Determining the scale in Lattice QCD

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– QCDSF-UKQCD Collaboration –

Vladivostok – Edinburgh – Toronto – RIKEN (Kobe) – Leipzig – FZ (Jülich) – Liverpool – DESY – Hamburg – Adelaide

Lattice 2015, Kobe, Japan

Wednesday 15/7/15 404 15:20

QCDSF related talks with $2+1$ flavours:

- Jack Dragos — Tuesday 406 17:30
  Improved Hadronic Matrix Element Determination Using the Variational Method
- Paul Rakow — Tuesday 402 18:10
  Dashen's theorem and electromagnetic contributions to pseudoscalar meson masses
- Gerrit Schierholz — Wednesday 404 17:30
  Light quark masses from infrared fixed point
- Ross Young — Thursday 403 8:30
  Applications of the Feynman – Hellmann theorem in hadron structure
- Arwed Schiller — Saturday 403 10:20
  Improving the lattice axial vector current

Introduction

- attempt to determine Wilson flow scales

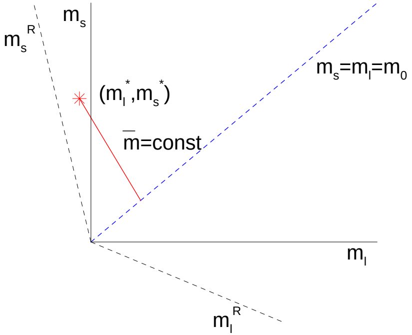
$$\sqrt{t_0^{\rm exp}} \qquad w_0^{\rm exp}$$

- strategy
- results

QCDSF strategy:

[arXiv:1102.5300]

eg $2 + 1$ simulations: many paths to approach the physical point [$m_u = m_d \equiv m_l$ case]



QCDSF: extrapolate from a point on the $SU(3)_F$ flavour symmetry line to the physical point

$$(m_0, m_0) \longrightarrow (m_l^*, m_s^*)$$

Choice here: keep the singlet quark mass $\overline{m}$ constant

$$\overline{m} = m_0 = \frac{1}{3}\,(2m_l + m_s)$$

Strategy

[arXiv:1102.5300]

- Develop $SU(3)_F$ flavour symmetry breaking expansion for masses, . . .
- Expansion in:

$SU(3)$ flavour symmetric point $\delta m_q = 0$

$$\delta m_q = m_q - \overline{m}\,, \quad \overline{m} = \tfrac{1}{3}(m_u + m_d + m_s) = m_0$$

- trivial constraint

$$\delta m_u + \delta m_d + \delta m_s = 0$$

Here we shall:

- Consider a flavour singlet quantity

$$X_S(m_u, m_d, m_s)$$

- Invariant under u, d, s permutations (by definition)
- Simple property:

Stationary point about the $SU(3)$ flavour symmetric line

Expanding a flavour singlet quantity about a point on the $SU(3)_F$-flavour line:

$$X_S(\overline{m}+\delta m_u, \overline{m}+\delta m_d, \overline{m}+\delta m_s)$$
$$= X_S(\overline{m},\overline{m},\overline{m}) + \left.\frac{\partial X_S}{\partial m_u}\right|_0 \delta m_u + \left.\frac{\partial X_S}{\partial m_d}\right|_0 \delta m_d + \left.\frac{\partial X_S}{\partial m_s}\right|_0 \delta m_s + O((\delta m_q)^2)$$

On the symmetric line:

$$\left.\frac{\partial X_S}{\partial m_u}\right|_0 = \left.\frac{\partial X_S}{\partial m_d}\right|_0 = \left.\frac{\partial X_S}{\partial m_s}\right|_0 ,$$

Together with $\delta m_u + \delta m_d + \delta m_s = 0$ this implies that

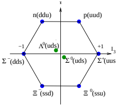
$$X_S(\overline{m}+\delta m_u, \overline{m}+\delta m_d, \overline{m}+\delta m_s) = X_S(\overline{m},\overline{m},\overline{m}) + O((\delta m_q)^2)$$

Singlet quantities I – hadronic – many possibilities

- Octet baryons: (centre of mass)

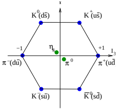
stable under QCD

$$X_N^2 = \frac{1}{6}(M_p^2 + M_n^2 + M_{\Sigma^+}^2 + M_{\Sigma^-}^2 + M_{\Xi^0}^2 + M_{\Xi^-}^2) = (1.160\,\mathrm{GeV})^2$$



