Hagedorn spectrum and equation of state of Yang-Mills theories[1]

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[1] M. Caselle, A. Nada, M. Panero, arXiv:1505.01106, accepted for publication in JHEP

Introduction and motivations

One of the main features of SU(N) non-abelian gauge theories is the existence of a deconfinement phase transition, i.e. a temperature above which gluons are "deconfined". Our goal is to study the thermodynamics of pure gauge theories in the confining phase when approaching the deconfinement transition from below.

In the confining phase the **only** degrees of freedom of the theory without quarks are the **glueballs**: looking at the thermodynamics in the confining phase we have a tool to explore the glueball spectrum of the theory.

Our main result is that the thermodynamics of the model can only be described assuming a **string-like** description of glueballs (and thus a Hagedorn spectrum). The fine details of the spectrum spectrum agree remarkably well with the predictions of the Nambu-Goto effective string. This turns out to be an highly non trivial test of the effective string picture of confinement.

This analysis was performed in the 3+1 dimensional SU(3) model in the pioneering work of Meyer[1]. Now, using high precision lattice data for SU(3)[2] and a new set of SU(2) data on (3+1) dimensions[3], we are in the position to refine the effective string analysis and test its predictive power. The present results confirm our previous findings[4] for (2+1) dimensional SU(N) theories (with $N = 2, 3, 4, 5, 6$).

[1] H. Meyer, *High-Precision Thermodynamics and Hagedorn Density of States*, 2009

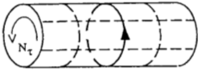
[2] Sz. Borsanyi *et al.*, *Precision SU(3) lattice thermodynamics for a large temperature range*, 2012

[3] M. Caselle, A. Nada, M. Panero, *Hagedorn spectrum and thermodynamics of SU(N) Yang-Mills theories*, arXiv:1505.01106

[4] M. Caselle *et al.*, *Thermodynamics of SU(N) Yang-Mills theories in 2+1 dimensions I - The confining phase*, 2011

Thermodynamic quantities

On a $N_t \times N_s^3$ lattice the volume is $V = (aN_s)^3$ (where a is the lattice spacing), while the temperature is determined by the inverse of the temporal extent (with periodic boundary conditions): $T = (aN_t)^{-1}$.



The thermodynamic quantities taken into account will be:

- the **pressure** p, that in the thermodynamic limit (i.e. for large and homogenous systems) can be written as

$$p \simeq \frac{T}{V} \log Z(T, V)$$

- the **trace of the energy-momentum tensor** Δ, that in units of T^4 is

$$\frac{\Delta}{T^4} = \frac{\epsilon - 3p}{T^4} = T \frac{\partial}{\partial T} \left(\frac{p}{T^4} \right).$$

Energy density $\epsilon = \Delta + 3p$ and entropy density $s = \frac{\epsilon + p}{T} = \frac{\Delta + 4p}{T}$ can be easily calculated.

Thermodynamics on the lattice

The **pressure** can be estimated by the means of the so-called "integral method"[1]:

$$p(T) \simeq \frac{T}{V} \log Z(T,V) = \frac{1}{a^4} \frac{1}{N_t N_s^3} \int_0^{\beta(T)} d\beta' \frac{\partial \log Z}{\partial \beta'}.$$

It can be written (relative to its $T = 0$ vacuum contribution) as

$$\frac{p(T)}{T^4} = -N_t{}^4 \int_0^{\beta} d\beta' \left[3(P_\sigma + P_\tau) - 6P_0\right]$$

where P_σ and P_τ are the expectation values of spacelike and timelike plaquettes respectively and P_0 is the expectation value at zero T.

