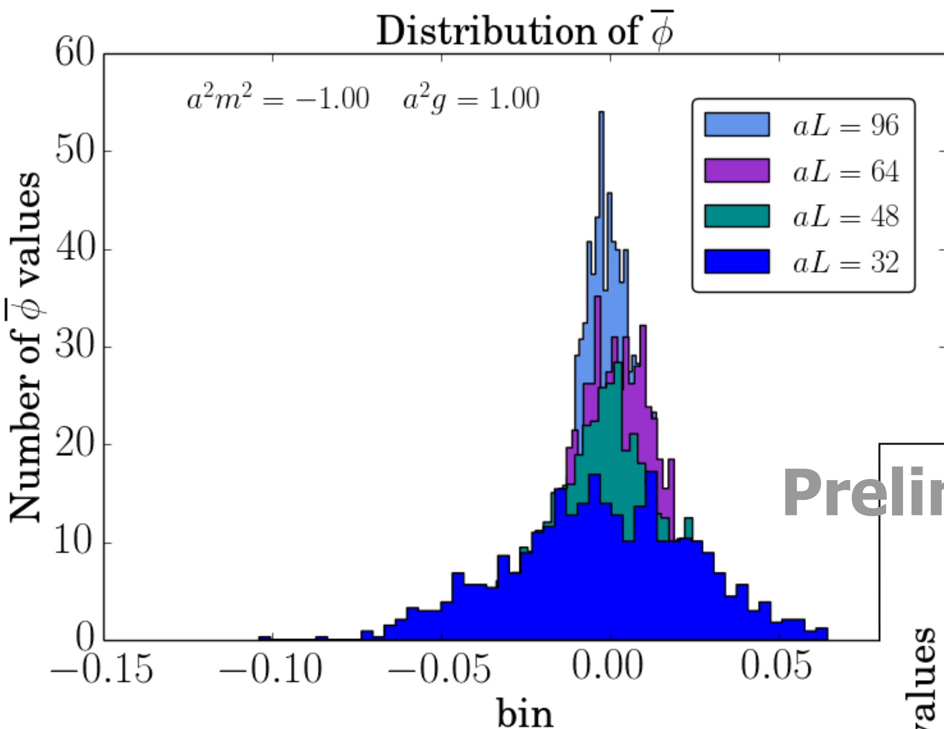
- experimented with 3 update algorithms and different combinations
  - Metropolis
  - microcanonical
  - embedded Wolff cluster

  and might implement multigrid method for fun

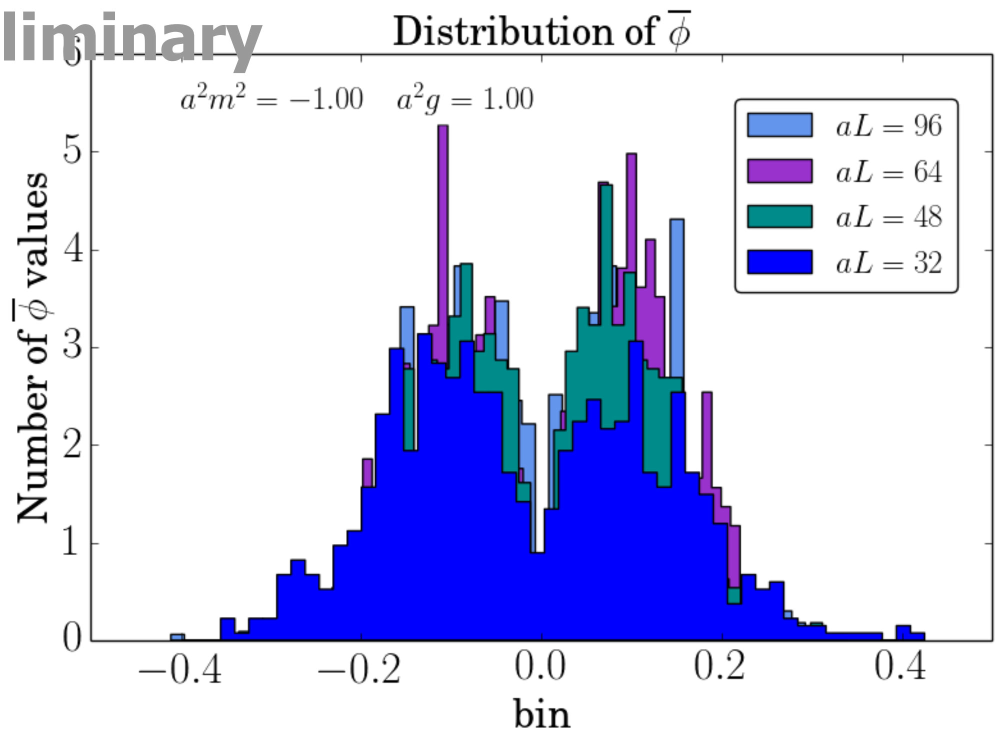
Morningstar (2007) hep-lat/0702020
Wolff, PRL 62 (1989) 361

