# QED Corrections to Hadronic Processes in Lattice QCD

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# 1. Introduction

N.Carrasco, V.Lubicz, G.Martinelli, CTS, N.Tantalo, C.Tarantino, M.Testa
(Phys.Rev.**D91** (2015) 074506)

- The precision of lattice calculations is such that Isospin-breaking effects, including electromagnetic corrections to hadronic masses, are now being calculated. For a review see A.Portelli at Lattice 2014. arXiv:1505.07057
  - A highlight has been the *Ab initio calculation of the neutron-proton mass difference* by the BMW collaboration. S.Borsanyi et al., arXiv:1406.4088
- In this talk I review our proposal to calculate electromagnetic corrections to matrix elements.
  - The new feature is the presence of infrared divergences.
- This is necessary for further progress in phenomenology, since the results of (some) weak matrix elements obtained from lattice QCD are now being quoted with $O(1\%)$ precision or better, e.g. FLAG Collaboration, arXiv:1310.8555

| $f_\pi$ | $f_K$ | $f_D$ | $f_{D_s}$ | $f_B$ | $f_{B_s}$ |
|---|---|---|---|---|---|
| 130.2(1.4) | 156.3(0.8) | 209.2(3.3) | 248.6(2.7) | 190.5(4.2) | 227.7(4.5) |

(results given in MeV)

- For illustration, we consider $f_\pi$ but the discussion is general; we do not use ChPT. For a ChPT based discussion of $f_\pi$, see J.Gasser & G.R.S.Zarnauskas, arXiv:1008.3479

## Infrared Divergences

- At $O(\alpha^0)$

$$\Gamma(\pi^+ \to \ell^+ \nu_\ell) = \frac{G_F^2\,|V_{ud}|^2 f_\pi^2}{8\pi}\, m_\pi\, m_\ell^2 \left(1 - \frac{m_\ell^2}{m_\pi^2}\right)^2 .$$

- At $O(\alpha)$ infrared divergences are present and we have to consider

$$\begin{aligned}\Gamma(\pi^+ \to \ell^+ \nu_\ell(\gamma)) &= \Gamma(\pi^+ \to \ell^+ \nu_\ell) + \Gamma(\pi^+ \to \ell^+ \nu_\ell \gamma) \\ &\equiv \Gamma_0 + \Gamma_1 \,,\end{aligned}$$

  where the suffix denotes the number of photons in the final state.
- Each of the two terms on the rhs is infrared divergent; the divergences cancel in the sum.
- The cancelation of infrared divergences between contributions with virtual and real photons is an old and well understood issue.

  F.Bloch and A.Nordsieck, PR 52 (1937) 54
- The question for our community is how best to combine this understanding with lattice calculations of non-perturbative hadronic effects.
- This is a generic problem if em corrections are to be included in the evaluation of a decay process.

- As techniques and resources improve in the future, it may be better to compute $\Gamma_1$ nonperturbatively over a larger range of photon energies.
- At present we do not propose to compute $\Gamma_1$ nonperturbatively. Rather we consider only photons which are sufficiently soft for the point-like (pt) approximation to be valid.
  - A cut-off $\Delta E$ of $O(10\text{-}20\,\mathrm{MeV})$ appears to be appropriate both experimentally and theoretically.

    F.Ambrosino et al., KLOE collaboration, hep-ex/0509045; arXiv:0907.3594
- We now write

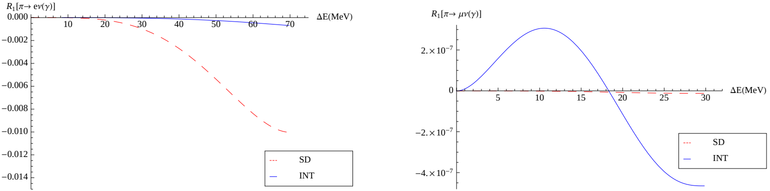
$$\boxed{\Gamma_0 + \Gamma_1(\Delta E) = \lim_{V\to\infty}(\Gamma_0 - \Gamma_0^{\mathrm{pt}}) + \lim_{V\to\infty}(\Gamma_0^{\mathrm{pt}} + \Gamma_1(\Delta E))\,.}$$

  - The second term on the rhs can be calculated in perturbation theory. It is infrared convergent, but does contain a term proportional to $\log\Delta E$.
  - The first term is also free of infrared divergences.
  - $\Gamma_0$ is calculated nonperturbatively and $\Gamma_0^{\mathrm{pt}}$ in perturbation theory.