The **trace of energy-momentum tensor** is simply

$$\frac{\Delta(T)}{T^4} = T \frac{\partial}{\partial T} \left(\frac{p}{T^4} \right) = -N_t{}^4 T \frac{\partial \beta}{\partial T} \left[3(P_\sigma + P_\tau) - 6P_0\right].$$

[1] J. Engels *et al.*, *Nonperturbative thermodynamics of SU(N) gauge theories*, 1990

Ideal glueball gas

The behaviour of the system is supposed to be dominated by a gas of non-interacting glueballs. The prediction of an ideal relativistic Bose gas can be used to describe the thermodynamics of such gas. Its partition function for 3 spatial dimensions is

$$\log Z = (2J+1)\frac{2V}{T}\left(\frac{m^2}{2\pi}\right)^2 \sum_{k=1}^{\infty}\left(\frac{T}{km}\right)^2 K_2\left(k\frac{m}{T}\right)$$

where m is the mass of the glueball, J is its spin and K_2 is the modified Bessel function of the second kind of index 2.
Observables such as Δ and p thus can be easily derived:

$$p = \frac{T}{V}\log Z = 2(2J+1)\left(\frac{m^2}{2\pi}\right)^2 \sum_{k=1}^{\infty}\left(\frac{T}{km}\right)^2 K_2\left(k\frac{m}{T}\right)$$

$$\Delta = \epsilon - 3p = 2(2J+1)\left(\frac{m^2}{2\pi}\right)^2 \sum_{k=1}^{\infty}\left(\frac{T}{km}\right) K_1\left(k\frac{m}{T}\right)$$

Test with the SU(2) model

The SU(2) model is a perfect laboratory to test these results.

- It is easy to simulate: very precise results may be obtained with a reasonable amount of computing power.
- The masses of several states of the glueball spectrum are known with remarkable accuracy.
- The deconfinement transition is of second order and thus it is expected to coincide with the Hagedorn temperature, i.e. $T_c \equiv T_H$.
- The infrared physics of the model is very similar to that of the SU(3) theory, with one important difference: the gauge group is real and thus only $C = 1$ glueballs exist. Thus the glueball exponential spectrum contains only **half** of the states with respect to SU(3).

Scale setting

The SU(2) scale setting is fixed by calculating the string tension via the computation of **Polyakov loop correlators** with the multilevel algorithm.

The range of the parameter β which has been considered ($\beta \in [2.27, 2.6]$) covers approximately the temperature region analyzed in the finite temperature simulations.

The string tension is obtained with a two-parameter fit of

$$V = -\frac{1}{N_t} \log\langle PP \rangle$$

with the first order effective string prediction for the potential

$$V = \sigma r + V_0 - \frac{\pi}{12r}.$$

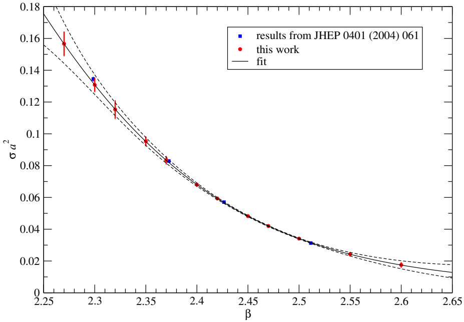
Higher order effective string corrections turned out to be negligible within the precision of our data.

Scale setting

The values of the string tension are interpolated by a fit to

$$\log(\sigma a^2) = \sum_{j=0}^{n_{\mathrm{par}}-1} a_j(\beta - \beta_0)^j \qquad \text{with } \beta_0 = 2.35 \text{ and } n_{\mathrm{par}} = 4$$

which yields a χ^2_{red} of 0.01. It is presented below along with older data[1].



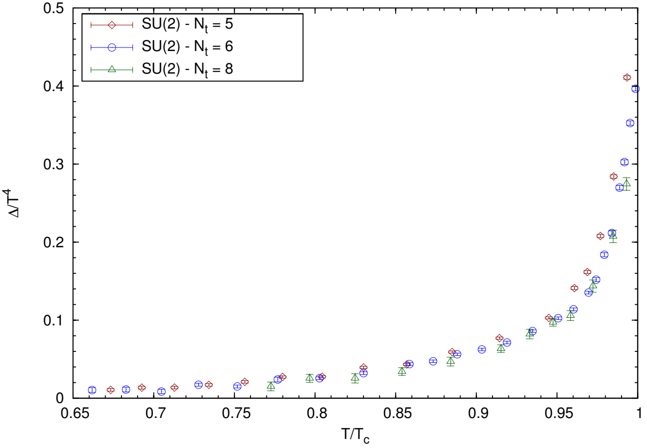
[1] B. Lucini, M. Teper, U. Wenger, *The high temperature phase transition in SU(N) gauge theories*, 2003