- currently running on scalar processor
  - few seconds to tens of minutes for L = 16, … , 192 for configurations
  - measurements < few seconds

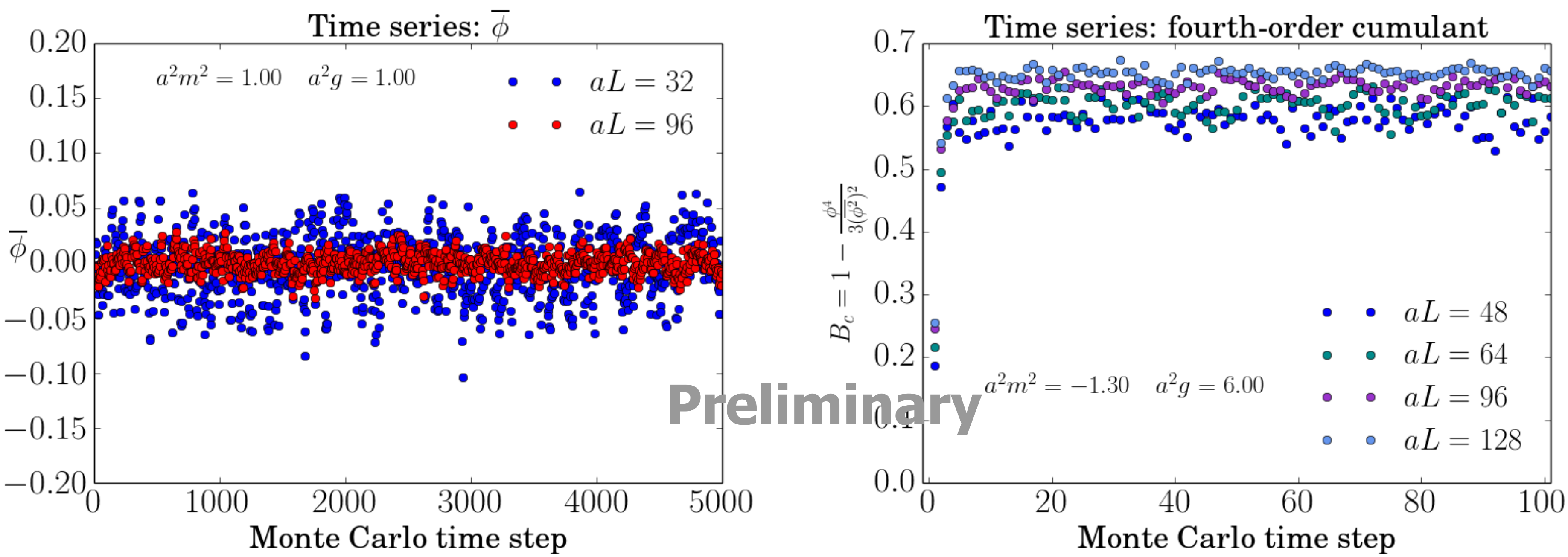
- primarily a dry-run test case for 2D O(3) model (code exists)



