Domain-wall/overlap, and topological insulators

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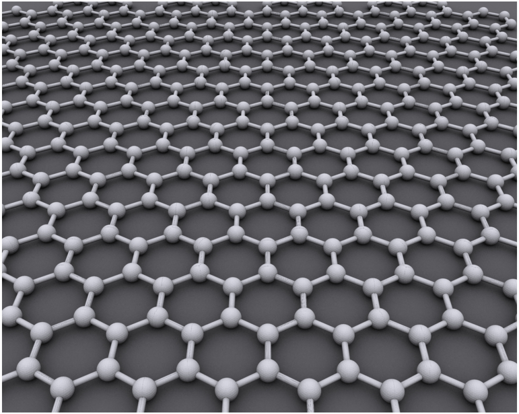
collaboration with T. Morimoto (RIKEN)

Relativistic fermions in cond-mat systems

- Graphene



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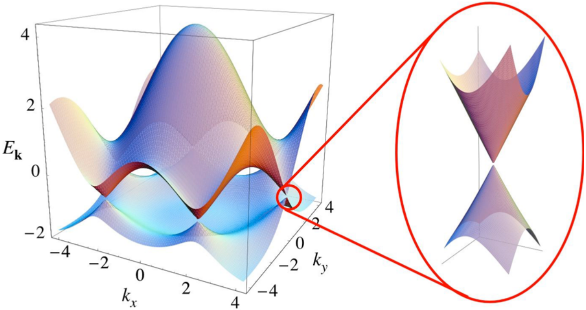

Relativistic fermions in cond-mat systems

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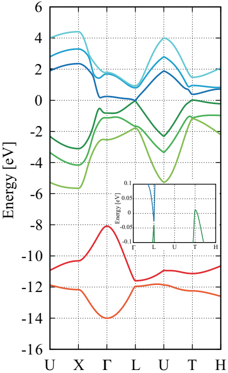

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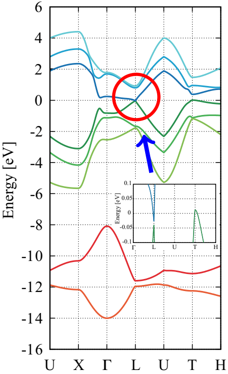

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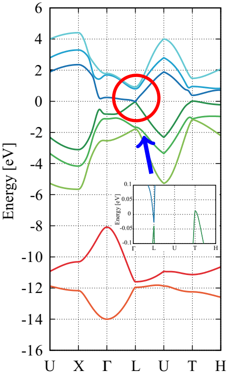
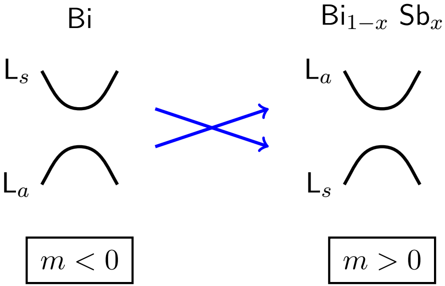


Mass = Band gap

"Mass" manipulation

- $\mathsf{Bi}_{1-x}\ \mathsf{Sb}_x$



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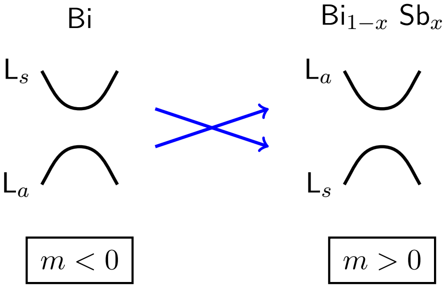

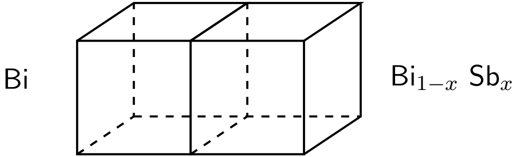

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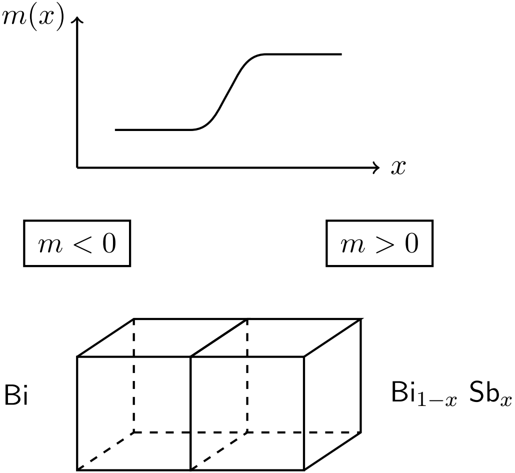