Lattice setup for finite temperature simulations

N_s^4 at $T = 0$	$N_s^3 \times N_t$ at $T \neq 0$	n_β	β-range	$n_{\rm conf}$
32^4	$60^3 \times 5$	17	$[2.25, 2.3725]$	1.5×10^5
40^4	$72^3 \times 6$	25	$[2.3059, 2.431]$	1.5×10^5
40^4	$72^3 \times 8$	12	$[2.439, 2.5124]$	10^5

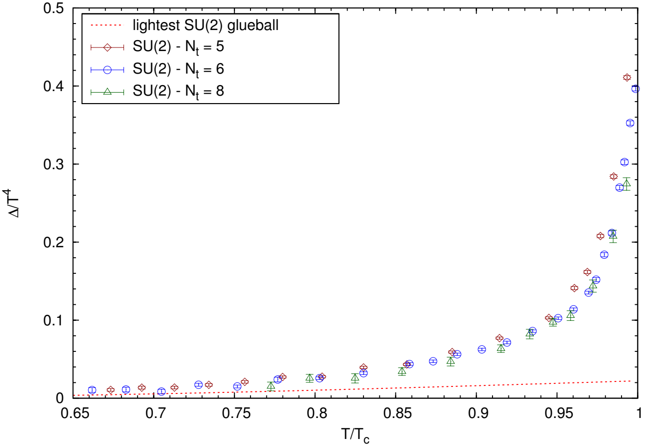
The first two columns show the lattice sizes (in units of the lattice spacing a) for the $T = 0$ and finite-temperature simulations, respectively. In the third column, n_β denotes the number of β-values simulated within the β-range indicated in the fourth column. Finally, in the fifth column we report the cardinality $n_{\rm conf}$ of the configuration set for the $T = 0$ and finite-T simulations.

SU(2): trace of energy-momentum tensor



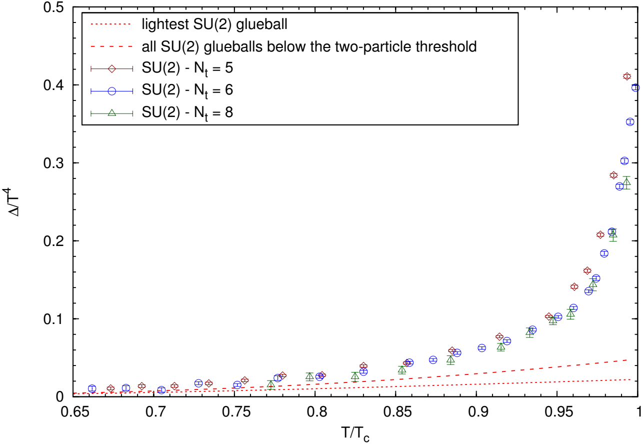
Despite the small values of N_t the data scale reasonably well.

SU(2): trace of energy-momentum tensor



Plot of the contribution of the lowest glueball state 0^{++} compared with the data.

SU(2): trace of energy-momentum tensor



The contribution of all SU(2) glueball states with mass $m < 2m_{0^{++}}$.

A few important observations

- Usually the thermodynamics of the system is saturated by the first state (or, in some cases, the few lowest states) of the spectrum due to the exponential dependence on the mass.
- The large gap between the $m_{0^{++}}$ and the $m < 2m_{0^{++}}$ curves and those between them and the data show that the spectrum must be of the **Hagedorn** type, i.e. that the number of states increases exponentially with the mass.
- A Hagedorn spectrum is typically the signature of a string-like origin of the spectrum.
- The thermal behaviour of the model in the confining phase is thus a perfect laboratory to study the nature of this spectrum and of the underlying string model.
- Effective string theory suggests that, with a very good approximation, this model should be a **Nambu-Goto** string.