- Pseudoscalar mesons: (centre of mass)

$$X_\pi^2 = \frac{1}{6}(M_{K^+}^2 + M_{K^0}^2 + M_{\pi^+}^2 + M_{\pi^-}^2 + M_{\overline{K}^0}^2 + M_{K^-}^2) = (0.4116\,\mathrm{GeV})^2$$



- Vector mesons: (centre of mass)

$$X_\rho^2 = \frac{1}{6}(M_{K^{*+}}^2 + M_{K^{*0}}^2 + M_{\rho^+}^2 + M_{\rho^-}^2 + M_{\overline{K}^{*0}}^2 + M_{K^{*-}}^2)$$

- Some other possibilities

$$X_S^2 = \begin{cases} \frac{1}{2}(M_\Sigma^2 + M_\Lambda^2) & S = \Lambda \\ M_{\Sigma^*}^2\,,\ \frac{1}{2}(M_\Delta^2 + M_{\Xi^*}^2) & S = \Sigma^*\,,\ \Delta \end{cases}$$

baryon decuplet, unstable under QCD

Singlet quantities II – gluonic – many possibilities

- Force scale:

$$X_{r_0}^2 = \frac{1}{r_0^2}$$

- Wilson flow scales:
  - $$X_{t_0}^2 = \frac{1}{t_0}$$
  - $$X_{w_0}^2 = \frac{1}{w_0^2}$$

- ‘Secondary scales’, physical value has to be determined

Check I – $SU(3)$ flavour breaking expansion — Gell-Mann–Okubo

eg pseudoscalar mesons:

$$\begin{aligned} M^2_{\pi^+}(= M^2_{\pi^-}) &= M^2_{0\pi} + \alpha_\pi(\delta m_u + \delta m_d) + O((\delta m_q)^2) \\ M^2_{K^+}(= M^2_{K^-}) &= M^2_{0\pi} + \alpha_\pi(\delta m_u + \delta m_s) + O((\delta m_q)^2) \\ M^2_{K^0}(= M^2_{\overline{K}^0}) &= M^2_{0\pi} + \alpha_\pi(\delta m_d + \delta m_s) + O((\delta m_q)^2) \end{aligned}$$

$$\begin{aligned} X^2_\pi &= \tfrac{1}{6}(M^2_{K^+} + M^2_{K^0} + M^2_{\pi^+} + M^2_{\pi^-} + M^2_{\overline{K}^0} + M^2_{K^-}) \\ &= M^2_{0\pi} + \alpha_\pi(\delta m_u + \delta m_d + \delta m_s) + O((\delta m_q)^2) \\ &= M^2_{0\pi} + O((\delta m_q)^2) \end{aligned}$$

$M_{0\pi} \equiv M_{0\pi}(\overline{m})$, $\alpha_\pi \equiv \alpha(\overline{m})$ only

Check II – using χ-PT

Choose your favourite χ-PT result

Expand about a $SU(3)_F$ flavour symmetric point:

$$X_S(m_u, m_d, m_s) = X_S(\overline{m}, \overline{m}, \overline{m}) + O((\delta m_q)^2)$$

eg Wilson flow scale: t_0

- Bär and Golterman [arXiv:1312.4999] derive $2+1$ flavour expression: $[\mu_P = (M_P/4\pi f_0)^2 \ln(M_p/\mu)^2]$

$$
\begin{aligned}
t_0 &= t_{0,\mathrm{ch}} \Big[1 + \frac{k_1}{(4\pi f_0)^2}(2M_K^2 + M_\pi^2) \\
&\quad + \frac{1}{(4\pi f_0)^2}\left((3k_2 - k_1)M_\pi^2\mu_\pi + 4k_2 M_K^2 \mu_K + \frac{k_1}{3}(M_\pi^2 - 4M_K^2)\mu_\eta + k_2 M_\eta^2 \mu_\eta \right) \\
&\quad + \frac{k_4}{(4\pi f_0)^4}(2M_K^2 + M_\pi^2)^2 + \frac{k_5}{(4\pi f_0)^4}(M_K^2 - M_\pi^2)^2 \Big]
\end{aligned}
$$