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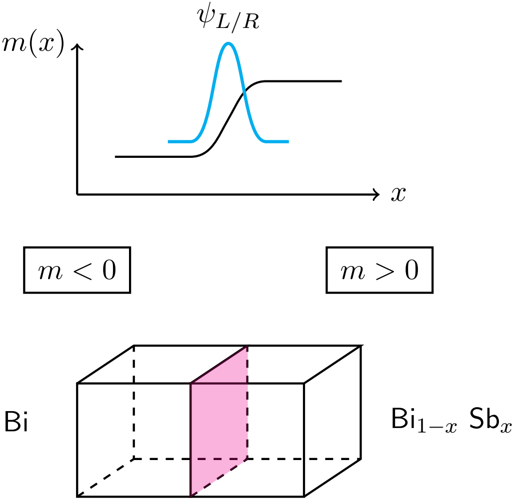
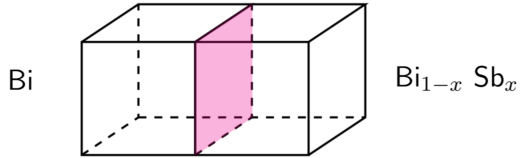


- A massless fermion carries topological charge (à la index thm)



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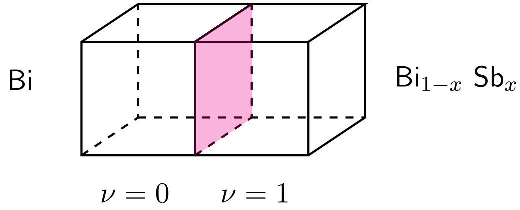

  - Topological number

$$\nu = \frac{1}{4\pi^2} \int \mathrm{Tr} \left(\mathcal{A} d\mathcal{A} + \frac{2}{3} \mathcal{A}^3 \right)$$

$\mathcal{A}$: Berry connection

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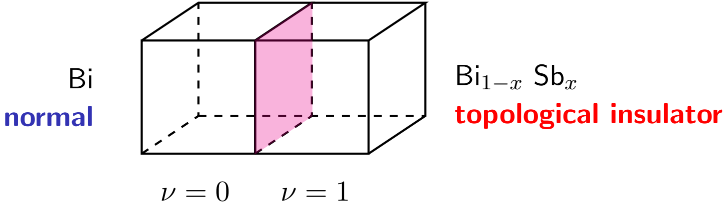

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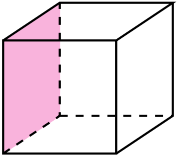

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topological insulator

$\nu = 0 \qquad \nu = 1$

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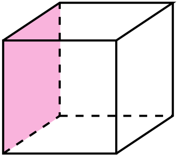

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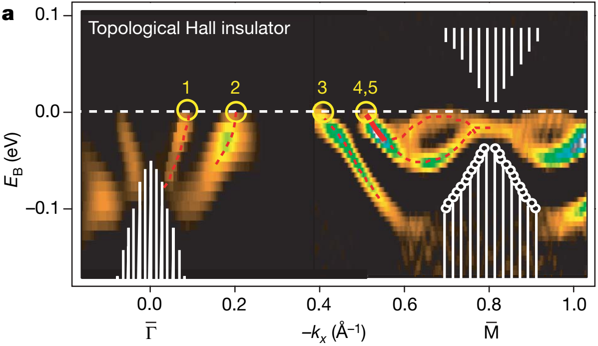
TI surface state = domain-wall fermion

Topological insulator

- Bi Sb, Bi Se [Hsieh et al.] [Zhang et al.]



