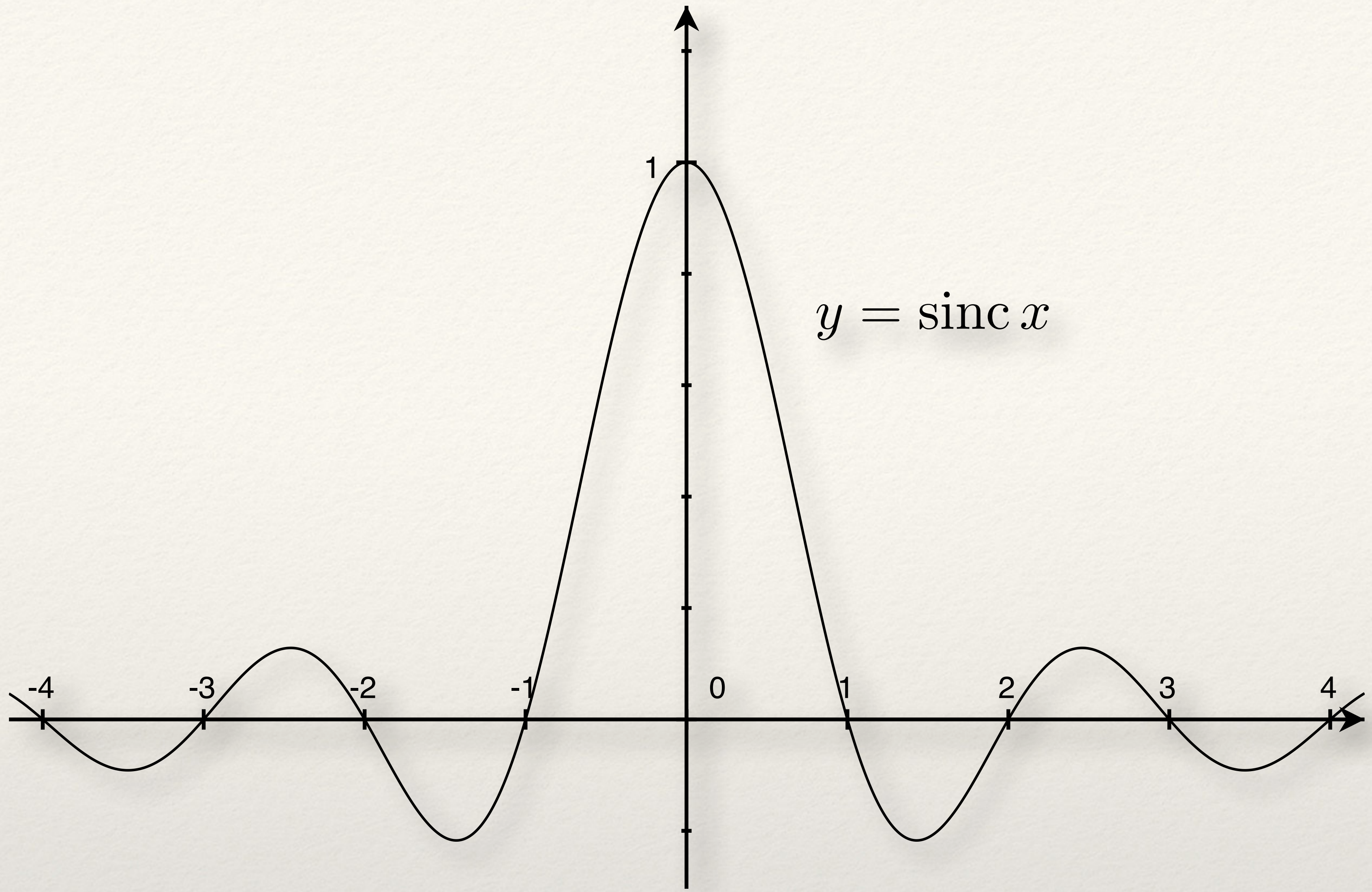

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Finite volume hadronic vacuum polarisation at arbitrary momenta