Preliminary



Loinaz & Willey, PRD 58 (1998) 076003



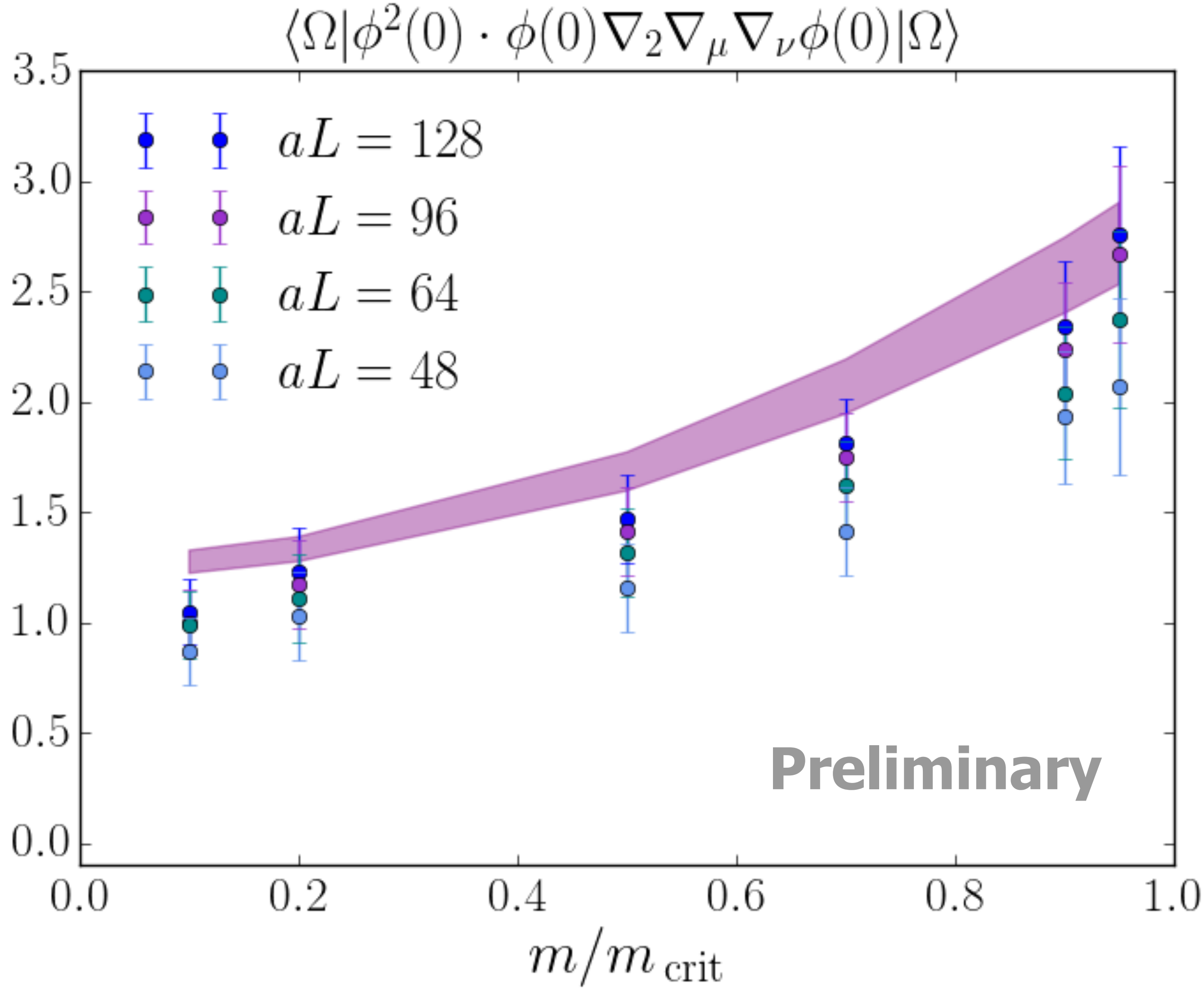
Time series: $\overline{\phi}$
$a^2m^2 = 1.00$ $a^2g = 1.00$
$aL = 32$
$aL = 96$
$\overline{\phi}$
Monte Carlo time step
Time series: fourth-order cumulant
$B_c = 1 - \frac{\overline{\phi^4}}{3(\overline{\phi^2})^2}$
$aL = 48$
$aL = 64$
$aL = 96$
$aL = 128$
$a^2m^2 = -1.30$ $a^2g = 6.00$
Monte Carlo time step
Preliminary