- The size of the neglected structure-dependent contributions can be estimated using ChPT.

J.Bijnens, G.Ecker and J.Gasser, hep-ph/9209261, V. Cirigliano and I. Rosell, arXiv:0707.3439

Pion




$$R_1^A(\Delta E) = \frac{\Gamma_1^A(\Delta E)}{\Gamma_0^{\alpha,\mathrm{pt}} + \Gamma_1^{\mathrm{pt}}(\Delta E)}\,, \qquad A = \{\mathrm{SD,INT}\}\,,$$

- For the $B$-meson, for which we cannot use ChPT, we have another small scale $< \Lambda_{\mathrm{QCD}}$, $m_{B^*} - m_B \simeq 45$ MeV so that we may expect that we will have to go to smaller $\Delta E$ in order to be able to neglect SD effects.

# Outline of Talk

$$\boxed{\Gamma_0 + \Gamma_1(\Delta E) = \lim_{V\to\infty}(\Gamma_0 - \Gamma_0^{\rm pt}) + \lim_{V\to\infty}(\Gamma_0^{\rm pt} + \Gamma_1(\Delta E))\,.}$$

1. Introduction
2. What is $G_F$ at $O(\alpha)$?
3. Proposed (and ongoing) calculation of $\Gamma_0 - \Gamma_0^{\rm pt}$
4. Calculation of $\Gamma_0^{\rm pt} + \Gamma_1(\Delta E)$
5. Summary and Conclusions

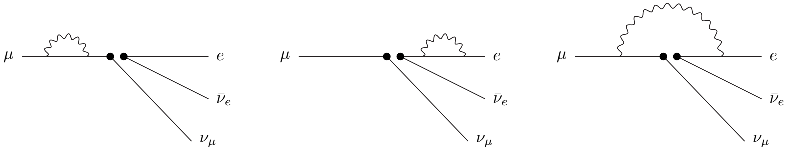
## 2. What is $G_F$ at $O(\alpha)$?

1. The results for the widths are expressed in terms of $G_F$, the Fermi constant $(G_F = 1.16632(2)\times 10^{-5}\,\mathrm{GeV}^{-2})$. This is obtained from the muon lifetime:

$$\frac{1}{\tau_\mu} = \frac{G_F^2 m_\mu^5}{192\pi^3}\left[1-\frac{8m_e^2}{m_\mu^2}\right]\left[1+\frac{\alpha}{2\pi}\left(\frac{25}{4}-\pi^2\right)\right].$$

S.M.Berman, PR 112 (1958) 267; T.Kinoshita and A.Sirlin, PR 113 (1959) 1652

- This expression can be viewed as the definition of $G_F$. Many EW corrections are absorbed into the definition of $G_F$; the explicit $O(\alpha)$ corrections come from the following diagrams in the effective theory:

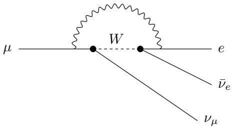


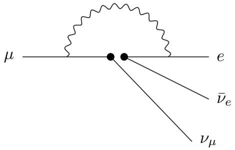

  together with the diagrams with a real photon.

- These diagrams are evaluated in the $W$-regularisation in which the photon propagator is modified by: A.Sirlin, PRD 22 (1980) 971

$$\frac{1}{k^2} \to \frac{M_W^2}{M_W^2-k^2}\,\frac{1}{k^2}. \qquad \left(\frac{1}{k^2} = \frac{1}{k^2-M_W^2} + \frac{M_W^2}{M_W^2-k^2}\,\frac{1}{k^2}\right)$$

- The $\gamma - W$ box diagram:




As an example providing some evidence & intuition that the W-regularization is useful consider the $\gamma - W$ box diagram.