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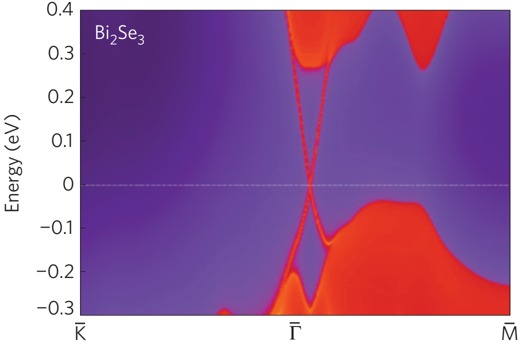
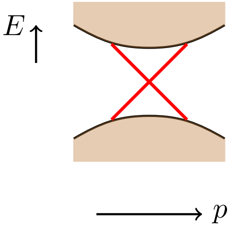


- Effective theory for (massless) surface states



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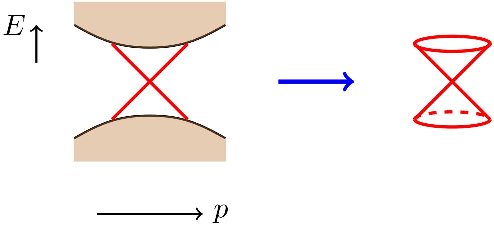

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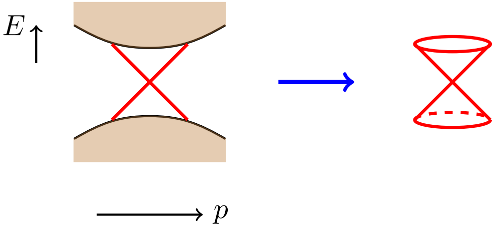

- Subtracting the bulk contribution [Neuberger]

Overlap formula

$$\mathcal{Z}_{\text{eff}} = \frac{\det D_{\text{open}}}{\det D_{\text{period}}} = \det D_{\text{eff}}$$

Overlap properties

Ginsparg–Wilson relation

$$\gamma_5 D + D\gamma_5 = aD\gamma_5 D$$

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- Discretization effect (finite a)

$$\gamma_5 D + D\Gamma_5 = 0 \quad \text{w/} \quad \Gamma_5 = \gamma_5 (1 - aD)$$

⟶ particle-antiparticle (hole) asymmetry $(d = 3 + 1)$

[Lüscher]

$$\psi \to e^{i\theta\Gamma_5}\psi\,, \qquad \bar{\psi} \to \bar{\psi}\, e^{i\theta\gamma_5}$$

Another expression

$$D + D^\dagger = aD^\dagger D \qquad \left(\gamma_5 D \gamma_5 = D^\dagger\right)$$

- A solution: $D = \dfrac{1}{a}(1 - V)\,, \quad V^\dagger = V^{-1}$

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- Particle-hole asymmetry in $d = 2+1$

  [Bietenholz-Nishimura] [Kikukawa-Neuberger]

$$\psi \;\to\; i\,\mathcal{R}\,V\,\psi\,, \quad \bar\psi \;\to\; i\,\bar\psi\,\mathcal{R}$$

or $$\psi \;\to\; i\,\mathcal{R}\,\psi\,, \quad \bar\psi \;\to\; i\,\bar\psi\,V\,\mathcal{R}$$

  - $\mathcal{R}$: reflection operator $(x,y,z) \;\to\; (-x,-y,-z)$

$$\mathcal{R}\,D\,\mathcal{R} = D^\dagger$$

How to detect it?

- Magneto-optical conductivity

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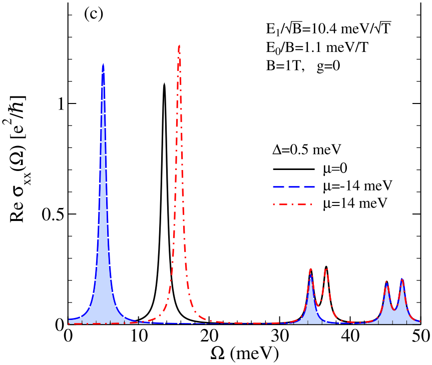