Operator mixing on the lattice

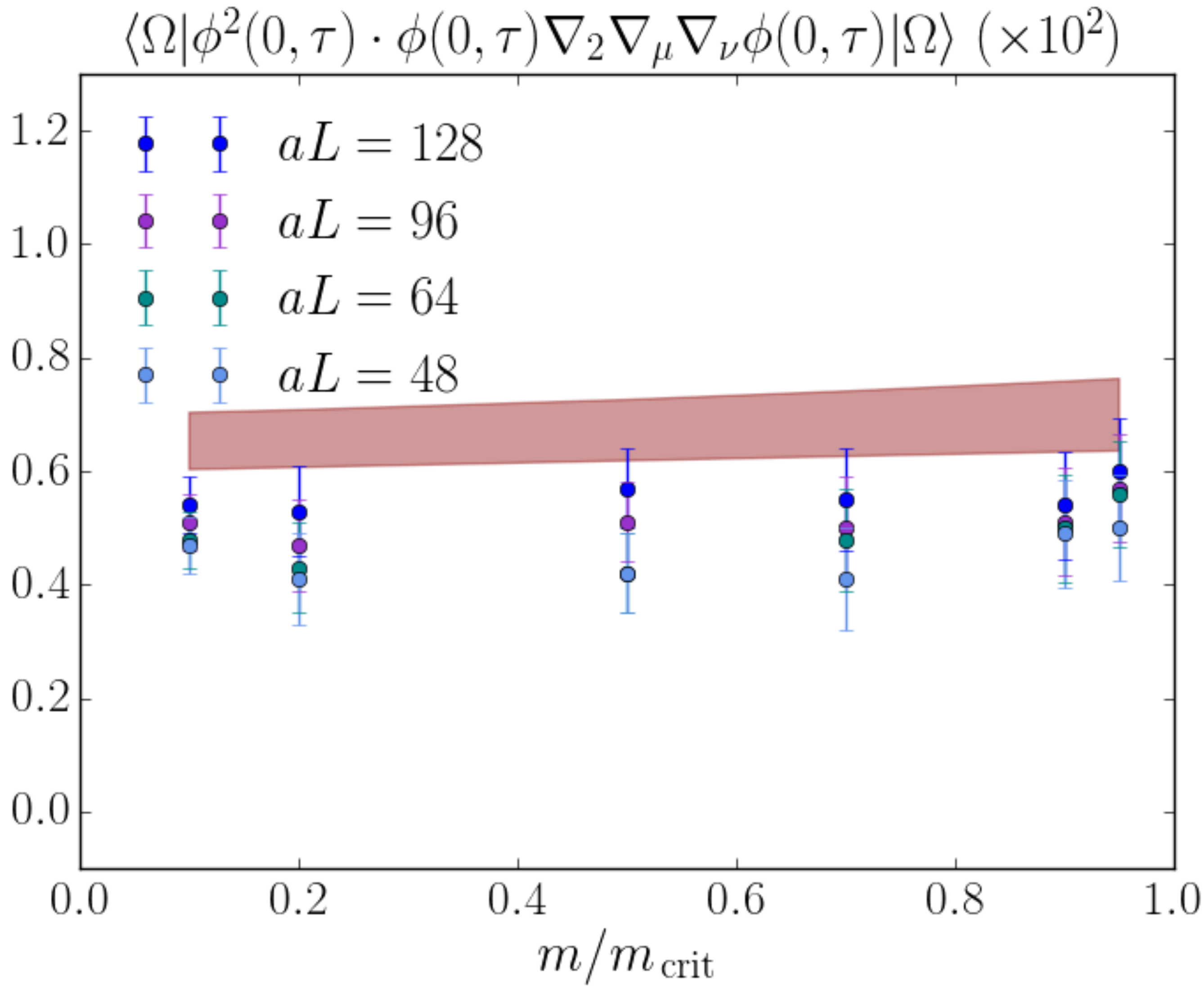
Recall our simple example

$$\langle \Omega | \phi^2(0) \cdot \phi(0) \nabla^2 \nabla_\mu \nabla_\nu \phi(0) | \Omega \rangle = \frac{\delta_{\mu\nu}}{8a^2} + \dots$$