$[k_1 \ \ldots \ k_5 \ parameters]$

- Manipulate to give

$$
t_0 = T(\overline{\chi}) \left[1 + \frac{1}{(4\pi f_0)^4}\left(\tfrac{5}{6}k_2 + \tfrac{1}{4}k_5''\right)(\underbrace{\chi_s - \chi_l}_{\delta m_s - \delta m_l})^2 + \cdots \right]
$$

where

$$
T(\overline{\chi}) = t_{0,\mathrm{ch}} \left[1 + \frac{3k_1}{(4\pi f_0)^2}\overline{\chi} + \frac{8k_2}{(4\pi f_0)^4}\overline{\chi}^2 \ln\frac{\overline{\chi}}{\Lambda^2} + \frac{9k_4'}{(4\pi f_0)^4}\overline{\chi}^2 \right]
$$

$[\chi_l = B_0 m_l, \ \chi_s = B_0 m_s, \ \overline{\chi} = \frac{1}{3}(2\chi_l + \chi_s)]$

- no linear term, first term is quadratic in the $SU(3)$ breaking

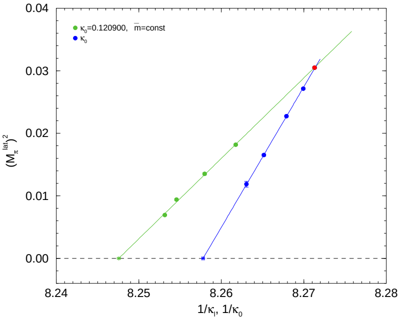
Lattice

- $O(a)$ NP improved clover action
  - tree level Symanzik glue
  - mildy stout smeared $2+1$ clover fermion
  - $\beta = 5.40, 5.50, 5.65, 5.80$ $[24^3 \times 48,\ 32^3 \times 64,\ 48^3 \times 96]$

-
$$m_q = \frac{1}{2}\left(\frac{1}{\kappa_q} - \frac{1}{\kappa_{0c}}\right)$$
κ_{0c} is chiral limit along symmetric line

-
$$\delta m_q = m_q - m_0 = \frac{1}{2}\left(\frac{1}{\kappa_q} - \frac{1}{\kappa_0}\right)$$

- typical $M_\pi^{lat\,2}$ values



Wilson Flow

[follows Lüscher arXiv:1006.4518]

$$\frac{dU_\mu(x,t)}{dt} = -\frac{\delta S_{\text{flow}}[U]}{\delta U_\mu(x,t)} U_\mu(x,t)\,, \qquad \text{with} \;\; U_\mu(x,0) = U_\mu(x)$$

- Observable:

$$F(t) \equiv t^2 \langle E(t) \rangle\,, \qquad \text{where} \;\; E(t) = \tfrac{1}{4} F^{a\,2}_{\mu\nu}(t)$$

- $\sqrt{t_0}$:

$$F(t)\big|_{t=t_0(c)} = c$$

w_0: [BMW arXiv:1203.4469]

$$t\frac{d}{dt}F(t)\bigg|_{t=w_0^2(c)} = c$$

$c = 0.3$ [conventional]

- Discretisation:

$$(\text{flow}, \text{gauge action}, \text{observable}) = (\text{Wilson}, \text{Symanzik [tree level]}, \text{Clover})$$

Runge-Kutta for flow equation

Improved scaling behaviour: $O(a^2)$ terms

- eg $\sqrt{t_0}$:

$$\left.\frac{F(t)}{1 + C_2 \frac{a^2}{t} + \ldots}\right|_{t=t_{0\,\mathrm{imp}}(c)} = c \quad \Rightarrow \quad t_{0\,\mathrm{imp}} = t_0 \left(1 + C_2 \frac{F_0}{F_0'} \frac{a^2}{t_0} + \ldots\right)$$

- invert, relabel $t_{0\,\mathrm{imp}} \to t_{0\,\mathrm{cont}}$ [MILC arXiv:1503.02769]