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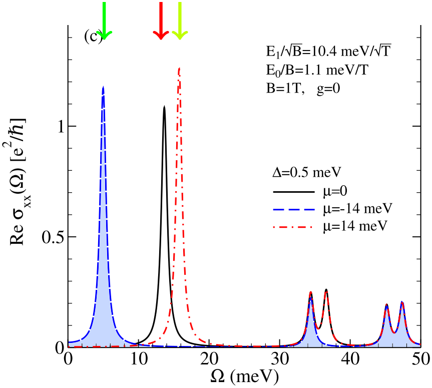

Asymmetric shift only for zero modes

- What's the role of dimensions and symmetry?
  - $d = 3 + 1 \longrightarrow$ chiral anomary
  - $d = 2 + 1 \longrightarrow$ parity anomary

Classification of topological phases

- Periodic table [Schnyder et al.] [Kitaev] (cf. [Altland–Zirnbauer])

		TRS	PHS	SLS	$d=1$	$d=2$	$d=3$
Standard	A (unitary)	0	0	0	-	$\mathbb{Z}$	-
(Wigner-Dyson)	AI (orthogonal)	+1	0	0	-	-	-
	AII (symplectic)	−1	0	0	-	$\mathbb{Z}_2$	$\mathbb{Z}_2$
Chiral	AIII (chiral unitary)	0	0	1	$\mathbb{Z}$	-	$\mathbb{Z}$
(sublattice)	BDI (chiral orthogonal)	+1	+1	1	$\mathbb{Z}$	-	-
	CII (chiral symplectic)	−1	−1	1	$\mathbb{Z}$	-	$\mathbb{Z}_2$
BdG	D	0	+1	0	$\mathbb{Z}_2$	$\mathbb{Z}$	-
	C	0	−1	0	-	$\mathbb{Z}$	-
	DIII	−1	+1	1	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
	CI	+1	−1	1	-	-	$\mathbb{Z}$

- Dimensions: $d = 0, 1, 2, 3, \dots$

- Symmetry: T/CP, C, chiral

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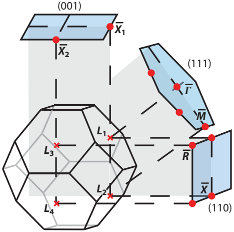
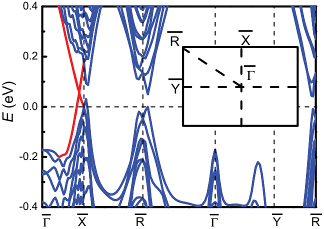
- Dimensions: $d = 0, 1, 2, 3, \ldots$
- Symmetry: T/CP, C, chiral $\longrightarrow$ **additional**: crystal structure

Symmetry-Protected Topological (SPT) phase

- Topological crystalline insulator [Liu–Duan–Fu]





  - Symmetry: x & y-reflection, C_2 rotation

- **Q.** Overlap? Ginsparg–Wilson relation?

Reflection symmetry

- x-reflection operator:

$$\mathcal{R}_x \, D(x,y,z) \, \mathcal{R}_x \;=\; D(-x,y,z)$$

- Invariant surface:

$$\mathcal{R}_x \, D(x,y,z) \, \mathcal{R}_x \;=\; D(x,y,z) \quad \text{at} \quad x = 0$$

Ginsparg–Wilson relation

- Chiral operator: $\Gamma_x = i\gamma_x \mathcal{R}_x$

$$\longrightarrow \qquad \Gamma_x D + D\Gamma_x = aD\Gamma_x D$$

  - Topological class: $\mathbb{Z}_2$ (parity) $\longrightarrow$ $\mathbb{Z}$ (chiral)

Summary

- Mass $=$ Band gap

  $\longrightarrow$ Domain-wall & overlap on the surface

- Ginsparg–Wilson relation and its consequence

  $\longrightarrow$ particle-hole asymmetry

- Ginsparg–Wilson formalism for the additional symmetry

  $\longrightarrow$ reflection symmetry and classification $\mathbb{Z}_2 \rightarrow \mathbb{Z}$

- Many possible applications

  $\longrightarrow$ index thm, admissibility

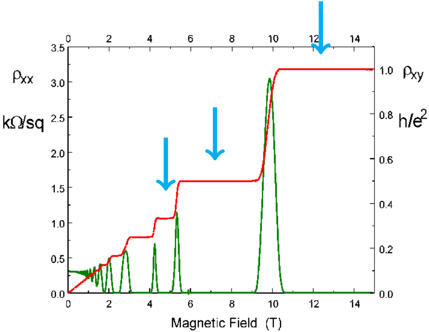
What is the topological phase?