- In the standard model (left-hand diagram) it contains both the $\gamma$ and $W$ propagators.
- In the effective theory this is preserved with the $W$-regularization where the photon propagator is proportional to

$$\frac{1}{k^2}\,\frac{1}{k^2 - M_W^2}$$

and the two diagrams are equal up to terms of $O(q^2/M_W^2)$, where $q$ is the momentum of the $e$ and $\nu_e$.

## 3. Proposed calculation of $\Gamma_0 - \Gamma_0^{\mathrm{pt}}$

- Most (but not all) of the EW corrections which are absorbed in $G_F$ are common to other processes (including pion decay) $\Rightarrow$ factor in the amplitude of $(1+3\alpha/4\pi(1+2\bar{Q})\log M_Z/M_W)$, where $\bar{Q}=\frac{1}{2}(Q_u+Q_d)=1/6$.

  A.Sirlin, NP B196 (1982) 83; E.Braaten & C.S.Li, PRD 42 (1990) 3888

- We therefore need to calculate the pion-decay diagrams in the effective theory with

$$H_{\mathrm{eff}}=\frac{G_F}{\sqrt{2}}\,V_{ud}^*\left(1+\frac{\alpha}{\pi}\log\frac{M_Z}{M_W}\right)(\bar{d}_L\gamma^\mu u_L)(\bar{\nu}_{\ell,L}\,\gamma_\mu\ell_L)$$

  in the $W$-regularization.

- Thus for example, with the Wilson action for both the gluons and fermions:

$$\begin{aligned}O_1^{\mathrm{W-reg}} &= \left(1+\frac{\alpha}{4\pi}\left(2\log a^2M_W^2-15.539\right)\right)O_1^{\mathrm{bare}}+\frac{\alpha}{4\pi}\,\Big(0.536\,O_2^{\mathrm{bare}}\\ &\qquad +1.607\,O_3^{\mathrm{bare}}-3.214\,O_4^{\mathrm{bare}}-0.804\,O_5^{\mathrm{bare}}\Big)\,,\end{aligned}$$

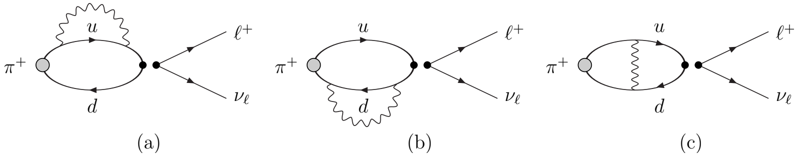
where

$$O_1=(\bar{d}\gamma^\mu(1-\gamma^5)u)\,(\bar{\nu}_\ell\gamma_\mu(1-\gamma^5)\ell)\qquad O_2=(\bar{d}\gamma^\mu(1+\gamma^5)u)\,(\bar{\nu}_\ell\gamma_\mu(1-\gamma^5)\ell)$$

$$O_3=(\bar{d}(1-\gamma^5)u)\,(\bar{\nu}_\ell(1+\gamma^5)\ell)\qquad O_4=(\bar{d}(1+\gamma^5)u)\,(\bar{\nu}_\ell(1+\gamma^5)\ell)$$

$$O_5=(\bar{d}\sigma^{\mu\nu}(1+\gamma^5)u)\,(\bar{\nu}_\ell\sigma_{\mu\nu}(1+\gamma^5)\ell)\,.$$

Consider now the evaluation of $\Gamma_0$.




- The correlation function for this set of diagrams is of the form:

$$C_1(t) = -\frac{1}{2}\int d^3\vec{x}\, d^4x_1\, d^4x_2\, \langle 0|T\big\{J_W^\nu(0)\, j_\mu(x_1) j_\mu(x_2)\phi^\dagger(\vec{x},-t)\big\}\,|\,0\rangle\; \Delta(x_1,x_2)\,,$$

  where $j_\mu(x) = \sum_f Q_f \bar{f}(x)\gamma_\mu f(x)$, $J_W$ is the weak current, $\phi$ is an interpolating operator for the pion and $\Delta$ is the photon propagator.