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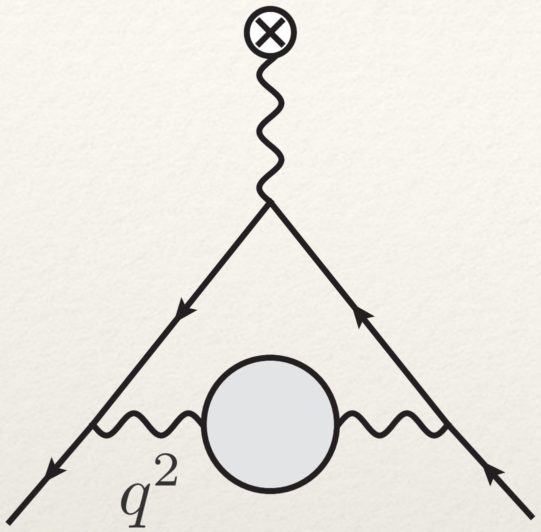
Motivations

HVP and momentum quantisation



$$= 4\alpha^2 \int_0^{+\infty} \mathrm{d}q^2\, f(q^2)[\Pi(q^2) - \Pi(0)]$$

- Finite volume: momentum quantisation.
- Dominated by momenta around $q^2 \sim m_\mu^2 \sim (100\ \mathrm{MeV})^2$.
- Typical finite-volume quantum: $(2\pi/10\ \mathrm{fm})^2 \sim (125\ \mathrm{MeV})^2$.
- Problem generally circumvented by modelling the HVP form factor in the low-q^2 region.

HVP and momentum quantisation

- In practice: one computes $\langle J_\mu(x) J_\nu(0) \rangle$ on the lattice and Fourier-transform the result.
- Numerically speaking, we could use continuous momenta in the Fourier transform.
- How wrong would that be?
- Procedure already used in:
  [Feng *et al.*, PRD 88(3), 034505, 2013] & [C. Lehner and T. Izubuchi, Lattice 2014]
  but assume an infinite time extent or infinite-volume data.

Sine cardinal interpolation (SCI)