- (Thermodynamic) phases $\longrightarrow$ **bulk** property
  - ex.) H_2O : ice/water/vapor
  - Phase transition : free energy singularity

- Topological phases $\longrightarrow$ **boundary** property
  - ex.) QHE : $\sigma_{\mathsf{H}} = \nu \dfrac{e^2}{h}, \quad \nu \in \mathbb{Z}$ (topological num.)
  - Phase transition : topology change

What is the topological phase?

- ex.) Quantum Hall state



Quantum Hall effect

$$\sigma_{xx} = 0\,, \quad \sigma_{xy} = \nu \frac{e^2}{h} \quad (\nu \in \mathbb{Z})$$

What is the topological phase?

- What is the effective field theory for QHE?
  - Dimension $d = 2 + 1$
  - Parity broken (due to B)

Chern–Simons theory

$$S_{\mathsf{CS}} = \frac{k}{4\pi} \int A dA$$

- Current: $J_\mu = \frac{\delta S_{\mathsf{CS}}}{\delta A^\mu} = \frac{k}{2\pi} \epsilon_{\mu\nu\rho} \partial^\nu A^\rho \longrightarrow J_\mu = \frac{k}{2\pi} \epsilon_{\mu\nu} E^\nu$

$$\sigma_{xx} = 0 \, , \quad \sigma_{xy} = \frac{k}{2\pi} \left(\frac{e^2}{\hbar} \right) = k \frac{e^2}{h} \quad (k \in \mathbb{Z})$$

What is the topological phase?

- Topological terms in QFT

$$\frac{1}{32\pi^2}\int d^4x\,\theta(x)\,\epsilon_{\mu\nu\rho\sigma}\,F^{\mu\nu}F^{\rho\sigma} \quad \longrightarrow \quad d=3 \text{ TI}$$

$$\frac{1}{192\pi}\int d^3x\,\epsilon_{\mu\nu\rho}\ \mathrm{Tr}\left[\omega^{\mu}\partial^{\mu}\omega^{\rho}+\frac{2}{3}\omega^{\mu}\omega^{\nu}\omega^{\rho}\right] \quad \longrightarrow \quad d=2 \text{ TSC}$$

$$\frac{1}{1534\pi^2}\int d^4x\,\theta(x)\,\epsilon_{\mu\nu\rho\sigma}\,R^{\alpha}{}_{\beta}{}^{\mu\nu}R^{\beta}{}_{\alpha}{}^{\rho\sigma} \quad \longrightarrow \quad d=3 \text{ TSC}$$

Wavefunction topology

- ex.) Massive Dirac fermion in $d = 1$

$$\mathcal{H}(k) = \begin{pmatrix} & m - ik \\ m + ik & \end{pmatrix}$$

- Topological #: the base (k-space) to the Hilbert space

$$\nu = \frac{1}{2\pi i} \int_{-\infty}^{\infty} dk \, \mathcal{A}_k = \frac{1}{2} \operatorname{sgn}(m) \qquad \text{w/} \quad \mathcal{A}_k = \psi^\dagger \partial_k \psi$$

- Topology change: $\Delta\nu = \pm 1$ at $m = 0$

Wavefunction topology

- ex.) Wilson fermion in $d = 4$

$$D(k) = m + \sum_{\mu=1}^{4} i\gamma_\mu \sin k_\mu + r \sum_{\mu=1}^{4} (1 - \cos k_\mu)$$

- Topological #: the base (Brillouin zone) to the Hilbert space

$$\nu = \int_{\mathsf{BZ}} \mathrm{tr}\, \mathcal{F} \wedge \mathcal{F} \qquad \text{w/} \qquad \mathcal{A} = \psi^\dagger d\psi$$

- Topology change:

$$\Delta\nu = +1, -4, +6, -4, +1 \quad \text{at} \quad m = 0, -2r, -4r, -6r, -8r$$

Wavefunction topology

- Topological #: the base to the Hilbert space
- Massless point $\longrightarrow$ topology change

- cf.) Atiyah–Bott–Shapiro, ADHM/Nahm, K-theory, etc.