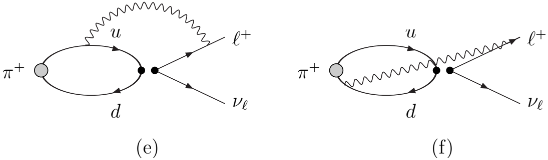
- Combining $C_1$ with the lowest order correlator:

$$C_0(t) + C_1(t) \simeq \frac{e^{-m_\pi t}}{2m_\pi} Z^\phi\, \langle\, 0\,| J_W^\nu(0)\,|\,\pi^+\rangle\,,$$

  where now $O(\alpha)$ terms are included.

- $e^{-m_\pi t} \simeq e^{-m_\pi^0 t}\,(1 - \delta m_\pi\, t)$ and $Z^\phi$ is obtained from the two-point function.

## Proposed calculation of $\Gamma_0 - \Gamma_0^{\text{pt}}$ (Cont)

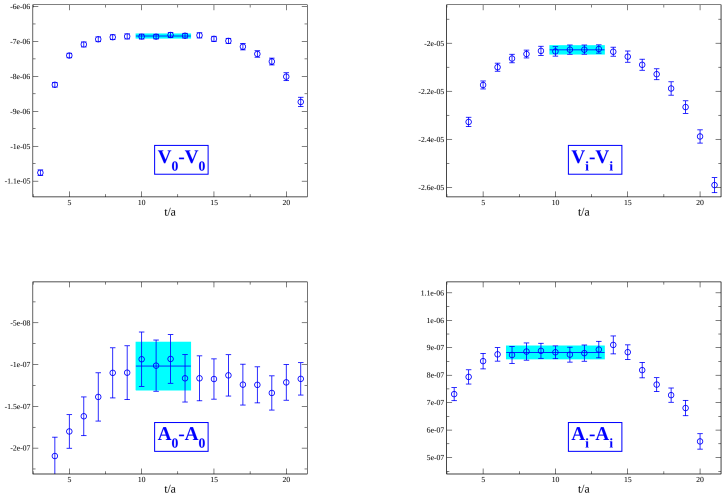



$$\begin{aligned}\bar{C}_1(t)_{\alpha\beta} &= -\int d^3\vec{x}\,d^4x_1\,d^4x_2\,\langle 0|T\left\{J_W^\nu(0)\,j_\mu(x_1)\,\phi^\dagger(\vec{x},-t)\right\}|\,0\rangle\,\Delta(x_1,x_2)\\ &\qquad\times\left(\gamma_\nu(1-\gamma^5)S(0,x_2)\gamma_\mu\right)_{\alpha\beta}\,e^{E_\ell\,t_2}\,e^{-i\vec{p}_\ell\cdot\vec{x}_2}\\ &\simeq Z_0^\phi\,\frac{e^{-m_\pi^0 t}}{2m_\pi^0}\,(\bar{M}_1)_{\alpha\beta}\end{aligned}$$

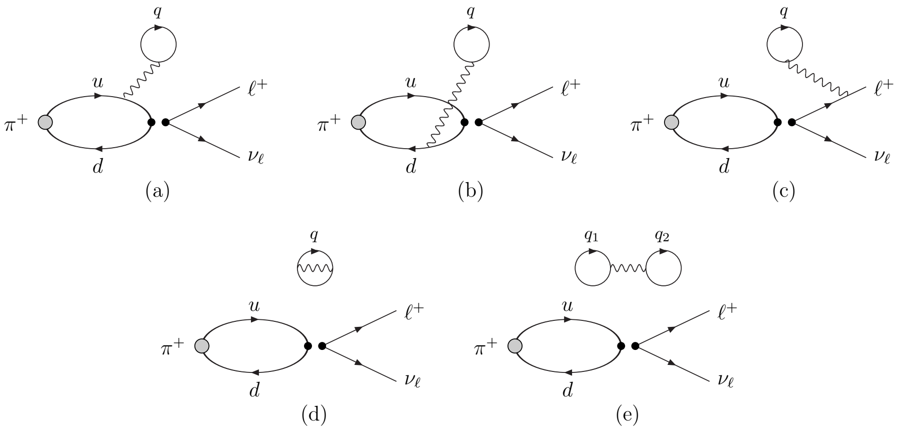
- Corresponding contribution to the amplitude is $\bar{u}_\alpha(p_{\nu_\ell})(\bar{M}_1)_{\alpha\beta}v_\beta(p_\ell)$.
- Diagrams (e) and (f) are not simply generalisations of the evaluation of $f_\pi$.
- The lepton's wave function renormalisation cancels in the difference $\Gamma_0 - \Gamma_0^{\text{pt}}$.
- We have to be able to isolate the finite-volume ground state (pion).
- The Minkowski $\leftrightarrow$ Euclidean continuation can be performed (the time integrations are convergent).