SCI definition

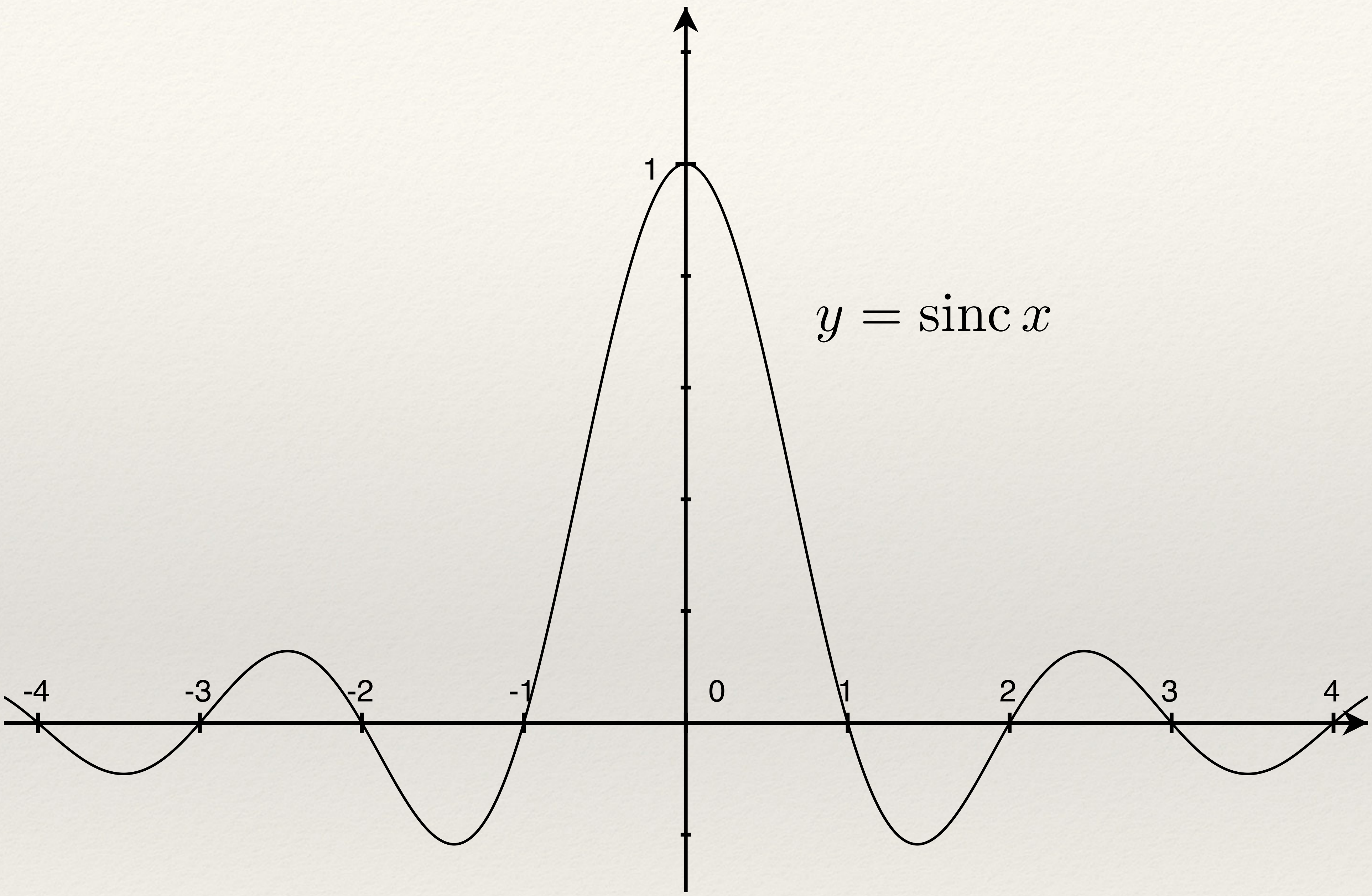
$$C_{\mu\nu}(x) = \langle J_\mu(x) J_\nu(0) \rangle$$

$$\int_{-L/2}^{L/2} \mathrm{d}^4x \, C_{\mu\nu}(x) e^{-iq\cdot x} = \frac{1}{L^4} \sum_{\bar{k}} \Pi_{\mu\nu}^{\mathrm{FV}}(\bar{k}) \int_{-L/2}^{L/2} \mathrm{d}^4x \, e^{i(\bar{k}-q)\cdot x}$$

$$= \sum_{\bar{k}} \Pi_{\mu\nu}^{\mathrm{FV}}(\bar{k}) \, \mathrm{sinc}\left[\frac{L}{2\pi}(q - \bar{k})\right]$$

$$\mathrm{sinc}(\xi) = \prod_{\mu} \frac{\sin(\pi \xi_\mu)}{\pi \xi_\mu}$$

SCI definition



sinc: "finite volume delta function"

SCI definition

- Physics:

$$\tilde{\Pi}_{\mu\nu}(q) = \sum_{\bar{k}} \Pi^{\mathrm{FV}}_{\mu\nu}(\bar{k}) \,\mathrm{sinc}\left[\frac{L}{2\pi}(q-\bar{k})\right]$$

How fast $\tilde{\Pi}_{\mu\nu}(q)$ converges to the infinite volume correlation function for $L \to +\infty$?

- Maths:

$$\tilde{f}(q,h) = \sum_{n\in\mathbb{Z}^D} f(hn)\,\mathrm{sinc}\left(\frac{q-hn}{h}\right)$$

What is the extrapolation error for h small?
Answer: **Sampling theory**

Shannon sampling theorem

Shannon, 1946:

If the Fourier transform of f has its support included in $(-\pi/h, \pi/h)^D$ then $\tilde{f}(q,h) = f(q)$.

Heuristic: a function with compact support can be exactly described by a discrete amount of information. We know that already: **Fourier series**.