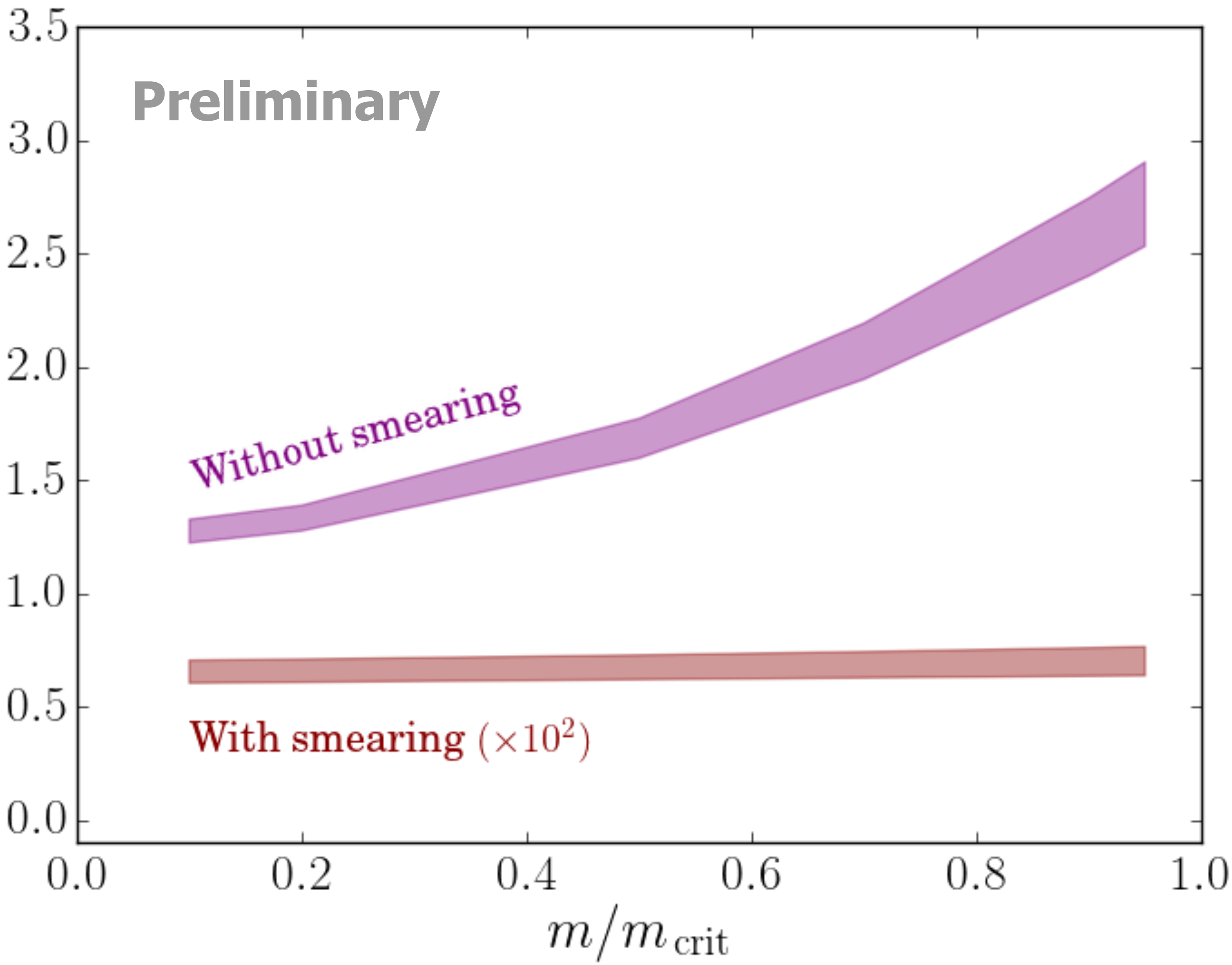
$$\langle \Omega | \phi^2(\tau, 0) \cdot \phi(\tau, 0) \nabla^2 \nabla_\mu \nabla_\nu \phi(\tau, 0) | \Omega \rangle = \delta_{\mu\nu} \frac{e^{-4\pi^2 \tau / a^2} - 1}{32 \pi \tau} + \dots$$



$\langle\Omega|\phi^2(0)\cdot\phi(0)\nabla_2\nabla_\mu\nabla_\nu\phi(0)|\Omega\rangle$
3.5
3.0
2.5
2.0
1.5
1.0
0.5
0.0
$aL = 128$
$aL = 96$
$aL = 64$
$aL = 48$
Preliminary
0.0
0.2
0.4
0.6
0.8
1.0
$m/m_{\rm crit}$



$\langle\Omega|\phi^2(0,\tau)\cdot\phi(0,\tau)\nabla_2\nabla_\mu\nabla_\nu\phi(0,\tau)|\Omega\rangle$ $(\times 10^2)$
$aL = 128$
$aL = 96$
$aL = 64$
$aL = 48$
1.2
1.0
0.8
0.6
0.4
0.2
0.0
0.0
0.2
0.4
0.6
0.8
1.0
$m/m_{\rm crit}$



Preliminary
Without smearing
With smearing $(\times 10^2)$
3.5
3.0
2.5
2.0
1.5
1.0
0.5
0.0
0.0
0.2
0.4
0.6
0.8
1.0
$m/m_{\rm crit}$

Outlook

2D and 4D φ^4 scalar field theory

- illustrative perturbative and nonperturbative calculations largely complete

2D non-linear sigma model

- code written (not debugged!)
- determine:
  - finite volume running coupling
  - twist-2 matrix elements
  - finite volume step-scaling procedure to match to...
- 2-loop perturbation theory for twist-2 matrix elements (underway)

Thank you

chris.monahan@utah.edu

With special thanks to:

Carleton DeTar
Kostas Orginos

LATTICE 2015 KOBE