Twisted-mass study, $24^3 \times 48$ lattice with $a = 0.086\,\mathrm{fm}$, $m_\pi \simeq 500\,\mathrm{MeV}$, 240 configs with 3 stochastic sources per configuration.

together with F.Sanfilippo and S.Simula



# There are also disconnected diagrams to be evaluated.



# 4. Calculation of $\Gamma^{\rm pt}=\Gamma_0^{\rm pt}+\Gamma_1^{\rm pt}$

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- The total width, $\Gamma^{\rm pt}$ was calculated in 1958/9 using a Pauli-Villars regulator for the UV divergences and $m_\gamma$ for the infrared divergences.

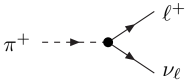
  S.Berman, PR **112** (1958) 267, T.Kinoshita, PRL **2** (1959) 477

  - This is a useful check on our perturbative calculation.

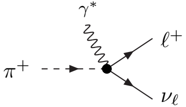
- In the perturbative calculation we use the following Lagrangian for the interaction of a point-like pion with the leptons:

$$\mathcal{L}_{\pi\text{-}\ell\text{-}\nu_\ell} \;=\; i\,G_F f_\pi V_{ud}^*\;\{(\partial_\mu-ieA_\mu)\pi\}\left\{\bar\psi_{\nu_\ell}\frac{1+\gamma_5}{2}\gamma^\mu\psi_\ell\right\}+\text{H.C.}\,.$$

The corresponding Feynman rules are:



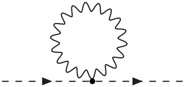

$$= -iG_F\,f_\pi V_{ud}^*\,p_\pi^\mu\,\tfrac{1+\gamma^5}{2}\gamma_\mu$$

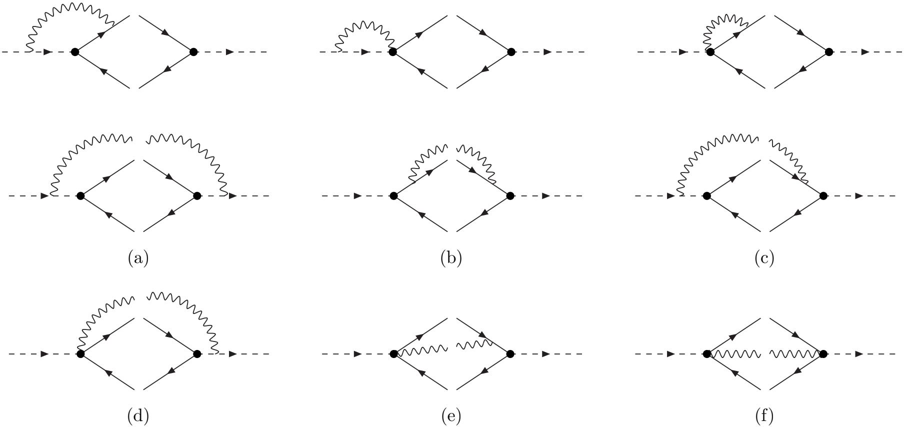



$$= ie\,G_F\,f_\pi V_{ud}^*\,g^{\mu\nu}\,\tfrac{1+\gamma^5}{2}\gamma_\mu$$

## Diagrams to be evaluated

and





- We find, for $E_\gamma<\Delta E$

$$\begin{aligned}\Gamma^{\rm pt}(\Delta E) \;=\; \Gamma_0^{\rm tree}\times\Bigg(1+\frac{\alpha}{4\pi}\Bigg\{&3\log\left(\frac{m_\pi^2}{M_W^2}\right)+\log\left(r_\ell^2\right)-4\log(r_E^2)+\frac{2-10r_\ell^2}{1-r_\ell^2}\log(r_\ell^2)\\ &-2\frac{1+r_\ell^2}{1-r_\ell^2}\log(r_E^2)\log(r_\ell^2)-4\frac{1+r_\ell^2}{1-r_\ell^2}\,\mathrm{Li}_2(1-r_\ell^2)-3\\ &+\Bigg[\frac{3+r_E^2-6r_\ell^2+4r_E(-1+r_\ell^2)}{(1-r_\ell^2)^2}\log(1-r_E)+\frac{r_E(4-r_E-4r_\ell^2)}{(1-r_\ell^2)^2}\log(r_\ell^2)\\ &-\frac{r_E(-22+3r_E+28r_\ell^2)}{2(1-r_\ell^2)^2}-4\frac{1+r_\ell^2}{1-r_\ell^2}\,\mathrm{Li}_2(r_E)\Bigg]\Bigg\}\Bigg)\,,\end{aligned}$$

where $r_E=2\Delta E/m_\pi$ and $r_\ell=m_\ell/m_\pi$.

- We believe that this is a new result.

- The total rate is readily computed by setting $r_E$ to its maximum value, namely $r_E = 1 - r_\ell^2$, giving

$$\Gamma^{\rm pt} = \Gamma_0^{\rm tree} \times \Bigg\{ 1 + \frac{\alpha}{4\pi} \left( 3\log\left(\frac{m_\pi^2}{M_W^2}\right) - 8\log(1-r_\ell^2) - \frac{3r_\ell^4}{(1-r_\ell^2)^2}\log(r_\ell^2) \right.$$
$$\left. -8\frac{1+r_\ell^2}{1-r_\ell^2}\,\mathrm{Li}_2(1-r_\ell^2) + \frac{13-19r_\ell^2}{2(1-r_\ell^2)} + \frac{6-14r_\ell^2-4(1+r_\ell^2)\log(1-r_\ell^2)}{1-r_\ell^2}\log(r_\ell^2) \right) \Bigg\} .$$

- This result agrees with the well known results in literature providing an important check of our calculation.
- Summary: The perturbative calculation of $\Gamma_0^{\rm pt} + \Gamma_1(\Delta E)$ is done.

## 5. Summary and Conclusions

- Lattice calculations of some physical quantities are approaching $O(1\%)$ precision $\Rightarrow$ we need to include isospin-breaking effects, including electromagnetic effects, to make the tests of the SM even more stringent.

- For decay widths and scattering cross sections including em effects introduces infrared divergences.

- We propose a method for dealing with these divergences, illustrating the procedure by a detailed study of the leptonic (and semileptonic) decays of pseudoscalar mesons.

- Although challenging, the method is within reach of present simulations and we are now implementing the procedure in an actual numerical computation.
  - Power-like FV corrections, $O(1/(L\Lambda_{\rm QCD})^n)$, to be evaluated.
  - $O(\alpha\alpha_s)$ matching factors to be studied.

## 5. Summary and Conclusions (cont.)



- One can certainly envisage relaxing the condition $\Delta E \ll \Lambda_{\rm QCD}$, including the emission of real photons with energies which do resolve the structure of the initial hadron. Such calculations can be performed in Euclidean space under the same conditions as above, i.e. providing that there is a mass gap.
  - In that case we generalise the master formula to

$$\Gamma_0 + \Gamma_1(\Delta E) = \lim_{V\to\infty}(\Gamma_0 - \Gamma_0^{\rm pt}) + \lim_{V\to\infty}(\Gamma_1(\Delta E) - \Gamma_1^{\rm pt}(\Delta E)) + \lim_{V\to\infty}(\Gamma_0^{\rm pt} + \Gamma_1^{\rm pt}(\Delta E)).$$

  - The important point is to organise the calculation into terms, each of which is infrared convergent.
  - $\Gamma_0^{\rm pt} + \Gamma_1^{\rm pt}(\Delta E)$ (in infinite volume) is done.
  - At present we are exploring how best to calculate

$$\lim_{V\to\infty}(\Gamma_0 - \Gamma_0^{\rm pt})$$

  and exploratory numerical calculations are underway.

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