Home-brewed extension

On some blackboard in Edinburgh, 2015:

If the Fourier transform of f decays faster than any power of $|x|^{-1}$, $\tilde{f}(q,h) - f(q)$ *decays faster than any power of* h.

Heuristic: Poisson summation formula.

Any function infinitely differentiable with integrable derivatives (Sobolev functions) verifies the hypothesis.

MSW theorem

Mc Namee, Steinger and Whitney, 1971 (1D):

If f can be analytically continued to the band $|\Im(q)| < E$,
then $\tilde{f}(q,h) - f(q) = \mathrm{O}(e^{-\frac{\pi E}{h}})$.

Heuristic: reconstruct the cardinal series through residue theorem applied to a nicely chosen function.

The hypothesis implies that the Fourier transform of f decays exponentially (Paley-Wiener theorems).

Application to the HVP

HVP tensor in finite volume

- In a hypercubic volume, Hodge decomposition plus a pinch of group theory:

(*cf.* also [D. Bernecker & H. B. Meyer, EPJ A 47(11), pp. 148–16, 2011])

$$C_{\mu\nu}(x) = (\partial_\mu \partial_\nu - \delta_{\mu\nu}\Delta)F(x) + \delta_{\mu\nu} z$$

z : dimension 6 constant (background density)

- Fourier transform using an arbitrary momentum:

$$\tilde{\Pi}_{\mu\nu}(q) = (q_\mu q_\nu - \delta_{\mu\nu} q^2)\tilde{\Pi}(q^2) + \delta_{\mu\nu} L^4 z \,\mathrm{sinc}\left(\frac{qL}{2\pi}\right)$$

This defines the SCI $\tilde{\Pi}$ of the HVP form factor.

SCI and zero-mode subtraction

- For simplicity: $q = (q_0, 0, 0, 0)$ and $\mu = \nu = j$ spatial.
- Traditional computation of $\tilde{\Pi}(q^2)$:

$$\frac{\tilde{\Pi}_{jj}(q_t)}{q_t^2} = \tilde{\Pi}(q_t^2) + \frac{L^4 z}{q_t^2}\,\mathrm{sinc}\left(\frac{q_t L}{2\pi}\right)$$

The zero-mode term diverges for $q_t \to 0$!

- Subtracted zero-mode $\overline{\Pi}_{\mu\nu}(q) = \tilde{\Pi}_{\mu\nu}(q) - \tilde{\Pi}_{\mu\nu}(0)$:

$$\frac{\overline{\Pi}_{jj}(q_t)}{q_t^2} = \tilde{\Pi}(q_t^2) + \frac{L^4 z}{q_t^2}\left[\mathrm{sinc}\left(\frac{q_t L}{2\pi}\right) - 1\right]$$

The zero-mode term converges to $-L^6 z/24$.

SCI error

- Let's assume that HVP finite-volume effects are decaying exponentially with $M_\pi L$.
- Spectral representation: $\Pi_{\mu\nu}(q)$ has an analytical continuation in the band $|\Im(q_0)| < 2M_\pi$.
- MWS theorem: $\tilde{\Pi}(q^2)$ is the infinite-volume form factor up to O($e^{-M_\pi L}$) effects.
- "We didn't make things worse"

SCI and moments

- One can compute consistently continuous moments in finite-volume:

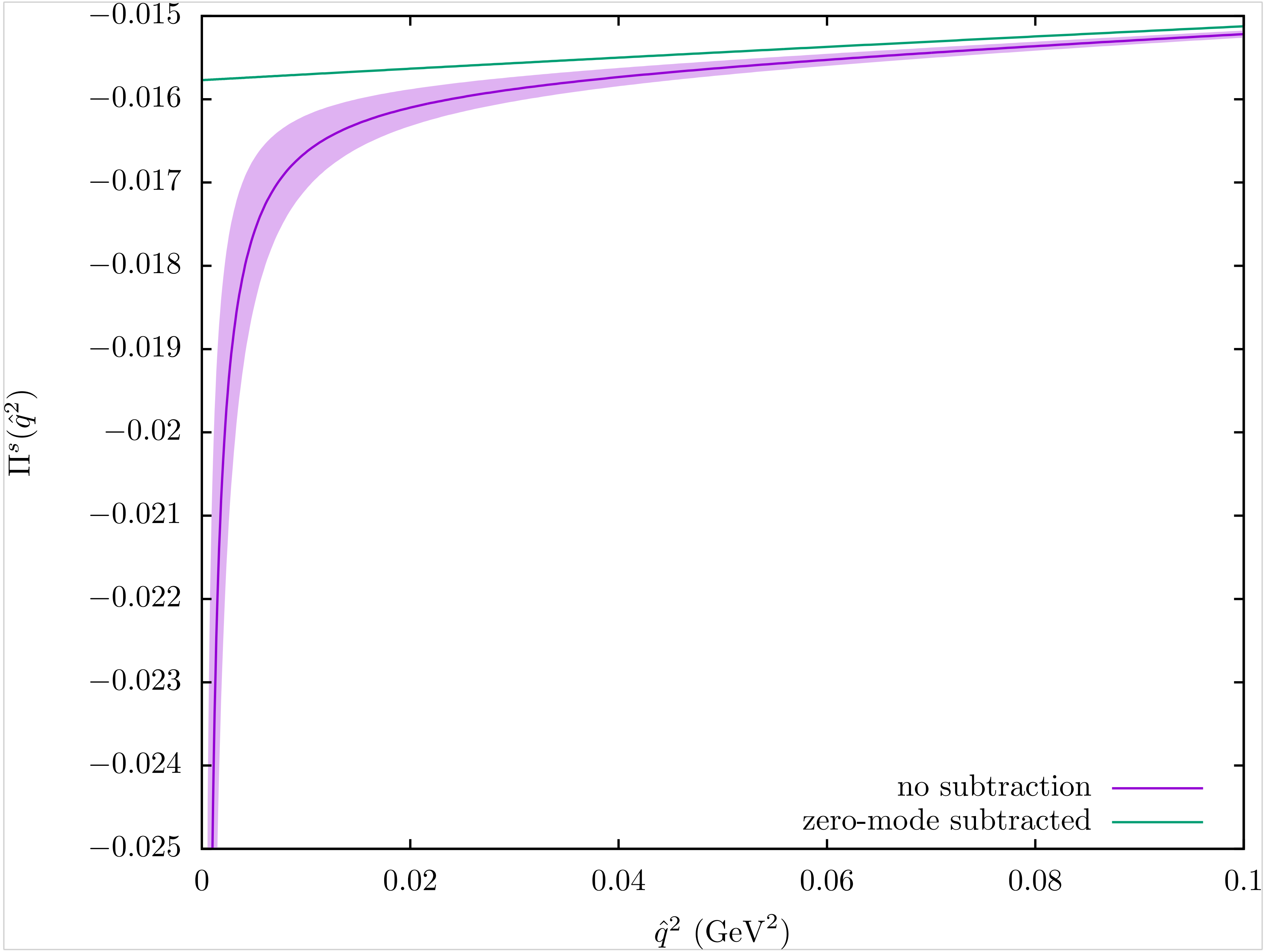
$$-\frac{1}{2}\int_{-L/2}^{L/2} \mathrm{d}^4x\, x_0^2 C_{jj}(x) = \tilde{\Pi}(0) - \frac{\pi^2 L^6 z}{24}$$

$$\frac{1}{24}\int_{-L/2}^{L/2} \mathrm{d}^4x\, x_0^4 C_{jj}(x) = \tilde{\Pi}'(0) + \frac{\pi^4 L^8 z}{1920}$$