Let us see the consequences of this assumption.

A **closed string model** for the full glueball spectrum that follows the original work of Isgur and Paton[1][2] can be introduced to account for the values of thermodynamic quantities near the transition. In the closed-string approach glueballs are described in the limit of large masses as **"rings of glue"**, that is closed tubes of flux modelled by closed bosonic string states.

The **mass spectrum** of a closed strings gas in D spacetime dimensions is given by

$$m^2 = 4\pi\sigma \left(n_L + n_R - \frac{D-2}{12} \right)$$

where $n_L = n_R = n$ are the total contribution of left- and right-moving phonons on the string.

[1] N. Isgur and J. Paton, *A Flux Tube Model for Hadrons in QCD*, 1985

[2] R. Johnson and M. Teper, *String models of glueballs and the spectrum of SU(N) gauge theories in (2+1)-dimensions*, 2002

Every glueball state corresponds to a given phonon configuration, but associated to each fixed n there are multiple different states whose number is given by $\pi(n)$, i.e. the **partitions** of n.

The **density of states** $\rho(n)$ is expressed through the square of $\pi(n)$

$$\rho(n) = \pi(n_L)\pi(n_R) = \pi(n)^2 \simeq 12\,(D-2)^{\frac{D-1}{2}} \left(\frac{1}{24n}\right)^{\frac{D+1}{2}} \exp\left(2\pi\sqrt{\frac{2(D-2)n}{3}}\right).$$

Spectral density

The spectral density as a function of the mass (i.e. $\hat{\rho}(m)\mathrm{d}m = \rho(n)\mathrm{d}n$) can be expressed as

$$\hat{\rho}(m) = \frac{(D-2)^{D-1}}{m}\left(\frac{\pi T_H}{3m}\right)^{D-1} e^{m/T_H}$$

where the **Hagedorn temperature**[1] is defined as

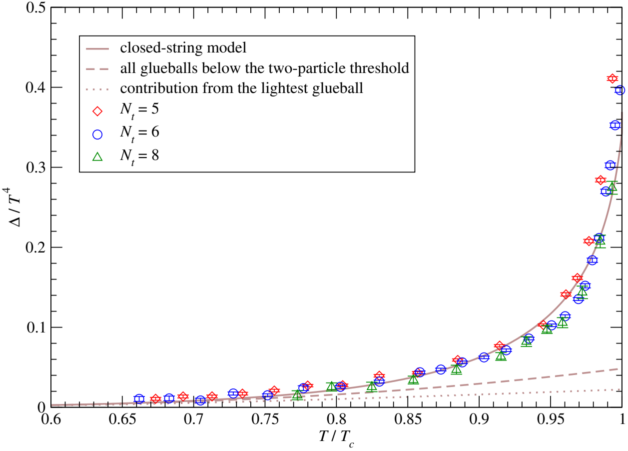
$$T_H = \sqrt{\frac{3\sigma}{\pi(D-2)}}.$$

Finally, the spectral density is used to account for all the states above the mass threshold $2m_{0^{++}}$:

$$\Delta = \sum_{m<2m_{0^{++}}} (2J+1)\Delta(m,T) + \int_{2m_{0^{++}}}^{\infty} \mathrm{d}m\, \hat{\rho}(m)\, \Delta(m,T)$$

[1] R. Hagedorn, Nuovo Cim. Suppl. **3**, 147 (1965)

SU(2): trace of energy-momentum tensor



SU(2) *vs.* SU(3)

The SU(3) case was studied for the first time in 2009 in the pioneering work of Meyer[1]. Now, using high precision lattice data[2] we are in the position to test the Hagedorn behaviour in a very stringent way.

With respect to SU(2)

- SU(3) has **complex** representations, thus glueballs have both $C = +1/-1$ and the spectrum contains **approximately twice** the number of glueballs than in the SU(2) case.
- SU(3) has a **first order** deconfining transition, so $T_c < T_H$.