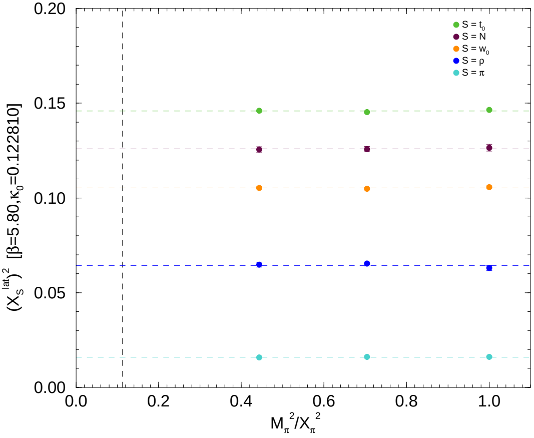
$$t_0 = t_{0\,\mathrm{cont}} \left(1 - C_2 \frac{F_{0\,\mathrm{cont}}}{F'_{0\,\mathrm{cont}}} \frac{a^2}{t_{0\,\mathrm{imp}}} + \ldots\right)$$

[$F_x = F(t_x)$, $F'_x = t dF(t)/dt|_{t_x}$, where $x = 0$ or $0\,\mathrm{cont}$]

- at tree level for $(fgo) = (WSC)$, $C_2 = -7/72$ [Fodor et al arXiv:1406.0827]

so expect gradient to be +ve

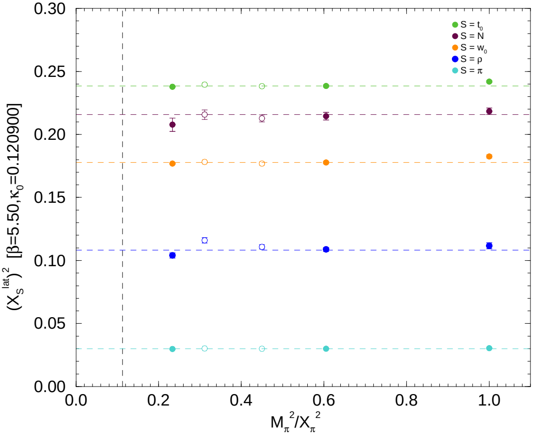
X_S^2 determination I:



- $\beta = 5.80$, $\kappa_0 = 0.122810$
- $X_{t_0}^2$, $X_{w_0}^2$, X_{π}^2, X_{ρ}^2, $X_N^2 \approx X_{\Lambda}^2$ along the $\overline{m} =$ const. line

[in plot $M_\pi \sim$ 420 MeV - 275 MeV]

X_S^2 determination II:



- $\beta = 5.50$, $\kappa_0 = 0.120900$
- $X_{t_0}^2$, $X_{w_0}^2$, X_π^2, X_ρ^2, $X_N^2 \approx X_\Lambda^2$ along the $\overline{m} = \text{const.}$ line

[in plot $M_\pi \sim$ 460 MeV - 225 MeV]

Alternatively:

- we have

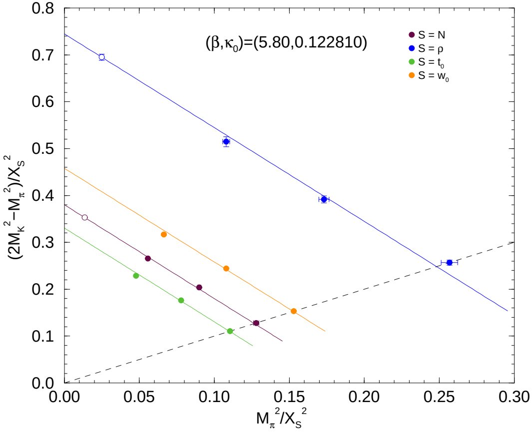
$$\frac{X_\pi^2}{X_s^2} = \frac{2M_K + M_\pi^2}{X_s^2}$$

- giving

$$\frac{2M_K^2 - M_\pi^2}{X_S^2} = \underbrace{\frac{X_\pi^2}{X_s^2}}_{\text{const}} - \underbrace{2}_{\text{const}} \frac{M_\pi^2}{X_s^2}$$

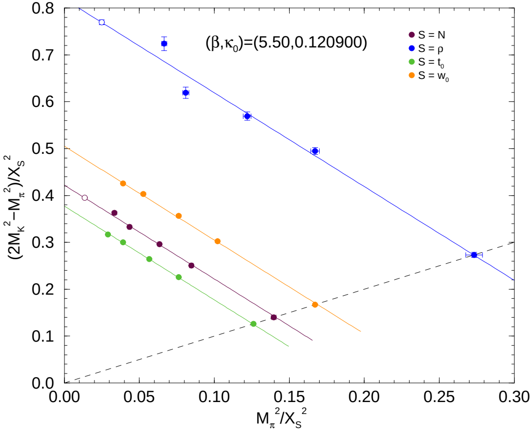
for $S = N, \rho, t_0, w_0$

Path in quark mass plane I:



- $S = N,\ \rho,\ t_0,\ w_0$
- $(\beta, \kappa_0) = (5.80, 0.122810)$

Path in quark mass plane II:



- $S = N,\ \rho,\ t_0,\ w_0$
- $(\beta, \kappa_0) = (5.50, 0.12090)$

Conclusion:

- Results for $\beta = 5.80 - 5.40$ and a variety of κ_0
- All constant — Gell-Mann–Okubo

Goal:

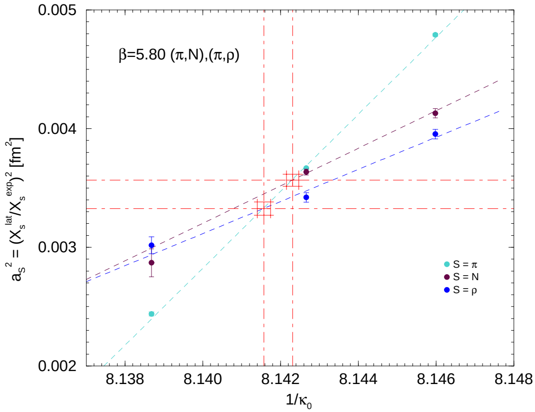
- Use $X_S^{\exp}$ to determine scale

$$a_S^2 = \frac{X_S^{lat\ 2}}{X_S^{\exp\ 2}}$$

- Vary κ_0 – when pairs a_S, $a_{S'}$ cross gives common lattice spacing a
    - apply in particular to

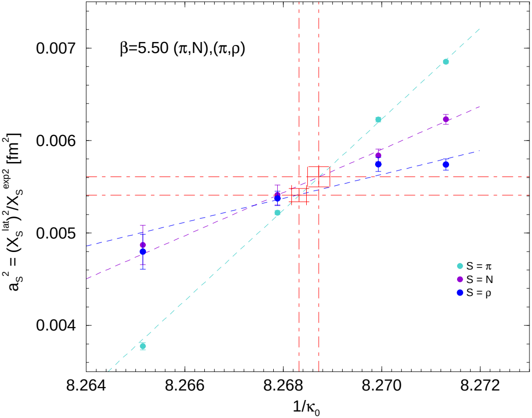
$$(\pi, N), (\pi, \rho)$$

Crossing of X_S^2 I:



- $\beta = 5.80$

Crossing of X_S^2 II:



- $\beta = 5.50$

Conclusion:

- Determination of (κ_0, a) for $\beta = 5.80 - 5.40$

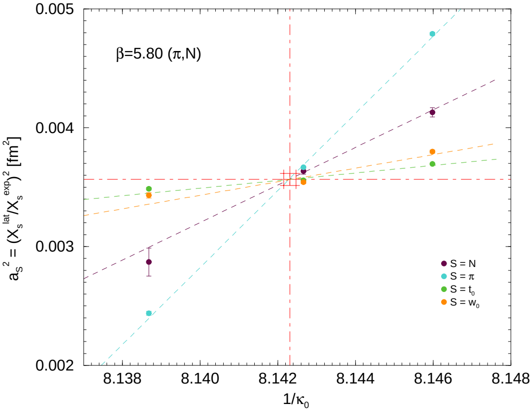
Goal:

- Use these crossings to match to X_{t_0} and X_{w_0}
- eg

$$w_0^{\mathrm{exp}\ 2} \equiv \frac{1}{X_{w_0}^{\mathrm{exp}\ 2}} = \frac{a^2}{X_{w_0}^{lat\ 2}}$$