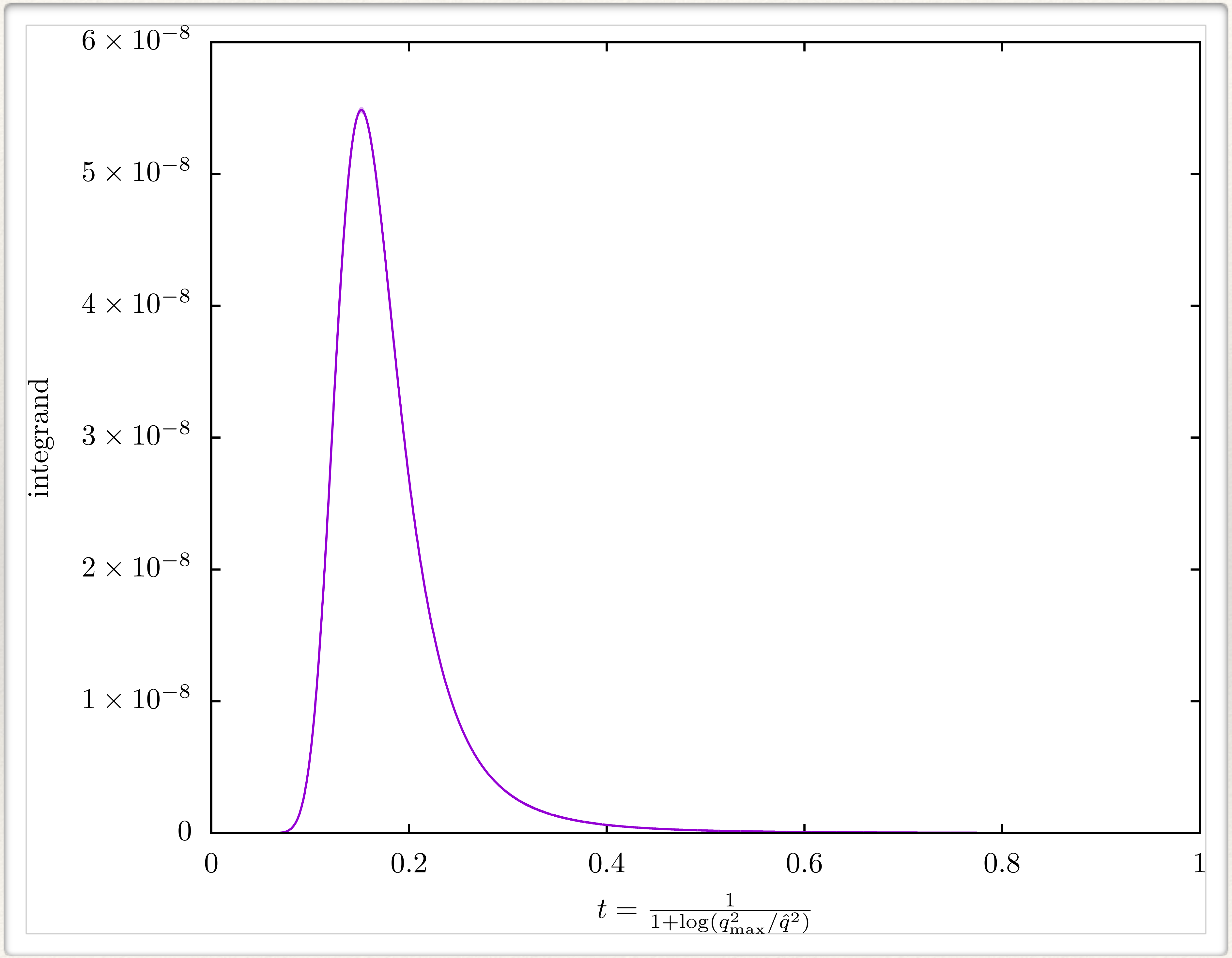
etc…

- Continuous moments are valid in finite volume up to exponentially small finite-volume effects