In the effective string framework we can safely fix T_H at the expected Nambu-Goto value, i.e. $T_H = \sqrt{3\sigma/2\pi} \simeq 0.691\sqrt{\sigma}$. Lorentz invariance of the effective string tells us that this should be a very good approximation of the exact result.

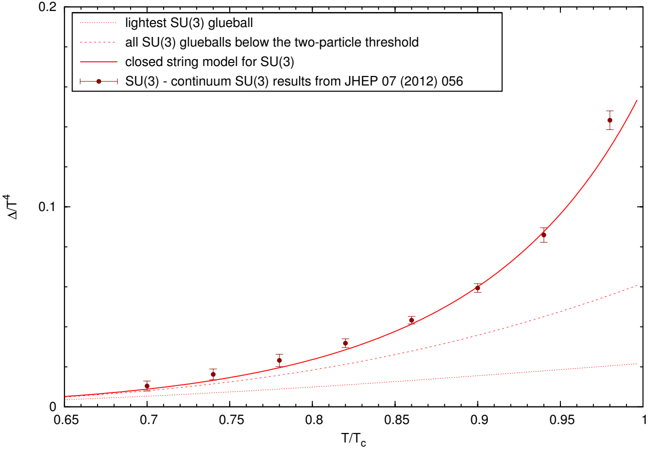
The relation between T_H and T_c is:

$$\frac{T_H}{T_c} = 1.098$$

[1] H. Meyer, *High-Precision Thermodynamics and Hagedorn Density of States*, 2009

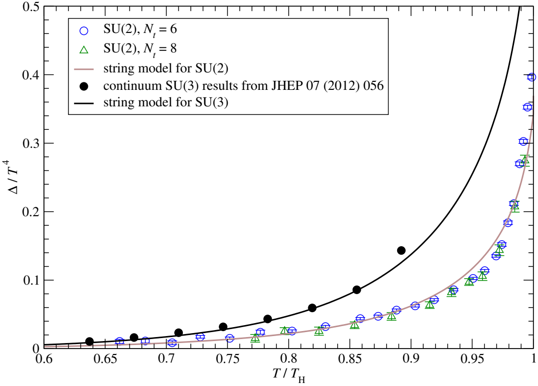
[2] Sz. Borsanyi *et al.*, *Precision SU(3) lattice thermodynamics for a large temperature range*, 2012

SU(3): trace of energy-momentum tensor



Also in this case the $m < 2m_{0^{++}}$ sector of the glueball spectrum is not enough to fit the behaviour of Δ/T^4, while including the whole Hagedorn spectrum we find again a remarkable agreement (with no free parameter!)

SU(2) *vs.* SU(3)



The doubling of the Hagedorn spectrum is clearly visible in the data!

Conclusions

- The thermodynamics of SU(2) and SU(3) Yang-Mills theories in $D = (3+1)$ is well described by a gas of **non-interacting** glueballs.
- The agreement is obtained only assuming a **Hagedorn spectrum** for the glueballs.
- The fine details of the spectrum, in particular the Hagedorn temperature, agree well with the predictions of the **Nambu-Goto effective string**.
- The results agree with previous findings[1] in $D = (2+1)$ SU(N) Yang Mills theories with $N = 2, 3, 4, 5, 6$.

[1] M. Caselle *et al.*, *Thermodynamics of SU(N) Yang-Mills theories in 2+1 dimensions I - The confining phase*, 2011

For SU(N) pure gauge theories on the lattice the dynamics is described by the standard Wilson action

$$S_W = \beta \sum_{p=sp,tp} (1 - \frac{1}{N} \mathrm{ReTr} U_p)$$

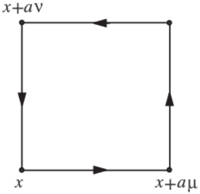
where U_P is the product of four U_μ SU(N) variables on the space-like or time-like plaquette P and $\beta = \frac{2N}{g^2}$.