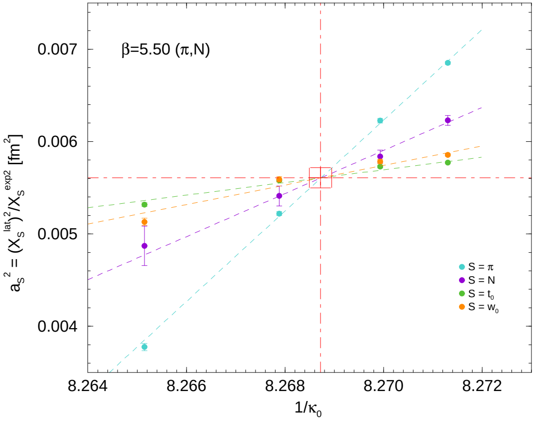
- determines $\sqrt{t_0^{\mathrm{exp}}}$, w_0^{exp} $(n_f = 2 + 1)$

Crossing of X_S^2 with X_{t_0}, X_{w_0} I:



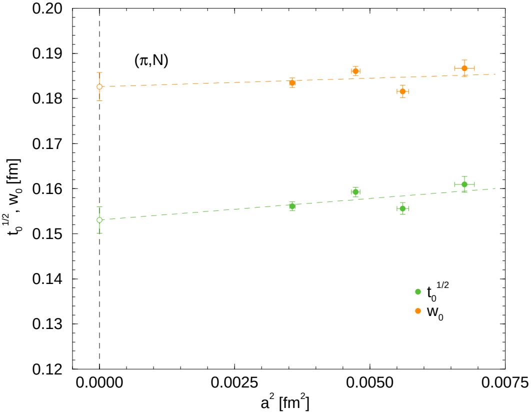
- $\beta = 5.80$
- only consider (π, N) here

Crossing of X_S^2 with $X_{t_0}^2$, $X_{w_0}^2$ II:

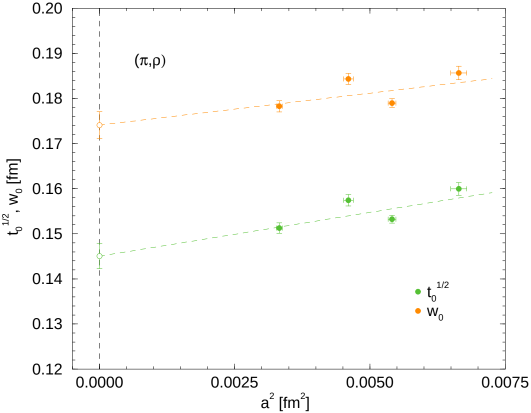


- $\beta = 5.50$
- only consider (π, N) here

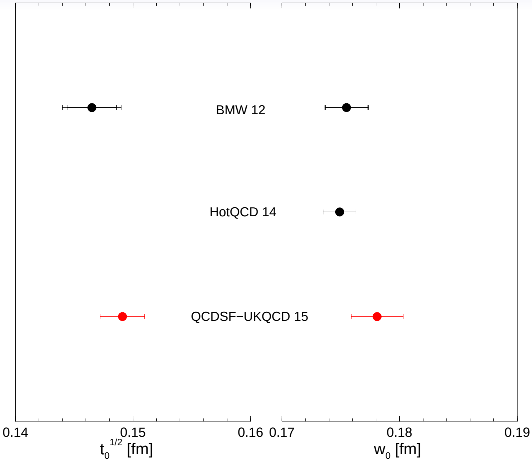
Continuum extrapolation: (π, N)



Continuum extrapolation: (π, ρ)



Comparison with other $n_f = 2 + 1$ results



[BMW 12 arXiv:1203.4469; HotQCD 14 arXiv:1407.6387]

- weighted average: $\sqrt{t_0^{\text{exp}}} \sim 0.149(2)(?)\,\text{fm}$, $w_0^{\text{exp}} \sim 0.178(2)(?)\,\text{fm}$
- preliminary, presently only statistical errors
- n_f dependence?

Conclusions

- Programme:
  Tune strange and light quark masses to their physical values simultaneously by keeping
  $$\overline{m} = \frac{1}{3}\,(2m_l + m_s) = \text{const.}$$
- $M_\pi \searrow$; $M_K \nearrow$
- $X_S(\kappa_0)$ (singlet quantities) remain constant from $SU(3)$ flavour symmetric line — Gell-Mann–Okubo
  - Use X_S^{exp} to determine $a_S(\kappa_0)$ scale
  - Vary κ_0 – determine when (X_π, X_N), (X_π, X_ρ) cross (common a)
  - Arrange so X_{t_0}, X_{w_0} also cross here – determines $\sqrt{t_0^{\text{exp}}}$, w_0^{exp} [fm]