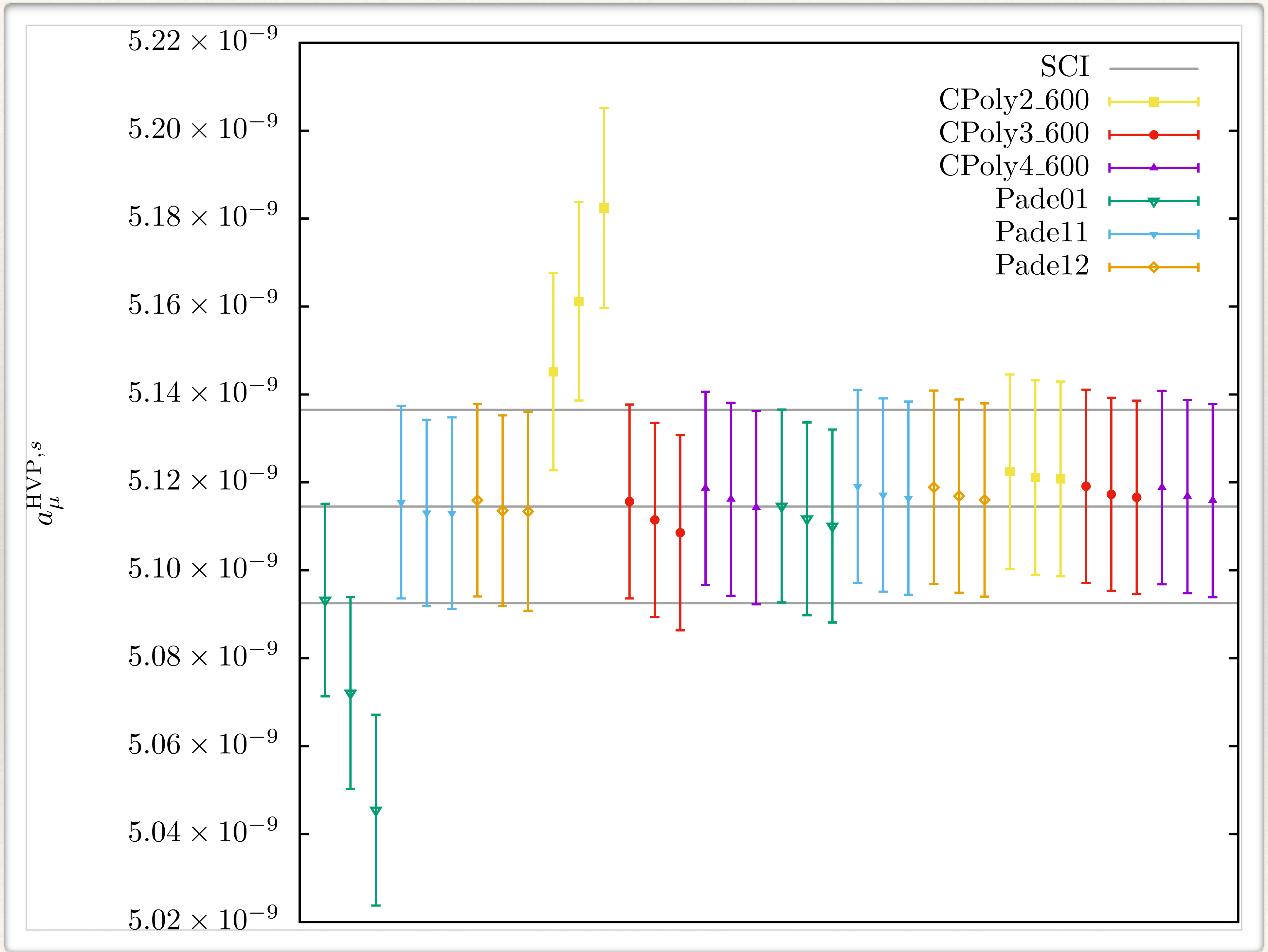
Preliminary results



Preliminary results



Preliminary results



Conclusions

- It is possible to have easily a model-independent, smooth description of the HVP form factor at low q^2.
- The SCI and MWS theorem give the mathematical foundations for “continuous-momenta” methods
- The resulting interpolation error is comparable to the existing physical finite-volume effects.
- This method is in very good agreement with a multitude of low-q^2 models.
- Apologies for the sloppiness, all the mathematical details will be released in a future write-up.

Outlook

- High-precision data needed: what about light quarks?
- Quantitative finite-volume effects
- Applicable in other lattice computations?

Thank you!