The partition function is

$$Z = \int \prod_{x,\mu} \mathrm{d}U_\mu(x) e^{-S_W}$$

the expectation value of an observable A

$$\langle A \rangle = \frac{1}{Z} \int \prod_{n,\mu} \mathrm{d}U_\mu(n)\, A(U_\mu(n))\, e^{-S_W}$$

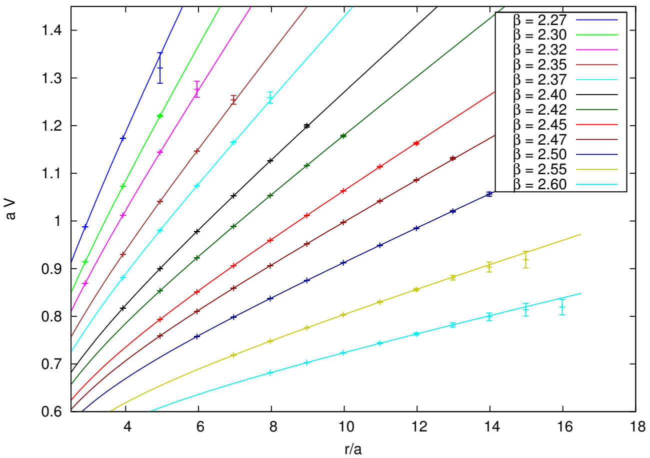


Scale setting

β	$r_{\min}/a$	σa^2	aV_0	χ^2_{red}
2.27	2.889	0.157(8)	0.626(14)	0.6
2.30	2.889	0.131(4)	0.627(30)	0.1
2.32	3.922	0.115(6)	0.627(32)	2.3
2.35	3.922	0.095(3)	0.623(20)	0.2
2.37	3.922	0.083(3)	0.621(18)	1.0
2.40	4.942	0.068(1)	0.617(10)	1.4
2.42	4.942	0.0593(4)	0.613(5)	0.1
2.45	4.942	0.0482(2)	0.608(4)	0.4
2.47	4.942	0.0420(4)	0.604(5)	0.3
2.50	5.954	0.0341(2)	0.599(2)	0.1
2.55	6.963	0.0243(13)	0.587(11)	0.2
2.60	7.967	0.0175(16)	0.575(16)	0.3

Results for the string tension in units of the inverse squared lattice spacing at different values of the Wilson action parameter β (first column). V was extracted from Polyakov loop correlators on lattices of temporal extent $L_t = 32a$.

Scale setting

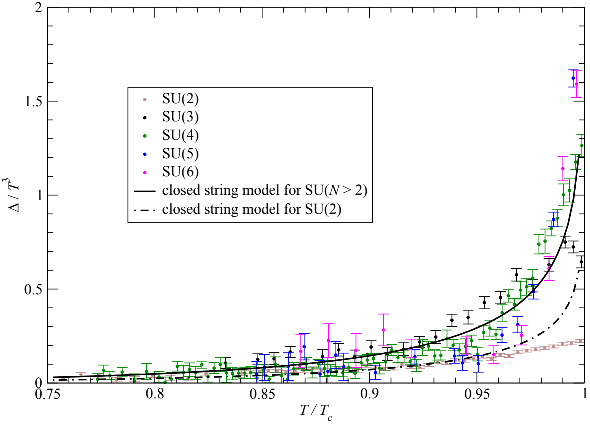


The same picture is confirmed by a study performed a few years ago[1] in (2+1) dimensional SU(N) Yang-Mills theories for $N = 2, 3, 4, 6$. Also in this case:

- a Hagedorn spectrum was mandatory to fit the thermodynamic data
- there was a jump between the SU(2) and the SU($N > 2$) case due to the doubling of the spectrum
- we had to fix the Hagedorn temperature to the Nambu-Goto value which, due to the different number of transverse degrees of freedom is different from the (3+1) dimensional one: $T_H = \sqrt{3\sigma/\pi} \simeq 0.977\sqrt{\sigma}$

[1] M. Caselle *et al.*, *Thermodynamics of SU(N) Yang-Mills theories in 2+1 dimensions I - The confining phase*, 2011

SU(N) Yang-Mills theories in $(2+1)$ dimensions



SU(2) pressure